Image Processing 1, Exercise 2

We expect students to be able to: 1) compute basic Fourier transforms; 2) characterize specific image processing systems. These exercises will help you develop these skills.

1 Fourier transform

[basic] Computation and manipulation of Fourier transforms. You are encouraged to use the Fourier tables provided on Moodle.

(a) Compute the Fourier transform $\mathcal{F}\{f\}$ of the following functions:

i. $f(x, y) = \operatorname{rect}\left(\frac{x}{5} - \frac{1}{2}\right) \operatorname{rect}\left(y - \frac{3}{2}\right)$

Solution:

Image space	Fourier space	Fourier property
$\operatorname{rect}\left(\frac{x}{5}\right)$	$5\mathcal{F}\{\operatorname{rect}(x)\}(5\omega_x)$	Scaling
$\operatorname{rect}\left(\frac{x}{5} - \frac{1}{2}\right) = \operatorname{rect}\left(\frac{x-\frac{5}{2}}{5}\right)$	$5e^{-\frac{5j\omega_x}{2}}\mathcal{F}\{\operatorname{rect}(x)\}(5\omega_x)$	Shift
$\operatorname{rect}\left(y - \frac{3}{2}\right)$	$e^{-\frac{3j\omega_y}{2}}\mathcal{F}\{\operatorname{rect}(y)\}(\omega_y)$	Shift
$\operatorname{rect}\left(\frac{x}{5} - \frac{1}{2}\right) \operatorname{rect}\left(y - \frac{3}{2}\right)$	$5 \operatorname{sinc}\left(\frac{5\omega_x}{2\pi}\right) e^{-\frac{5j\omega_x}{2}} \operatorname{sinc}\left(\frac{\omega_y}{2\pi}\right) e^{-\frac{3j\omega_y}{2}}$	Fourier table

ii. $f(\boldsymbol{x}) = e^{j\langle \boldsymbol{\omega_0}, \boldsymbol{x}\rangle} \operatorname{sinc}\left(\frac{\boldsymbol{x}}{2\pi}\right)$, $\boldsymbol{x} \in \mathbb{R}^2$

Solution: We define $g(\boldsymbol{x}) = \operatorname{sinc}\left(\frac{\boldsymbol{x}}{2\pi}\right)$. Then

$$\mathcal{F}\{g\}(\boldsymbol{\omega}) = (2\pi)^2 \operatorname{rect}(\boldsymbol{\omega})$$

$$\begin{aligned}\mathcal{F}\{f\}(\boldsymbol{\omega}) &= \mathcal{F}\{g\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0) \\ &= (2\pi)^2 \operatorname{rect}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)\end{aligned}$$

(b) Compute the inverse Fourier transform $\mathcal{F}^{-1}\{\hat{f}\}$ of the following functions:

i. $\hat{f}(\omega) = \frac{4}{4+\omega^2}$

Solution:

$$\begin{aligned}\mathcal{F}^{-1}\left\{\frac{4}{4+\omega^2}\right\}(x) &= \mathcal{F}^{-1}\left\{2\,\frac{2}{2^2+\omega^2}\right\}(x) \\ &\overset{\text{Tables}}{=} \mathrm{e}^{-2|x|}\end{aligned}$$

ii. $\hat{f}(\boldsymbol{\omega}) = e^{j\,\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle}\, |\det(\boldsymbol{A})|^{-1}\, \hat{g}(\boldsymbol{A}^{-T}\,\boldsymbol{\omega})$, where $\mathbf{A}$ is a (2×2) invertible matrix and $\mathbf{x}_0 \in \mathbb{R}^2$

Solution:
From the Fourier tables we have

$$\begin{aligned}\mathcal{F}^{-1}\{e^{j\,\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle}\, |\det(\boldsymbol{A})|^{-1}\, \hat{g}(\boldsymbol{A}^{-T}\,\boldsymbol{\omega})\}(\boldsymbol{x}) &\overset{\text{Affine}}{=} \mathcal{F}^{-1}\{e^{j\,\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle}\, \mathcal{F}\{g(\boldsymbol{A}\boldsymbol{x})\}\} \\ &\overset{\text{Shift}}{=} g(\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{x}_0))\end{aligned}$$

(c) Use the definition of Fourier transform to show that $\int_{\mathbb{R}} \operatorname{sinc}(x)\mathrm{d}x = 1$.

Solution: Recall the definition of the Fourier transform of a function $f(x)$:

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-j\omega x}\mathrm{d}x,$$

notice that $\hat{f}(0) = \int_{\mathbb{R}} f(x)\mathrm{d}x$.
Let $f(x) = \operatorname{sinc}(x)$ whose Fourier transform is $\operatorname{rect}(\frac{\omega}{2\pi})$, then

$$\int_{\mathbb{R}} \operatorname{sinc}(x)\mathrm{d}x = \operatorname{rect}(\frac{0}{2\pi}) = 1.$$

2 Characterization of linear shift-invariant systems

[basic] Practical examples of the characterization of linear shift invariant systems.

(a) A two-dimensional system has the impulse response $h(x, y) = \operatorname{rect}(x)e^{\mathrm{j}\omega_0 x}(\operatorname{rect} * \operatorname{rect})(y)$. Give the associated transfer function H.

Solution: The transfer function is the Fourier transform of the impulse response. The impulse response is separable. We introduce

$$f_1(x) = \operatorname{rect}(x)\, e^{\mathrm{j}\omega_0 x};\;\; f_2(y) = (\operatorname{rect} * \operatorname{rect})\,(y).$$

$$\begin{aligned} H(\omega_x, \omega_y) &= \mathcal{F}\{f_1\}(\omega_x)\,\mathcal{F}\{f_2\}(\omega_y) \\ &= \operatorname{sinc}\left(\frac{\omega_x - \omega_0}{2\,\pi}\right)\,\mathcal{F}\{f_2\}(\omega_y)\;\; \textit{(using modulation + rect FFT)} \\ &= \operatorname{sinc}\left(\frac{\omega_x - \omega_0}{2\,\pi}\right)\,\left(\operatorname{sinc}\left(\frac{\omega_y}{2\,\pi}\right)\right)^2\;\; \textit{(using convolution + rect FFT)} \end{aligned}$$

(b) A linear shift-invariant system $\mathcal{T}$ is such that

$$\mathcal{T}\{\sin(\omega\cdot)\}(x) = \sin\left(\omega\left(x - \frac{\pi}{2}\right)\right), \text{ and } \mathcal{T}\{\cos(\omega\cdot)\}(x) = \cos\left(\omega\left(x - \frac{\pi}{2}\right)\right)$$

for every $\omega \in \mathbb{R}$. Give its transfer function.

Solution: For linear shift-invariant systems, the transfer function is the eigenvalue associated to $e^{\mathrm{j}\omega x}$.

$$\begin{aligned} \mathcal{T}\{\mathrm{e}^{\mathrm{j}\,\omega\,\cdot}\}(x) &= \mathcal{T}\{\cos(\omega\,\cdot) + \mathrm{j}\,\sin(\omega\,\cdot)\}(x) \\ &= \cos(\omega\,\left(x - \frac{\pi}{2}\right)) + \mathrm{j}\,\sin(\omega\,\left(x - \frac{\pi}{2}\right)) \\ &= \mathrm{e}^{\mathrm{j}\,\omega\,\left(x - \frac{\pi}{2}\right)} \end{aligned}$$

$$\begin{aligned} H(\omega) &= \frac{\mathcal{T}\{\mathrm{e}^{\mathrm{j}\langle\omega,\cdot\rangle}\}}{\mathrm{e}^{\mathrm{j}\langle\omega,\cdot\rangle}}(x) \\ &= \mathrm{e}^{-\mathrm{j}\,\omega\,x}\,\mathrm{e}^{\mathrm{j}\,\omega\,\left(x - \frac{\pi}{2}\right)} \\ &= \mathrm{e}^{-\mathrm{j}\,\omega\,\frac{\pi}{2}} \end{aligned}$$

(c) The transfer function H of a linear shift-invariant system $\mathcal{T}$ is known to be $H(\omega) = e^{-j\omega x_0}$. What is the impulse response h of the system $\mathcal{T}$?

Solution:

$$\begin{aligned} h(x) &= \mathcal{F}^{-1}\{H\}(x) \\ &= \mathcal{F}^{-1}\{\mathrm{e}^{-\mathrm{j}\,(\cdot)\,x_0}\}(x) \\ &= \delta(x - x_0) \end{aligned}$$