Image Processing 1, Exercise 6

1 Linear Filters

[refresher] This exercise reviews the basic relation between an input and output signal and a transfer function by using the z-transform. It will help you understand the structure of a recursive filter which has an infinite impulse response.

(a) Let a filter be implicitly given by the recursive equation $y[k] = x[k] + a\, y[k-1]$, where x is the input sequence, y the output sequence with $k \in \mathbb{Z}$, $a \in \mathbb{C}$, and $|a| < 1$. Establish the transfer function H of the filter—as opposed to its frequency response.

Solution:

$$\begin{aligned} Y(z) &= X(z) + a\, z^{-1}\, Y(z) \\ &= \underbrace{\frac{1}{1 - a\, z^{-1}}}_{H(z)}\, X(z) \end{aligned}$$

(b) Use H to determine the explicit causal impulse response $h[k]$ in terms of $k \in \mathbb{Z}$.

Solution:

z	Space
$H(z) = \frac{1}{1-a\, z^{-1}}$	$h[k] = a^k\, u[k]$

2 Discrete Images and Linear Filtering

[intermediate] Characterization and application of a simple linear filter. You are expected to master this type of analysis. When possible, exploit the separability property and use the z-transform to make life easier.

Consider the following frequency response of a system:

$$H(e^{j\omega_1}, e^{j\omega_2}) = -4j[\sin(\omega_1 + \omega_2) - \sin(\omega_1 - \omega_2) - 3\sin(\omega_2)]$$

(a) Give the corresponding transfer function.

Solution: Recall $\sin\omega = \frac{e^{j\omega} - e^{-j\omega}}{2j}$ and $\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2j}$.

$$\begin{aligned} &H(e^{j\omega_1}, e^{j\omega_2}) = \\ &= -4j[(\frac{e^{j(\omega_1+\omega_2)} - e^{-j(\omega_1+\omega_2)}}{2j} - \frac{e^{j(\omega_1-\omega_2)} - e^{-j(\omega_1-\omega_2)}}{2j} - 3\frac{e^{j\omega_2} - e^{-j\omega_2}}{2j}] \\ &= -2(e^{j(\omega_1+\omega_2)} - e^{-j(\omega_1+\omega_2)} - e^{j(\omega_1-\omega_2)} + e^{-j(\omega_1-\omega_2)}) + 6(e^{j\omega_2} - e^{-j\omega_2}). \end{aligned}$$

$$\Rightarrow H(z_1, z_2) = -2(z_1 z_2 - z_1^{-1} z_2^{-1} - z_1 z_2^{-1} + z_1^{-1} z_2) + 6(z_2 - z_2^{-1})$$

(b) Give the impulse response as a sum of unit impulses.

Solution:

$$
\begin{aligned}
h[k_1,k_2] = \\
&-2(\delta[k_1+1,k_2+1]-\delta[k_1+1,k_2-1]+\delta[k_1-1,k_2+1]-\delta[k_1-1,k_2-1]) \\
&+6(\delta[k_1,k_2+1]-\delta[k_1,k_2-1])
\end{aligned}
$$

(c) Give the impulse response as an image.

Solution:

$$
h = \begin{bmatrix} -2 & 6 & -2 \\ 0 & \boxed{0} & 0 \\ 2 & -6 & 2 \end{bmatrix}
$$

(d) Give the mask as an image.

Solution:

$$
w = \begin{bmatrix} 2 & -6 & 2 \\ 0 & \boxed{0} & 0 \\ -2 & 6 & -2 \end{bmatrix}
$$

(e) It turns out that this filter is separable. Let $H_2(e^{j\omega}) = -8\mathrm{j}\sin(\omega)$, give $h_1[k]$ as a sum of unit impulses such that $h[k_1,k_2] = h_1[k_1]h_2[k_2]$.

Solution:

$$
\begin{aligned}
&H_2(e^{j\omega_2}) = -8\frac{e^{j\omega_2}-e^{-j\omega_2}}{2} = -4(e^{j\omega_2}-e^{-j\omega_2}) \\
&\Leftrightarrow H_2(z_2) = -4(z_2 - z_2^{-1}) \\
&\Leftrightarrow h_2[k_2] = -4(\delta[k_2+1]-\delta[k_2-1]) \\
&\Leftrightarrow h_1[k_1] = \frac{1}{2}\delta[k_1+1] - \frac{3}{2}\delta[k_1] + \frac{1}{2}\delta[k_1-1]
\end{aligned}
$$

(f) Consider the 2D, zero-padded image $f = \begin{bmatrix} \boxed{1} & 2 \end{bmatrix}$ and the 2D version of h_1, which we define as $g_1[k_1,k_2] = h_1[k_1]\delta[k_2]$. Compute the 2D convolution $(g_1 * f)$ and give the result as an image.

Solution:

$$
\begin{aligned}
&g_1 = \frac{1}{2}\begin{bmatrix} 1 & \boxed{\text{-3}} & 1 \end{bmatrix} \Rightarrow G_1(z_1,z_2) = \frac{1}{2}(z_1 - 3 + z_1^{-1}) \\
&f = \begin{bmatrix} \boxed{1} & 2 \end{bmatrix} \Rightarrow F(z_1,z_2) = 1 + 2z_1^{-1} \\
&F(z_1,z_2)G_1(z_1,z_2) = \frac{1}{2}(z_1 - 1 - 5z_1^{-1} + 2z_1^{-2}) \Rightarrow g_1 * f = \frac{1}{2}\begin{bmatrix} 1 & \boxed{\text{-1}} & -5 & 2 \end{bmatrix}
\end{aligned}
$$

(g) Let the 2D version of h_2 be defined as $g_2[k_1,k_2] = \delta[k_1]h_2[k_2]$. Compute $(g_2 * g_1) * f$ and give the result as an image.

Solution:

$$g_2 = 2\begin{bmatrix} -2 \\ \boxed{0} \\ 2 \end{bmatrix} \Rightarrow G_2(z_1, z_2) = 2(-2z_2 + 2z_2^{-1})$$

$$G_2(z_1, z_2)G_1(z_1, z_2)F(z_1, z_2) = (-2z_2 + 2z_2^{-1})(z_1 - 1 - 5z_1^{-1} + 2z_1^{-2})$$

$$\Rightarrow g_2 * g_1 * f = \begin{bmatrix} -2 & 2 & 10 & -4 \\ 0 & \boxed{0} & 0 & 0 \\ 2 & -2 & -10 & 4 \end{bmatrix}$$

(h) Compute $h * f$ and give the result as an image.

Solution: Same as previous question.

(i) Compute the modulus of $H(e^{j\omega_1}, e^{j\omega_2})$. Express the result using trigonometric functions.

Solution:
Notice $\hat{h} = H(e^{j\omega_1}, e^{j\omega_2})$ is the Fourier Transform of h.

$$|\hat{h}(\omega_1, \omega_2)| = |\hat{h}(\omega_1)| \cdot |\hat{h}(\omega_2)| = |H_1(e^{j\omega_1})| \cdot |H_2(e^{j\omega_2})|$$

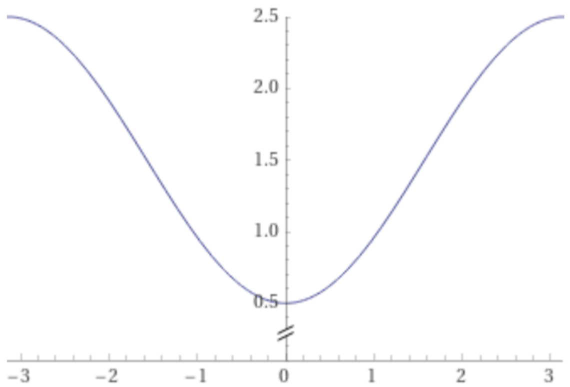
$$H_1(z) = \frac{1}{2}(z_1 - 3 + z_1^{-1}) \Leftrightarrow H_1(e^{j\omega_1}) = \frac{1}{2}(e^{j\omega_1} + e^{-j\omega_1} - 3) = \cos(w_1) - \frac{3}{2}$$

$$H_2(e^{j\omega_2}) = -8\mathrm{j}\sin(\omega_2)$$

$$\Rightarrow |H_1(e^{j\omega_1})| \cdot |H_2(e^{j\omega_2})| = |\cos(w_1) - \frac{3}{2}| \cdot |8\sin(\omega_2)|$$

(j) Indicate the type (i.e. lowpass, highpass, bandpass, allpass, bandcut, allcut) of the filter h_1 and of h_2.

Solution: h_1 is high pass.



h_2 is band pass.