Image Processing 1, Exercise 1

1 Inner Products

[basic] Computing the inner product of two functions is a basic skill in image processing. This exercise will make you practice and help you better understand the properties of inner products.

(a) Compute the inner product (which for each of the spaces can be found in the course notes) between the following functions.

i. $f, g \in L_2(\mathbb{R}^2)$,

$$f(x,y) = \mathrm{rect}\left(\frac{x}{2}\right)\mathrm{rect}\left(\frac{1}{2}+y\right); g(x,y) = \mathrm{rect}\left(\frac{y}{2}\right)\mathrm{rect}\left(\frac{1}{2}+x\right).$$

Hint: The function rect is defined in the table "Useful 1D Fourier-transform pairs" provided in the IP1 Appendix (on moodle).

ii. $f, g \in L_2(\mathbb{R}^2)$,

$$f(x,y) = \cos(2\pi x)\,\mathrm{rect}\left(\frac{x}{2}\right)\mathrm{rect}\,(y); g(x,y) = \mathrm{rect}\left(\frac{x}{3}\right)\mathrm{rect}\left(\frac{y}{2}\right).$$

iii. $f, g \in \ell_2(\mathbb{Z})$, where $\boxed{}$ here indicates the value of the sequence at the reference index $k = 0$.

$$f = \{\cdots\ 0\ 0\ 0\ \boxed{1}\ 2\ 5\ 1\ 2\ 0\ 0\ \cdots\}$$
$$g = \{\cdots\ 0\ 0\ 1\ \boxed{3}\ 2\ 4\ 0\ 9\ 2\ 0\ \cdots\}.$$

(b) Show that the following definitions of $\langle\cdot,\cdot\rangle$ do not represent $\mathcal{H}$-inner products.

i. $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1 y_1^*$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^2$, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$.
Hint: which vectors will have $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$?

ii. $\langle f, g\rangle = \int_0^1 \dot{f}(x)(\dot{g}(x))^* \mathrm{d}x$, where $f, g \in L_2(\mathbb{R})$ and $\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}x}$.

2 Linearity and shift-invariance properties

[basic] Linear and shift-invariant systems are fundamental to image processing. This exercise will help you determine if a given operator is shift-invariant and eventually linear.

(a) An optical system $\mathcal{T}$ is said to be *additive* if $\mathcal{T}\{f+g\} = \mathcal{T}\{f\} + \mathcal{T}\{g\}$, and *homogeneous* if $\mathcal{T}\{\lambda f\} = \lambda\mathcal{T}\{f\}$. Show that a system is linear iff it is additive and homogeneous.

(b) Let $\mathcal{T}_3$ be defined as the combination of $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $\mathcal{T}_3\{f\} = \mathcal{T}_2\{\mathcal{T}_1\{f\}\}$, where f is an image. Show that $\mathcal{T}_3$ is linear and shift invariant if $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear and shift invariant.

(c) Determine if the following standard operators $\mathcal{T}$ are linear, where $a = \{a_1, \ldots, a_N\}$:

i. Mean

$$\mathcal{T}\{a\} = \{\mu, \ldots, \mu\}, \quad \text{where } \mu = \sum_{k=1}^{N} a_k$$

ii. Median (N is an odd number.)

$$\mathcal{T}\{a\} = \{\zeta, \ldots, \zeta\}, \quad \text{where } \zeta = b_{(N+1)/2}$$

where $b_k = a_{\tau(k)}$, $b_1 \le \cdots \le b_N$, and $\tau(k)$ is a permutation that sorts the a_k.

iii. Gamma correction

$$\mathcal{T}\{a\} = \{a_1^\gamma, \ldots, a_N^\gamma\} \quad \text{where } \gamma > 0\ (\gamma \neq 1) \text{ is a constant.}$$

iv. Re-ordering

$$\mathcal{T}\{a\} = \{a_N, a_{N-1} \ldots, a_2, a_1\}$$

v. Binarization

$$\mathcal{T}\{a\} = \{B_T(a_1), \ldots, B_T(a_N)\},$$

$$\text{where } B_T(x) = \begin{cases} 0 & x < T \\ 1 & x \geq T, \end{cases} \quad \text{and } T \text{ is a constant.}$$

(d) Determine if the following linear systems $\mathcal{T}$ are shift-invariant

i. $\mathcal{T}\{f\}(x, y) = f(x + 3/2, y - 5)$

ii. $\mathcal{T}\{f\}(x, y) = f(-x, -y)$

iii. $\mathcal{T}\{f\}(x, y) = f(2x, y/2)$