

Image Processing 1, Exercise 3

1 The Dirac Delta

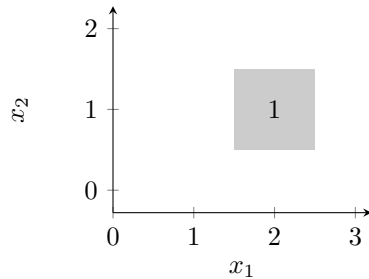
[basic] This exercise is an exploration of convolutions with the 2D Dirac delta to help you understand and build your intuition about 2D convolutions. As usual, you can rely on the tables in the back of your course notes.

Let $\mathbf{x}$ be a vector in $\mathbb{R}^2$.

- (a) Let $f(\mathbf{x}) = \delta\left(\mathbf{x} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$. Sketch $\text{rect} * f$.

Solution: Use the convolution property of the Dirac delta! Let us denote $\delta(\mathbf{x} - \mathbf{x}_0) = \delta_{\mathbf{x}_0}$. Then the convolution property tells us: $(g * \delta_{\mathbf{x}_0})(\mathbf{x}) = g(\mathbf{x} - \mathbf{x}_0)$.

Thus we have $(\text{rect} * f)(\mathbf{x}) = \text{rect}\left(\mathbf{x} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$. Hence the support of $\text{rect} * f$ is a unit square centered at $(2, 1)$.

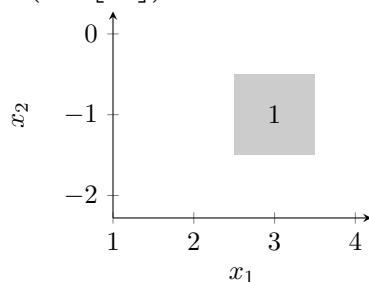


- (b) Let $f(\mathbf{x}) = \delta\left(-\mathbf{x} - \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$. Use the Fourier transform to show that $\delta(\mathbf{x}) = \delta(-\mathbf{x})$ and use this fact to sketch $\text{rect} * f$.

Solution:

Image space	Fourier space
$f(\mathbf{x})$	$\hat{f}(\boldsymbol{\omega})$
$f(-\mathbf{x})$	$\hat{f}(-\boldsymbol{\omega})$
$\delta(\mathbf{x})$	1
$\delta(-\mathbf{x})$	1

Therefore, we deduce that $\delta(\mathbf{x}) = \delta(-\mathbf{x})$. Hence, $\delta\left(-\mathbf{x} - \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right) = \delta\left(\mathbf{x} + \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right) = \delta\left(\mathbf{x} - \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$. Then use the same strategy as in (a).

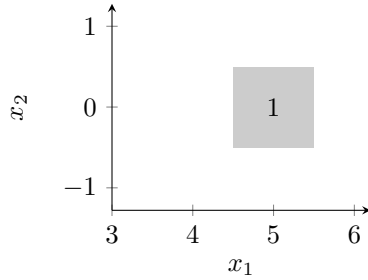


- (c) Fix $\mathbf{x}_0 \in \mathbb{R}^2$. Prove that $f * \delta(\cdot - \mathbf{x}_0) = f(\cdot - \mathbf{x}_0)$ using the Fourier transform.

Solution: By the convolution theorem, we have $f * \delta(\cdot - \mathbf{x}_0) \xleftrightarrow{\mathcal{F}} \hat{f}(\boldsymbol{\omega})e^{-j\langle \boldsymbol{\omega}, \mathbf{x}_0 \rangle}$. Taking the inverse Fourier transform and using the translation/modulation property of the Fourier transform (Chapter 1, Slide 26) we find $f * \delta(\cdot - \mathbf{x}_0) = f(\cdot - \mathbf{x}_0)$.

- (d) Let $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Use the result of (c) to compute $g = \delta(\cdot - \mathbf{x}_1) * \delta(\mathbf{x}_2 + \cdot)$. Then, plot $g * \text{rect}$.

Solution: Note that $\delta(\mathbf{x}_2 + \cdot) = \delta(\cdot + \mathbf{x}_2) = \delta(\cdot - (-\mathbf{x}_2))$. Therefore, using the result of (c) we find $g(\mathbf{x}) = \delta\left(\mathbf{x} - \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right)$. The plot of $g * \text{rect}$ is below.



- (e) Let $f \in L_1(\mathbb{R}^d)$, which ensures that the Fourier transform $\hat{f}$ is a continuous function. In this part you will prove the identity

$$(2\pi)^d \hat{f}(\boldsymbol{\omega}) \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0) = (2\pi)^d \hat{f}(\boldsymbol{\omega}_0) \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0), \quad (1)$$

which is the dual form of the Multiplication identity on Slide 23 in Chapter 1. We will proceed in two steps:

- (i) First prove that $f * e^{j\langle \boldsymbol{\omega}_0, \cdot \rangle} = \hat{f}(\boldsymbol{\omega}_0) e^{j\langle \boldsymbol{\omega}_0, \cdot \rangle}$ by an explicit calculation.
- (ii) Prove the identity (1) using the result of (i). *Hint: Take the inverse Fourier transform of the left-hand side of (1).*

Solution: (i) Since $f \in L_1(\mathbb{R}^d)$, we see that

$$(f * e^{j\langle \boldsymbol{\omega}_0, \cdot \rangle})(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{y}) e^{j\langle \boldsymbol{\omega}_0, \mathbf{x} - \mathbf{y} \rangle} d\mathbf{y} = e^{j\langle \boldsymbol{\omega}_0, \mathbf{x} \rangle} \int_{\mathbb{R}^d} f(\mathbf{y}) e^{-j\langle \boldsymbol{\omega}_0, \mathbf{y} \rangle} d\mathbf{y} = e^{j\langle \boldsymbol{\omega}_0, \mathbf{x} \rangle} \hat{f}(\boldsymbol{\omega}_0).$$

(ii) Taking the inverse Fourier transform of the left-hand side is $f * e^{j\langle \boldsymbol{\omega}_0, \cdot \rangle}$ (convolution theorem). By (i) it equals $\hat{f}(\boldsymbol{\omega}_0) e^{j\langle \boldsymbol{\omega}_0, \cdot \rangle}$. Taking the Fourier transform of this yields the right-hand side.

2 Aliasing

[intermediate] Aliasing must be considered in all practical systems that involve sampling continuous signals, e.g., cameras and scanning microscopes. In this exercise, we illustrate the effect of aliasing in the space domain and analyse the effect in the Fourier domain.

The image $f(\mathbf{x}) = \cos(2\pi \frac{3}{2} x_1) + \cos(2\pi \frac{5}{3} x_2)$, with $\mathbf{x} \in \mathbb{R}^2$, is sampled at $\mathbf{x} = \mathbf{k}T$ with $\mathbf{k} \in \mathbb{Z}^2$. The results for various values of T are shown in Figure 1.

Remark: the origin of the coordinate system is at the top left corner of the image, x_1 -axis goes horizontally to the right hand side and x_2 -axis goes vertically to the bottom of the page.

- (a) Match each of the images in Figure 1 with the correct value of T from the list $T = 1/6, 1, 2, 3$, and 6.

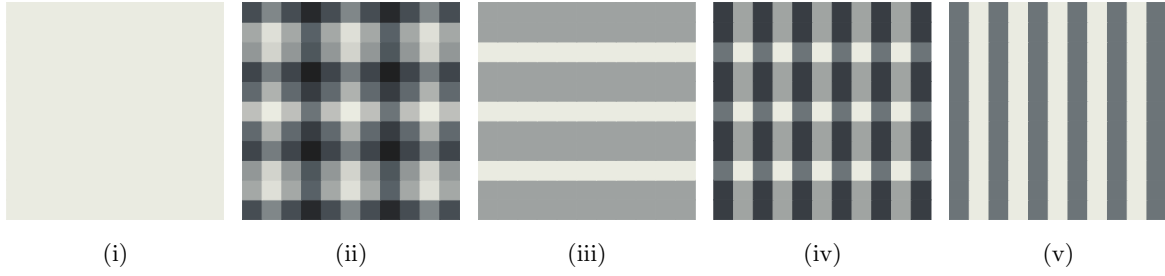


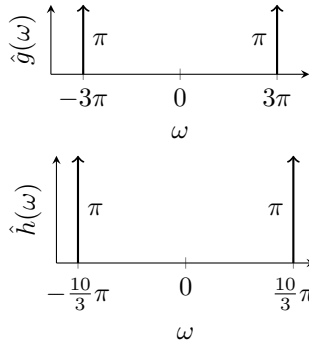
Figure 1: Sampled versions of $f(\mathbf{x})$.

Solution: Answer: $T = 6, 1/6, 2, 1$, and 3 .

How to solve: Start with (i), (iii), and (v), since these are the ones that are constant in either the x_1 or x_2 direction. For example, only $T = 6$ makes both $\cos(2\pi \frac{3}{2} k_1 T)$ and $\cos(2\pi \frac{5}{3} k_2 T)$ constant, so (i) is 6. Similar logic works for (iii) and (v). Then, note that (iv) has a period of 2 in the x_1 direction, only one of the remaining T values gives this. Image (ii) follows by process of elimination.

- (b) Determine the Fourier transforms of $g(x) = \cos(2\pi \frac{3}{2} x)$ and $h(x) = \cos(2\pi \frac{5}{3} x)$, then plot them in the range $[-4\pi, 4\pi]$.

Solution: From the 1D Fourier transform table: $\hat{g}(\omega) = \pi(\delta(\omega + 3\pi) + \delta(\omega - 3\pi))$ and $\hat{h}(\omega) = \pi(\delta(\omega + \frac{10}{3}\pi) + \delta(\omega - \frac{10}{3}\pi))$.



(Note that Fourier transforms are usually complex-valued, so we would have to either plot the real and imaginary parts separately, or plot the magnitude and phase separately. But, for these functions, the Fourier transform happened to be real.)

- (c) Let $g_1(x)$ and $h_1(x)$ be the signals formed by multiplying functions g and h from (b) respectively with a Dirac comb (with sampling step $T = 1$). Determine the Fourier transforms of g_1 and h_1 , plot them in the range $[-4\pi, 4\pi]$.

Solution: $\hat{g}_1(\omega) = \sum_{n \in \mathbb{Z}} \pi(\delta(\omega - 2\pi n + 3\pi) + \delta(\omega - 2\pi n - 3\pi)) \stackrel{*}{=} \sum_{n \in \mathbb{Z}} 2\pi\delta(\omega - 2\pi n + \pi)$ and $\hat{h}_1(\omega) = \sum_{n \in \mathbb{Z}} \pi(\delta(\omega - 2\pi n + \frac{4}{3}\pi) + \delta(\omega - 2\pi n - \frac{4}{3}\pi))$.

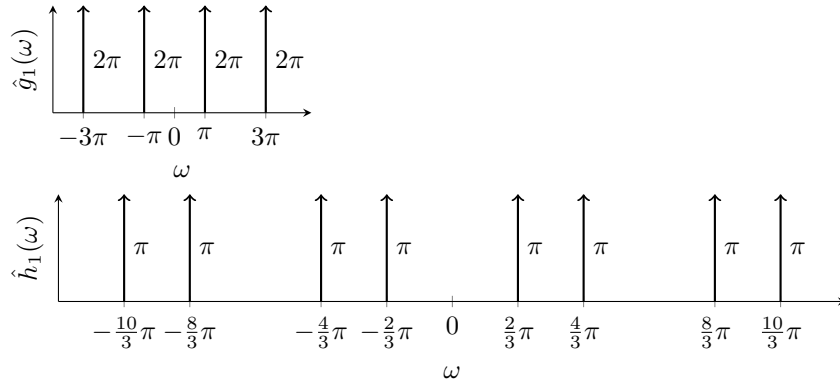
Explanation for the $*$ step:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \delta(\omega - 2\pi n + 3\pi) &= \sum_{n \in \mathbb{Z}} \delta(\omega - 2\pi(n+1) + 3\pi) \\ &= \sum_{n \in \mathbb{Z}} \delta(\omega - 2\pi n + \pi). \end{aligned}$$

and

$$\begin{aligned}\sum_{n \in \mathbb{Z}} \delta(\omega - 2\pi n - 3\pi) &= \sum_{n \in \mathbb{Z}} \delta(\omega - 2\pi(n - 2) - 3\pi) \\ &= \sum_{n \in \mathbb{Z}} \delta(\omega - 2\pi n + \pi).\end{aligned}$$

If your answers look different from these, remember that you can freely make variable substitutions in the sums like above, so your answers might be completely correct. The best way to check your answer is to plot it and compare with the solution plots.



- (d) Based on your previous answers, which (if either) of g and h can be reconstructed exactly from its samples with $T = 1$?

Solution: Neither, since both exhibit aliasing (that is, the central part of $\hat{g}_1(\omega)$ doesn't match that of $\hat{g}(\omega)$, same for $\hat{h}_1(\omega)$).

- (e) What is the range of T such that both g and h can be perfectly reconstructed from their samples?

Solution: According to the Nyquist criterion, $\omega_{\max} < \frac{\pi}{T}$ is required to ensure alias-free. $\omega_g = 3\pi, \omega_h = \frac{10}{3}\pi$ so $\omega_{\max} = \frac{10}{3}\pi$ and $T < \frac{\pi}{\omega_{\max}} = \frac{3}{10}$. Hence the range of T is $(0, \frac{3}{10})$.