

Image Processing 1, Exercise 1

1 Inner Products

[basic] Computing the inner product of two functions is a basic skill in image processing. This exercise will make you practice and help you better understand the properties of inner products.

- (a) Compute the inner product (which for each of the spaces can be found in the course notes) between the following functions.

- i. $f, g \in L_2(\mathbb{R}^2)$,

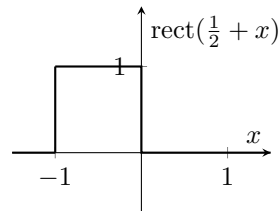
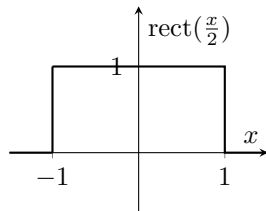
$$f(x, y) = \text{rect}\left(\frac{x}{2}\right) \text{rect}\left(\frac{1}{2} + y\right); g(x, y) = \text{rect}\left(\frac{y}{2}\right) \text{rect}\left(\frac{1}{2} + x\right).$$

Hint: The function rect is defined in the table “Useful 1D Fourier-transform pairs” provided in the IP1 Appendix (on moodle).

Solution: Directly compute the inner products using the definition associated to the Hilbert space of the input. You can find these definitions in the course notes.

$$\begin{aligned}\langle f, g \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g^*(x, y) \, dx dy \\ &= \int_{-\infty}^{\infty} \text{rect}\left(\frac{x}{2}\right) \text{rect}\left(\frac{1}{2} + x\right) \, dx \int_{-\infty}^{\infty} \text{rect}\left(\frac{y}{2}\right) \text{rect}\left(\frac{1}{2} + y\right) \, dy\end{aligned}$$

The common non zero interval of $\text{rect}\left(\frac{x}{2}\right)$ and $\text{rect}\left(\frac{1}{2} + x\right)$ is $[-1, 0]$:



Hence,

$$\langle f, g \rangle = \int_{-1}^0 1 \cdot dx \int_{-1}^0 1 \cdot dy = 1 \cdot 1 = 1$$

- ii. $f, g \in L_2(\mathbb{R}^2)$,

$$f(x, y) = \cos(2\pi x) \text{rect}\left(\frac{x}{2}\right) \text{rect}(y); g(x, y) = \text{rect}\left(\frac{x}{3}\right) \text{rect}\left(\frac{y}{2}\right).$$

Solution:

$$\begin{aligned}
 \langle f, g \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g^*(x, y) \, dx dy \\
 &= \int_{-\infty}^{\infty} \cos(2\pi x) \operatorname{rect}\left(\frac{x}{2}\right) \, dx \int_{-\infty}^{\infty} \operatorname{rect}(y) \, dy \\
 &= \int_{-1}^1 \cos(2\pi x) \, dx \int_{-1/2}^{1/2} 1 \, dy \\
 &= \left[\frac{1}{2\pi} \sin(2\pi x) \right]_{-1}^1 \cdot 1 \\
 &= 0
 \end{aligned}$$

iii. $f, g \in \ell_2(\mathbb{Z})$, where $\boxed{}$ here indicates the value of the sequence at the reference index $k = 0$.

$$\begin{aligned}
 f &= \{\dots 0 \ 0 \ 0 \ \boxed{1} \ 2 \ 5 \ 1 \ 2 \ 0 \ 0 \ \dots\} \\
 g &= \{\dots 0 \ 0 \ 1 \ \boxed{3} \ 2 \ 4 \ 0 \ 9 \ 2 \ 0 \ \dots\}.
 \end{aligned}$$

Solution:

$$\begin{aligned}
 \langle f, g \rangle &= \sum_{k=-\infty}^{\infty} f[k] g^*[k] \\
 &= \dots + 0 \cdot 0 + 0 \cdot 1 + \boxed{1 \cdot 3} + 2 \cdot 2 + 5 \cdot 4 + 1 \cdot 0 + 2 \cdot 9 + 0 \cdot 2 + 0 \cdot 0 + \dots \\
 &= 0 + \boxed{3} + 4 + 20 + 0 + 18 + 0 = 45
 \end{aligned}$$

(b) Show that the following definitions of $\langle \cdot, \cdot \rangle$ do not represent $\mathcal{H}$ -inner products.

i. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1^*$, where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$.

Hint: which vectors will have $\langle \mathbf{x}, \mathbf{x} \rangle = 0$?

Solution: Positive-definiteness is not satisfied since we can find a counter-example:

$$\begin{aligned}
 \text{for } \mathbf{x} &= (0, 1) \neq \mathbf{0}, \langle (0, 1), (0, 1) \rangle &= 0 \\
 &&\neq 0
 \end{aligned}$$

Note that in an exam, showing at least one property does not hold would be perfectly sufficient! For the sake of completeness, here is how you can check the other properties.
Linearity (satisfied):

$$\begin{aligned}
 \langle a_1 \mathbf{x} + a_2 \mathbf{z}, \mathbf{y} \rangle &= (a_1 x_1 + a_2 z_1) y_1^* \\
 &= a_1 x_1 y_1^* + a_2 z_1 y_1^* \\
 &= a_1 \langle \mathbf{x}, \mathbf{y} \rangle + a_2 \langle \mathbf{z}, \mathbf{y} \rangle
 \end{aligned}$$

Symmetry (satisfied):

$$\begin{aligned}
 \langle \mathbf{x}, \mathbf{y} \rangle^* &= (x_1 y_1^*)^* \\
 &= y_1 x_1^* \\
 &= \langle \mathbf{y}, \mathbf{x} \rangle
 \end{aligned}$$

ii. $\langle f, g \rangle = \int_0^1 \dot{f}(x) (\dot{g}(x))^* dx$, where $f, g \in L_2(\mathbb{R})$ and $\dot{f} = \frac{df}{dx}$.

Solution:

Positive-definiteness:

We can find an easy counter-example: any function that is constant on the interval $[0, 1]$ and square integrable will work. For example we can pick $c \in \mathbb{R} \setminus \{0\}$ and consider:

$$f(x) = \begin{cases} c & \text{for } x \in [-1, 2], \\ 0 & \text{otherwise.} \end{cases}$$

Again for the exam this would be perfectly valid answer! You can check if linearity and symmetry hold for this case or not, following the example above.

2 Linearity and shift-invariance properties

[basic] Linear and shift-invariant systems are fundamental to image processing. This exercise will help you determine if a given operator is shift-invariant and eventually linear.

- (a) An optical system $\mathcal{T}$ is said to be *additive* if $\mathcal{T}\{f + g\} = \mathcal{T}\{f\} + \mathcal{T}\{g\}$, and *homogeneous* if $\mathcal{T}\{\lambda f\} = \lambda \mathcal{T}\{f\}$. Show that a system is linear iff it is additive and homogeneous.

Solution: Necessity: assume that the system $\mathcal{T}$ is linear, and show that it is additive and homogeneous.

$$(\mathcal{T}\{a_1 f + a_2 g\} = a_1 \mathcal{T}\{f\} + a_2 \mathcal{T}\{g\}) \stackrel{a_1=1, a_2=1}{\Rightarrow} (\mathcal{T}\{f + g\} = \mathcal{T}\{f\} + \mathcal{T}\{g\})$$

$$(\mathcal{T}\{a_1 f + a_2 g\} = a_1 \mathcal{T}\{f\} + a_2 \mathcal{T}\{g\}) \stackrel{a_1=\lambda, a_2=0}{\Rightarrow} (\mathcal{T}\{\lambda f\} = \lambda \mathcal{T}\{f\})$$

Sufficiency: assume that the system $\mathcal{T}$ is additive and homogeneous, and show that it is linear.

$$\begin{aligned} \mathcal{T}\{(a_1 f) + (a_2 g)\} &\stackrel{\text{additivity}}{=} \mathcal{T}\{a_1 f\} + \mathcal{T}\{a_2 g\} \\ &\stackrel{\text{homogeneity}}{=} a_1 \mathcal{T}\{f\} + a_2 \mathcal{T}\{g\} \end{aligned}$$

- (b) Let $\mathcal{T}_3$ be defined as the combination of $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $\mathcal{T}_3\{f\} = \mathcal{T}_2\{\mathcal{T}_1\{f\}\}$, where f is an image. Show that $\mathcal{T}_3$ is linear and shift invariant if $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear and shift invariant.

Solution: Linearity is satisfied since

$$\begin{aligned} \mathcal{T}_3\{\lambda f + g\} &= \mathcal{T}_2\{\mathcal{T}_1\{\lambda f + g\}\} \\ &\stackrel{\mathcal{T}_1 \text{ is linear}}{=} \mathcal{T}_2\{\lambda \mathcal{T}_1\{f\} + \mathcal{T}_1\{g\}\} \\ &\stackrel{\mathcal{T}_2 \text{ is linear}}{=} \lambda \mathcal{T}_2\{\mathcal{T}_1\{f\}\} + \mathcal{T}_2\{\mathcal{T}_1\{g\}\} \\ &= \lambda \mathcal{T}_3\{f\} + \mathcal{T}_3\{g\} \end{aligned}$$

Shift-invariance is satisfied since

$$\begin{aligned} \mathcal{T}_3\{f(\cdot - x_0)\}(x) &= \mathcal{T}_2\{\mathcal{T}_1\{f(\cdot - x_0)\}\}(x) \\ &\stackrel{\mathcal{T}_1 \text{ is shift-invariant}}{=} \mathcal{T}_2\{\mathcal{T}_1\{f\}(\cdot - x_0)\}(x) \\ &\stackrel{\mathcal{T}_2 \text{ is shift-invariant}}{=} \mathcal{T}_2\{\mathcal{T}_1\{f\}\}(x - x_0) \\ &= \mathcal{T}_3\{f\}(x - x_0) \end{aligned}$$

(c) Determine if the following standard operators $\mathcal{T}$ are linear, where $a = \{a_1, \dots, a_N\}$:

i. Mean

$$\mathcal{T}\{a\} = \{\mu, \dots, \mu\}, \quad \text{where } \mu = \sum_{k=1}^N a_k$$

Solution:

$$\begin{aligned} T\{\alpha a + \beta b\} &= \left\{ \sum_{k=1}^N (\alpha a_k + \beta b_k), \dots, \sum_{k=1}^N (\alpha a_k + \beta b_k) \right\} \\ &= \left\{ \sum_{k=1}^N \alpha a_k + \sum_{k=1}^N \beta b_k, \dots, \sum_{k=1}^N \alpha a_k + \sum_{k=1}^N \beta b_k \right\} \\ &= \alpha \left\{ \sum_{k=1}^N a_k, \dots, \sum_{k=1}^N a_k \right\} + \beta \left\{ \sum_{k=1}^N b_k, \dots, \sum_{k=1}^N b_k \right\} \\ &= \alpha T\{a\} + \beta T\{b\}, \end{aligned}$$

with the last equality proving linearity.

ii. Median (N is an odd number.)

$$\mathcal{T}\{a\} = \{\zeta, \dots, \zeta\}, \quad \text{where } \zeta = b_{(N+1)/2}$$

where $b_k = a_{\tau(k)}$, $b_1 \leq \dots \leq b_N$, and $\tau(k)$ is a permutation that sorts the a_k .

Solution: Median: this operation is non-linear, which we can show with a counter-example. We take

$$a = \{3, 2, 5\}, \quad b = \{3, 1, 1\}, \quad \text{and} \quad c = a + b = \{6, 3, 6\}.$$

Applying the operator to each of them, we have

$$T\{a\} = \{3, 3, 3\}, \quad T\{b\} = \{1, 1, 1\}, \quad \text{and} \quad T\{a + b\} = T\{c\} = \{6, 6, 6\}.$$

However,

$$T\{a\} + T\{b\} = \{4, 4, 4\} \neq T\{a + b\}.$$

iii. Gamma correction

$$\mathcal{T}\{a\} = \{a_1^\gamma, \dots, a_N^\gamma\} \quad \text{where } \gamma > 0 \ (\gamma \neq 1) \text{ is a constant.}$$

Solution: Gamma correction: this operation is non-linear, which we can show with a counter-example. We take

$$a = \{3, 2, 5\}, \quad b = \{3, 1, 1\}, \quad \text{and} \quad c = a + b = \{6, 3, 6\}.$$

Applying the operator to each of them, with, say, $\gamma = 2$, we have

$$T\{a\} = \{9, 4, 25\}, \quad T\{b\} = \{9, 1, 1\}, \quad \text{and} \quad T\{a + b\} = T\{c\} = \{36, 9, 36\}.$$

However,

$$T\{a\} + T\{b\} = \{18, 5, 26\} \neq T\{a + b\}.$$

iv. Re-ordering

$$\mathcal{T}\{a\} = \{a_N, a_{N-1}, \dots, a_2, a_1\}$$

Solution:

$$\begin{aligned} T\{\alpha a + \beta b\} &= \{\alpha a_N + \beta b_N, \dots, \alpha a_1 + \beta b_1\} \\ &= \alpha \{a_N, \dots, a_1\} + \beta \{b_N, \dots, b_1\} \\ &= \alpha T\{a\} + \beta T\{b\}, \end{aligned}$$

with the last equality proving linearity.

v. Binarization

$$\mathcal{T}\{a\} = \{B_T(a_1), \dots, B_T(a_N)\},$$

$$\text{where } B_T(x) = \begin{cases} 0 & x < T \\ 1 & x \geq T, \end{cases} \quad \text{and } T \text{ is a constant.}$$

Solution: Binarization: this operation is non-linear, which we can show with a counter-example. We take

$$a = \{3, 2, 5\}, \quad b = \{3, 1, 1\}, \quad \text{and} \quad c = a + b = \{6, 3, 6\}.$$

Applying the operator to each of them, with, say, $T = 1.5$, we have

$$T\{a\} = \{1, 1, 1\}, \quad T\{b\} = \{1, 0, 0\}, \quad \text{and} \quad T\{a + b\} = T\{c\} = \{1, 1, 1\}.$$

However,

$$T\{a\} + T\{b\} = \{2, 1, 1\} \neq T\{a + b\}.$$

(d) Determine if the following linear systems $\mathcal{T}$ are shift-invariant

i. $\mathcal{T}\{f\}(x, y) = f(x + 3/2, y - 5)$

Solution: Shift-invariant.

$$\begin{aligned} \mathcal{T}\{f(\cdot - x_0, \cdot - y_0)\} &= f\left(x - x_0 + \frac{3}{2}, y - y_0 - 5\right) \\ &= \mathcal{T}\{f\}(x - x_0, y - y_0) \end{aligned}$$

ii. $\mathcal{T}\{f\}(x, y) = f(-x, -y)$

Solution: Not shift-invariant. Can show a counter example, or show algebraically: To make things clear, let $g(x, y) = f(x - x_0, y - y_0)$

$$\begin{aligned} \mathcal{T}\{f(\cdot - x_0, \cdot - y_0)\} &= \mathcal{T}\{g(\cdot)\}(x, y) \\ &= g(-x, -y) \\ &= f(-x - x_0, -y - y_0) \end{aligned}$$

$$\begin{aligned} \mathcal{T}\{f\}(x - x_0, y - y_0) &= f(-x + x_0, -y + y_0) \\ &\neq \mathcal{T}\{f(\cdot - x_0, \cdot - y_0)\} \end{aligned}$$

iii. $\mathcal{T}\{f\}(x, y) = f(2x, y/2)$

Solution: Not shift-invariant. Can show a counter example, or show algebraically:

$$\mathcal{T}\{f(\cdot - x_0, \cdot - y_0)\} = f\left(2x - x_0, \frac{y}{2} - y_0\right)$$

$$\begin{aligned}\mathcal{T}\{f\}(x - x_0, y - y_0) &= f\left(2x - 2x_0, \frac{y - y_0}{2}\right) \\ &\neq \mathcal{T}\{f(\cdot - x_0, \cdot - y_0)\}\end{aligned}$$

For the curious, the effects of the transforms are illustrated on an example in Figure 1.

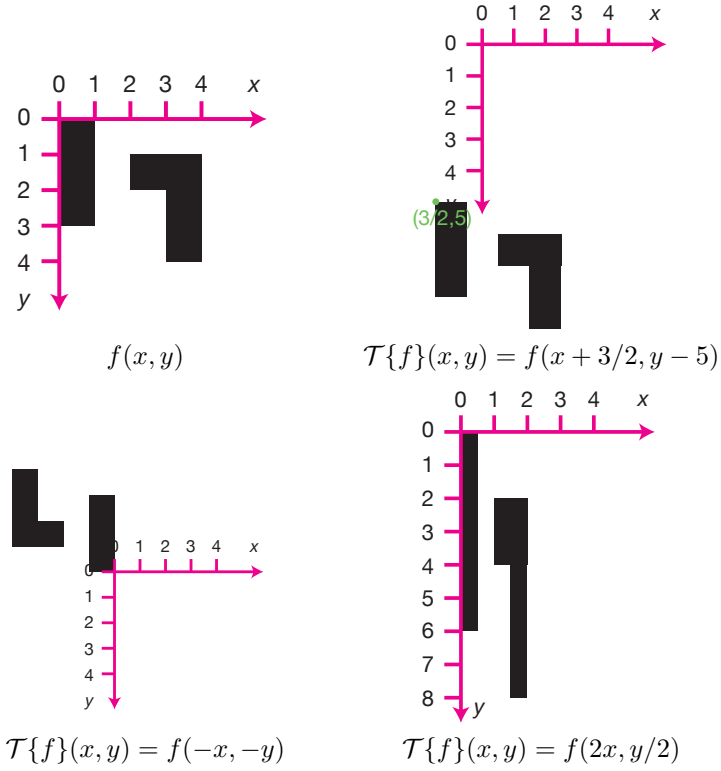


Figure 1: Application of the transformations of question (d) on a test image f .