Image Processing 1, Exercise 3

1 The Dirac Delta

[basic] This exercise is an exploration of convolutions with the 2D Dirac delta to help you understand and build your intuition about 2D convolutions. As usual, you can rely on the tables in the back of your course notes.

Let $\boldsymbol{x}$ be a vector in $\mathbb{R}^2$.

(a) Let $f(\boldsymbol{x}) = \delta\left(\boldsymbol{x} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$. Sketch $\mathrm{rect} * f$.

(b) Let $f(\boldsymbol{x}) = \delta\left(-\boldsymbol{x} - \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$. Use the Fourier transform to show that $\delta(\boldsymbol{x}) = \delta(-\boldsymbol{x})$ and use this fact to sketch $\mathrm{rect} * f$.

(c) Fix $\boldsymbol{x}_0 \in \mathbb{R}^2$. Prove that $f * \delta(\cdot - \boldsymbol{x}_0) = f(\cdot - \boldsymbol{x}_0)$ using the Fourier transform.

(d) Let $\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Use the result of (c) to compute $g = \delta(\cdot - \boldsymbol{x}_1) * \delta(\boldsymbol{x}_2 + \cdot)$. Then, plot $g * \mathrm{rect}$.

(e) Let $f \in L_1(\mathbb{R}^d)$, which ensures that the Fourier transform $\hat{f}$ is a continuous function. In this part you will prove the identity

$$(2\pi)^d \hat{f}(\boldsymbol{\omega})\delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0) = (2\pi)^d \hat{f}(\boldsymbol{\omega}_0)\delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0), \tag{1}$$

which is the dual form of the Multiplication identity on Slide 23 in Chapter 1. We will proceed in two steps:

(i) First prove that $f * e^{\mathrm{j}\langle \boldsymbol{\omega}_0, \cdot \rangle} = \hat{f}(\boldsymbol{\omega}_0) e^{\mathrm{j}\langle \boldsymbol{\omega}_0, \cdot \rangle}$ by an explicit calculation.

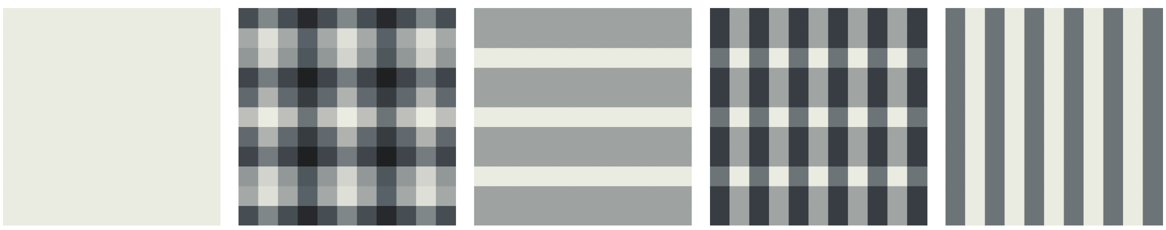
(ii) Prove the identity (1) using the result of (i). *Hint: Take the inverse Fourier transform of the left-hand side of (1).*

2 Aliasing

[intermediate] Aliasing must be considered in all practical systems that involve sampling continuous signals, e.g., cameras and scanning microscopes. In this exercise, we illustrate the effect of aliasing in the space domain and analyse the effect in the Fourier domain.

The image $f(\boldsymbol{x}) = \cos\left(2\pi \frac{3}{2} x_1\right) + \cos\left(2\pi \frac{5}{3} x_2\right)$, with $\boldsymbol{x} \in \mathbb{R}^2$, is sampled at $\boldsymbol{x} = \boldsymbol{k}T$ with $\boldsymbol{k} \in \mathbb{Z}^2$. The results for various values of T are shown in Figure 1.

Remark: the origin of the coordinate system is at the top left corner of the image, x_1-axis goes horizontally to the right hand side and x_2-axis goes vertically to the bottom of the page.



(i) (ii) (iii) (iv) (v)

Figure 1: Sampled versions of $f(\boldsymbol{x})$.

(a) Match each of the images in Figure 1 with the correct value of T from the list $T = 1/6$, 1, 2, 3, and 6.

(b) Determine the Fourier transforms of $g(x) = \cos(2\pi \frac{3}{2}\, x)$ and $h(x) = \cos(2\pi \frac{5}{3}\, x)$, then plot them in the range $[-4\pi, 4\pi]$.

(c) Let $g_1(x)$ and $h_1(x)$ be the signals formed by multiplying functions g and h from (b) respectively with a Dirac comb (with sampling step $T = 1$). Determine the Fourier transforms of g_1 and h_1, plot them in the range $[-4\pi, 4\pi]$.

(d) Based on your previous answers, which (if either) of g and h can be reconstructed exactly from its samples with $T = 1$?

(e) What is the range of T such that both g and h can be perfectly reconstructed from their samples?