Image Processing 1, Exercise 5

1 z-Transform and the Kronecker-Delta

[refresher] Determination of specific z-transforms.

(a) Give a sequence that has z^2 as its z-transform.

Solution:

z	Space	Property
z^{-k_0}	$\delta[k-k_0]$	Kronecker
z^2	$\delta[k+2]$	

Please refer to the table of 1D z-transform pairs in the Appendix.

(b) Complete the following table

One-dimensional sequence	Expression for $k \in \mathbb{Z}$	z-transform
$\{\ldots,0,0,0,0,0,\boxed{2},0,0,0,0,0,\ldots\}$	$2\,\delta[k]$	**?**
$\{\ldots,0,0,0,0,1,\boxed{0},0,0,0,0,0,\ldots\}$	**?**	z
?	$\delta[k+1]+5\,\delta[k-3]$	$z+5\,z^{-3}$

Solution:

One-dimensional sequence	Expression for $k \in \mathbb{Z}$	z-transform
$\{\ldots,0,0,0,0,0,\boxed{2},0,0,0,0,0,\ldots\}$	$2\,\delta[k]$	2
$\{\ldots,0,0,0,0,1,\boxed{0},0,0,0,0,0,\ldots\}$	$\delta[k+1]$	z
$\{\ldots,0,0,0,0,1,\boxed{0},0,0,5,0,0,\ldots\}$	$\delta[k+1]+5\,\delta[k-3]$	$z+5\,z^{-3}$

2 Basic properties of the z-Transform

[basic] Derivations of basic properties of the z-transform. Use the definition of the z-transform.

(a) Show that the 2D z-transform $A(z_1,z_2)$ of the real sequence $a[k_1,k_2]$ is such that $A(z_1,z_2) = (A(z_1^*,z_2^*))^*$ for $z_1,z_2 \in \mathbb{C}\setminus\{0\}$.

Solution:

$$\begin{aligned}
(A(z_1^*,z_2^*))^* &= \left(\sum_{k_1,k_2\in\mathbb{Z}} a[k_1,k_2]\,(z_1^*)^{-k_1}\,(z_2^*)^{-k_2}\right)^* \\
&= \sum_{k_1,k_2\in\mathbb{Z}} \underbrace{(a[k_1,k_2])^*}_{a[k_1,k_2]}\,z_1^{-k_1}\,z_2^{-k_2} \\
&= A(z_1,z_2)
\end{aligned}$$

(b) Let $A(z_1,z_2)$ be the 2D z-transform of the nonnegative sequence $a \in \ell_1(\mathbb{Z}^2)$. Use $A(\cdot,\cdot)$ to express $\|a\|_{\ell_1(\mathbb{Z}^2)}$.

Solution:

$$
\begin{aligned}
A(z_1, z_2) &= \sum_{k_1,k_2\in\mathbb{Z}} \underbrace{a[k_1,k_2]}_{\geq 0}\, z_1^{-k_1}\, z_2^{-k_2} \\
&= \sum_{k_1,k_2\in\mathbb{Z}} |a[k_1,k_2]|\, z_1^{-k_1}\, z_2^{-k_2}
\end{aligned}
$$

$$
\begin{aligned}
A(1,1) &= \sum_{k_1,k_2\in\mathbb{Z}} |a[k_1,k_2]| \\
&= \|a\|_{\ell_1(\mathbb{Z}^2)}
\end{aligned}
$$

3 Digital Filtering

[intermediate] The z-transform is a powerful tool for the characterization of digital filters. In this exercise, you will use it to obtain the frequency response of filters to determine if they are LP, BP, HP, etc.

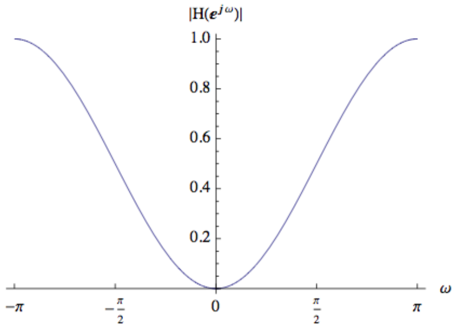
(a) Give the impulse response h_1 of the filter described by the mask $w_1 = \frac{1}{4}\begin{bmatrix} -1 & \boxed{2} & -1 \end{bmatrix}$ as a sum of unit impulses. Compute its z-transform. Plot the magnitude of the frequency response of this filter in the interval $[-\pi, \pi]$. Is this filter all-pass (AP), low-pass (LP), band-pass (BP), high-pass (HP), band-cut (BC), or all-cut (AC)?

Solution:

$$h_1[k] = -\frac{1}{4}\delta[k+1] + \frac{1}{2}\delta[k] - \frac{1}{4}\delta[k-1]$$

$$
\begin{aligned}
H(z) &= -\frac{1}{4}\,z^{-1} + \frac{1}{2} - \frac{1}{4}\,z \\
&= \frac{1}{2} - \frac{1}{2}\,\frac{z+z^{-1}}{2}
\end{aligned}
$$

$$H(\mathrm{e}^{\mathrm{j}\,\omega}) = \frac{1}{2} - \frac{1}{2}\cos(\omega)$$



$$\left(\left(\left|H(\mathrm{e}^{\mathrm{j}\,0})\right| = 0\right) \text{ and } \left(\left|H(\mathrm{e}^{\mathrm{j}\,\pi})\right| = 1\right)\right) \Rightarrow \text{HP}$$

(b) We define the 2D impulse response $h[k_1, k_2] = h_1[k_1]h_1[k_2]$, with h_1 from (a). Provide an expression for $h[k_1, k_2]$ as the sum of unit impulses. Compute the 2D z-transform of this filter. Is this filter all-pass (AP), low-pass (LP), band-pass (BP), high-pass (HP), band-cut (BC), or all-cut (AC)?

Solution:

$$\begin{aligned} h[k_1,k_2] = & \frac{1}{16}\delta[k_1+1,k_2+1] - \frac{1}{8}\delta[k_1,k_2+1] + \frac{1}{16}\delta[k_1-1,k_2+1] \\ & -\frac{1}{8}\delta[k_1+1,k_2] + \frac{1}{4}\delta[k_1,k_2] - \frac{1}{8}\delta[k_1-1,k_2] \\ & +\frac{1}{16}\delta[k_1+1,k_2-1] - \frac{1}{8}\delta[k_1,k_2-1] + \frac{1}{16}\delta[k_1-1,k_2-1] \end{aligned}$$

Since the filter is separable, the 2D z-transform is the product of the 1D z-transforms of the 1D filters. That is,

$$H(z_1,z_2) = \left(\frac{1}{2} - \frac{1}{2}\,\frac{z_1+z_1^{-1}}{2}\right)\left(\frac{1}{2} - \frac{1}{2}\,\frac{z_2+z_2^{-1}}{2}\right)$$

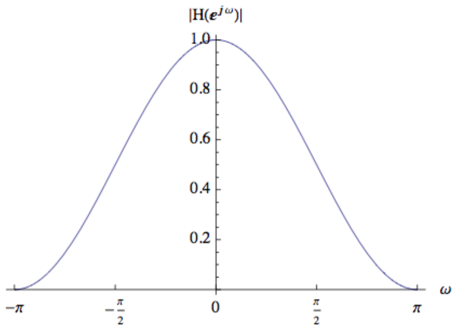
Since this filter is the product of two 1D HP filters, the 2D filter is HP.

(c) The frequency response of a discrete filter is given by $H(\mathrm{e}^{\mathrm{j}\,\omega}) = \left(\cos(\frac{\omega}{2})\right)^2$, with $\omega \in \mathbb{R}$. Determine the corresponding filter mask w. Always indicate the origin. Specify the type of this filter (AP, LP, BP, HP, BC, AC). To justify your answer, plot $|H|$ in the main frequency domain.

Solution:

$$\begin{aligned} H(\mathrm{e}^{\mathrm{j}\,\omega}) &= \left(\cos(\frac{\omega}{2})\right)^2 \\ &= \left(\frac{\mathrm{e}^{\mathrm{j}\,\frac{\omega}{2}} + \mathrm{e}^{-\mathrm{j}\,\frac{\omega}{2}}}{2}\right)^2 \\ &= \frac{1}{4}\,\mathrm{e}^{-\mathrm{j}\,\omega} + \frac{1}{2} + \frac{1}{4}\,\mathrm{e}^{\mathrm{j}\,\omega} \end{aligned}$$

$$w = \frac{1}{4}\left[\begin{array}{ccc} 1 & \boxed{2} & 1 \end{array}\right]$$



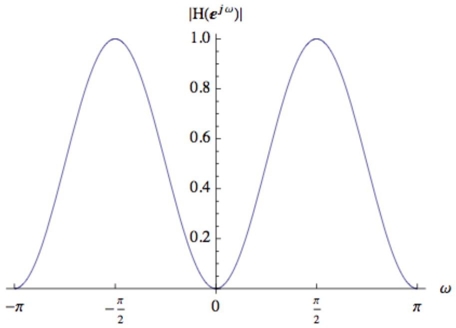
$$\left(\left(\left|H(\mathrm{e}^{\mathrm{j}\,0})\right| = 1\right) \text{ and } \left(\left|H(\mathrm{e}^{\mathrm{j}\,\pi})\right| = 0\right)\right) \Rightarrow \text{LP}$$

(d) Give the z-transform of the filter described by the impulse response $h = \frac{1}{4}\left[\begin{array}{ccccc} -1 & 0 & \boxed{2} & 0 & -1 \end{array}\right]$. Plot the magnitude of the frequency response of this filter in the interval $[-\pi, \pi]$. The impulse response of this filter is very similar to the mask of the filter of the previous question (a), up to the introduction of a few zero elements. But is it the same type of filter (AP, LP, BP, HP, BC, AC)?

Solution:

$$\begin{aligned} H(z) &= -\frac{1}{4}\,z^{-2} + \frac{1}{2} - \frac{1}{4}\,z^2 \\ &= \frac{1}{2} - \frac{1}{2}\,\frac{z^2+z^{-2}}{2} \end{aligned}$$

$$H(\mathrm{e}^{\mathrm{j}\,\omega}) \quad = \quad \frac{1}{2} - \frac{1}{2}\,\cos(2\,\omega)$$



$$\left(\left(\left|H(\mathrm{e}^{\mathrm{j}\,0})\right| = 0\right) \text{ and } \left(\left|H(\mathrm{e}^{\mathrm{j}\,\frac{\pi}{2}})\right| = 1\right) \text{ and } \left(\left|H(\mathrm{e}^{\mathrm{j}\,\pi})\right| = 0\right)\right) \Rightarrow \mathrm{BP}$$