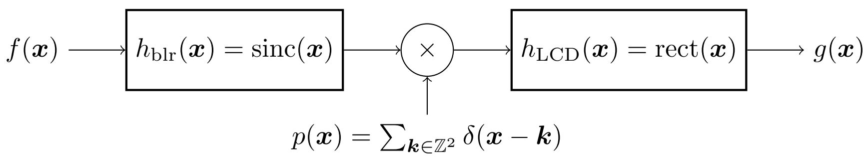
Image Processing 1, Exercise 4

1 Acquisition System

[basic] In this exercise, we show you how to model a real acquisition system using basic modules from the course. You will learn how to express an acquisition system in three different manners: a block diagram, a space-domain formula, and a frequency-domain formula.

The acquisition system of some camera is blurring an otherwise perfect image $f \in \mathbb{R}^2$. The blurring is modeled by an ideal filter with impulse response $h_{\mathrm{blr}}(x_1, x_2) = \mathrm{sinc}(x_1)\mathrm{sinc}(x_2)$. The blurred image is sampled by the sampling function $p(x) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \delta(\boldsymbol{x} - \boldsymbol{k})$. Finally, the sampled image is displayed on an LCD display which has $h_{\mathrm{LCD}}(x_1, x_2) = \mathrm{rect}(x_1)\,\mathrm{rect}(x_2)$ as its impulse response. The displayed image is $g \in \mathbb{R}^2$.

(a) Draw a block diagram of the whole system, you can refer to page 2-20 of the lecture slides of Chapter 2a for an example of a block diagram.



Solution:
Note: We are not being very formal with block diagrams in this course. Basically: signals are on the lines. Boxes representing filtering operations, with either the impulse response or transfer function of the filter written in the box. Arithmetic operations are in circles.

(b) Express g in terms of f.

Solution:

$$g(\boldsymbol{x}) = \left(\left(\sum_{\boldsymbol{k} \in \mathbb{Z}^2} (f * \mathrm{sinc})\,(\cdot)\delta(\cdot - \boldsymbol{k}) \right) * \mathrm{rect}(\cdot) \right) (\boldsymbol{x}).$$

Note: we read this off the block diagram, or we can get it from the description of the system given to use in the problem. See how the block diagram goes left to right, while the equation goes inside to outside.

(c) Express $\hat{g}$ in terms of $\hat{f}$.

Solution:

$$\hat{g}(\boldsymbol{\omega}) = \left(\sum_{\boldsymbol{n} \in \mathbb{Z}^2} \hat{f}(\boldsymbol{\omega} - 2\pi\boldsymbol{n})\mathrm{rect}\left(\frac{\boldsymbol{\omega} - 2\pi\boldsymbol{n}}{2\pi} \right) \right) \mathrm{sinc}\left(\frac{\boldsymbol{\omega}}{2\pi} \right).$$

Explanation: We use Fourier transform properties from the table along with rearranging some terms:
Note that $g = q * \mathrm{rect}$ with

$$q(\boldsymbol{x}) = (f * \mathrm{sinc})\,(\boldsymbol{x}) \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \delta(\boldsymbol{x} - \boldsymbol{k})$$

By the convolution theorem, $\hat{g}(\boldsymbol{\omega}) = \hat{q}(\boldsymbol{\omega})\widehat{\mathrm{rect}}(\boldsymbol{\omega})$. We have

$$\begin{aligned}
\hat{q} &= \mathcal{F}\left\{(f * \mathrm{sinc})\,(\cdot) \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \delta(\cdot - \boldsymbol{k})\right\} \\
&= \frac{1}{(2\pi)^2}\mathcal{F}\{f * \mathrm{sinc}\} * (2\pi)^2 \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \delta(\cdot - 2\pi\boldsymbol{n}) \quad \text{(Multiplication Property)} \\
&= \mathcal{F}\{f * \mathrm{sinc}\} * \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \delta(\cdot - 2\pi\boldsymbol{n}) \\
&= (\hat{f}(\cdot)\,\widehat{\mathrm{sinc}}(\cdot)) * \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \delta(\cdot - 2\pi\boldsymbol{n}) \quad \text{(Convolution Theorem)} \\
&= (\hat{f}(\cdot)\ \mathrm{rect}(\cdot/(2\pi))) * \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \delta(\cdot - 2\pi\boldsymbol{n}) \\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}(\boldsymbol{\omega} - 2\pi\boldsymbol{n})\,\mathrm{rect}\left(\frac{\boldsymbol{\omega} - 2\pi\boldsymbol{n}}{2\pi}\right)
\end{aligned}$$

Finally, since $\widehat{\mathrm{rect}}(\boldsymbol{\omega}) = \mathrm{sinc}(\boldsymbol{\omega}/(2\pi))$, we find

$$\hat{g}(\boldsymbol{\omega}) = \left(\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}(\boldsymbol{\omega} - 2\pi\boldsymbol{n})\,\mathrm{rect}\left(\frac{\boldsymbol{\omega} - 2\pi\boldsymbol{n}}{2\pi}\right)\right)\mathrm{sinc}\left(\frac{\boldsymbol{\omega}}{2\pi}\right).$$

2 Lloyd-Max

[intermediate] In the lecture, you have learned that the Lloyd-Max quantizer achieves minimum error. In this exercise, you will prove that it is also unbiased.

Let $\tilde{f} = Q(f)$ be a quantized version of the image f, where Q is the Lloyd-Max quantizer. Show that $E\{f\} = E\{\tilde{f}\}$; that is, applying Lloyd-Max quantization does not change the mean of an image.

Solution:

$$\begin{aligned}
E\{\tilde{f}\} &= \int_{t_0}^{t_K} Q(f)\,p(f)\,\mathrm{d}f \\
&= \sum_{k=0}^{K-1} \int_{t_k}^{t_{k+1}} r_k\,p(f)\,\mathrm{d}f \\
&= \sum_{k=0}^{K-1} \frac{\int_{t_k}^{t_{k+1}} f\,p(f)\,\mathrm{d}f}{\int_{t_k}^{t_{k+1}} p(f)\,\mathrm{d}f} \int_{t_k}^{t_{k+1}} p(f)\,\mathrm{d}f \\
&= \sum_{k=0}^{K-1} \int_{t_k}^{t_{k+1}} f\,p(f)\,\mathrm{d}f \\
&= \int_{t_0}^{t_K} f\,p(f)\,\mathrm{d}f \\
&= E\{f\}
\end{aligned}$$

3 Difference Equations

[intermediate] We recall the 1D z-transform from signals and systems and then use the 2D z-transform to analyze a 2D difference equation. Difference equations express the current output of a system recursively as a weighted sum of previous inputs and outputs. They are convenient for designing fast algorithms. Inverting the z-transform then let you find the underlying impulse response.

(a) Let f be defined recursively as $f[k] = \delta[k] + a\,f[k-1]$. Use the z-transform to compute an explicit expression for $f[k]$ assuming that it is causal (ROC $= \{z : |z| > |a|\}$).

Solution: We take the z-transform, solve for $F(z)$, then use inspection to get $f[k]$.

$$\begin{aligned} f[k] = \delta[k] + af[k-1] \overset{z}{\longleftrightarrow} F(z) &\overset{(a)}{=} 1 + a\,z^{-1}F(z) \\ &\overset{(b)}{\to} F(z) = \frac{1}{1 - a\,z^{-1}} \\ &\overset{(c)}{\overset{z}{\longleftrightarrow}} f[k] = u[k]a^k, \end{aligned}$$

where (a) comes from the delay property of the z-transform; (b) comes from rearranging terms; and (c) from the z-transform of the causal exponential in the table.

(b) Let $f[\boldsymbol{k}] = u[k_1+3]u[4-k_2]e^{2k_2-k_1}$. Give its $\boldsymbol{z}$-transform.

Solution:

$$\begin{aligned} f[\boldsymbol{k}] &= u[k_1+3]u[4-k_2]e^{2k_2-k_1} \\ &\overset{(a)}{=} (e^{-1})^{-3}\ \underbrace{(e^{-1})^{k_1+3}u[k_1+3]}_{\text{shifted causal exponential}}\ (e^{-2})^{-4}\ \underbrace{(e^{-2})^{4-k_2}u[4-k_2]}_{\text{shifted anti-causal exponential}} \\ &\overset{(b)}{\overset{z}{\longleftrightarrow}} e^3\frac{z_1^3}{1-e^{-1}z_1^{-1}} \cdot -e^{10}\frac{z_2^{-5}}{1-e^2z_2^{-1}}, \quad \text{ROC: } e^{-1} < |z_1| \quad \text{and} \quad |z_2| < e^2 \end{aligned}$$

where (a) comes from rearranging terms and (b) from the z-transforms of the causal and anti-causal exponentials given in the notes and the separability property of the z-transform. The ROC is the intersection of the two individual ROCs.

(c) Let $f[\boldsymbol{k}] = \delta[k_1,k_2] + af[k_1-1,k_2] + bf[k_1,k_2-1] - abf[k_1-1,k_2-1]$. Use the z-transform to compute an explicit expression for $f[\boldsymbol{k}]$, again assuming that it is causal.

Solution: We follow a similar procedure as in 1D.

$$\begin{aligned} f[\boldsymbol{k}] &= \delta[k_1,k_2] + af[k_1-1,k_2] + bf[k_1,k_2-1] - abf[k_1-1,k_2-1] \\ &\overset{(a)}{\overset{z}{\longleftrightarrow}} F[\boldsymbol{z}] = 1 + az_1^{-1}F[\boldsymbol{z}] + bz_2^{-1}F[\boldsymbol{z}] - abz_1^{-1}z_2^{-1}F[\boldsymbol{z}] \\ &\overset{(b)}{\to} F[\boldsymbol{z}] = \frac{1}{1 - az_1^{-1} - bz_2^{-1} + abz_1^{-1}z_2^{-1}} \\ &\overset{(c)}{=} \frac{1}{(1-az_1^{-1})(1-bz_2^{-1})} \\ &\overset{(d)}{\overset{z}{\longleftrightarrow}} f[\boldsymbol{k}] = a^{k_1}u[k_1]b^{k_2}u[k_2], \end{aligned}$$

where (a) comes from the delay property of the z-transform; (b) is a rearrangement; (c) comes from factoring the denominator; and (d) comes from separability.