

1 Statistical Independence and uncorrelation

A) For each distribution indicate whether the variables X and Y are correlated or uncorrelated and whether they are dependent or independent. How did you arrive to those conclusions? For each distribution make a sketch of the conditional probability of X given a few instances of Y , $P(X|Y = y_i)$.

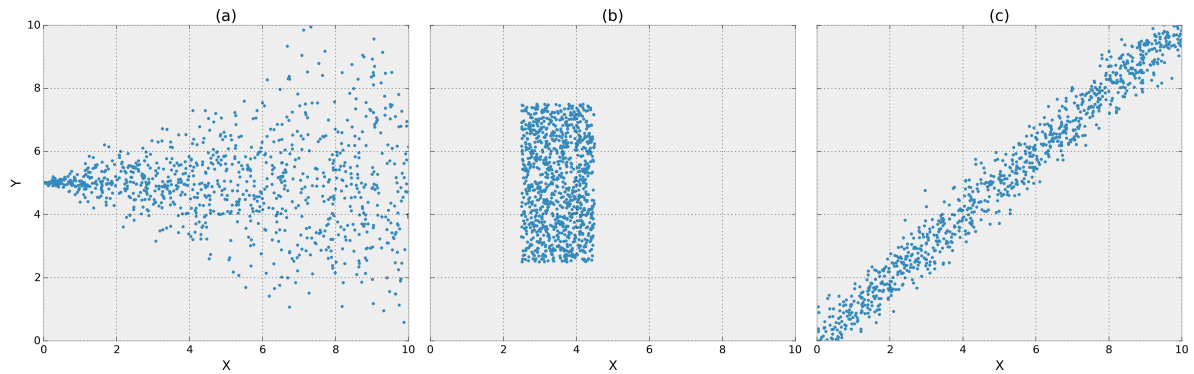


Figure 1: The three different datasets.

B) Consider two discrete random variables x, y with the following joint distribution table.

$p(x, y)$	$y = 1$	$y = 2$
$x = 1$	0.08	0.12
$x = 2$	0.12	0.18
$x = 3$	0.2	0.3

Table 1: Joint Distribution Table.

Compute the marginals and answer the two following questions: Are these random variables statistically independent? Are they correlated?

2 Optimal Fit and Conditional Distribution

Let us consider the data drawn in Figure . Let us assume that both clusters have approximately the same amount of data points.

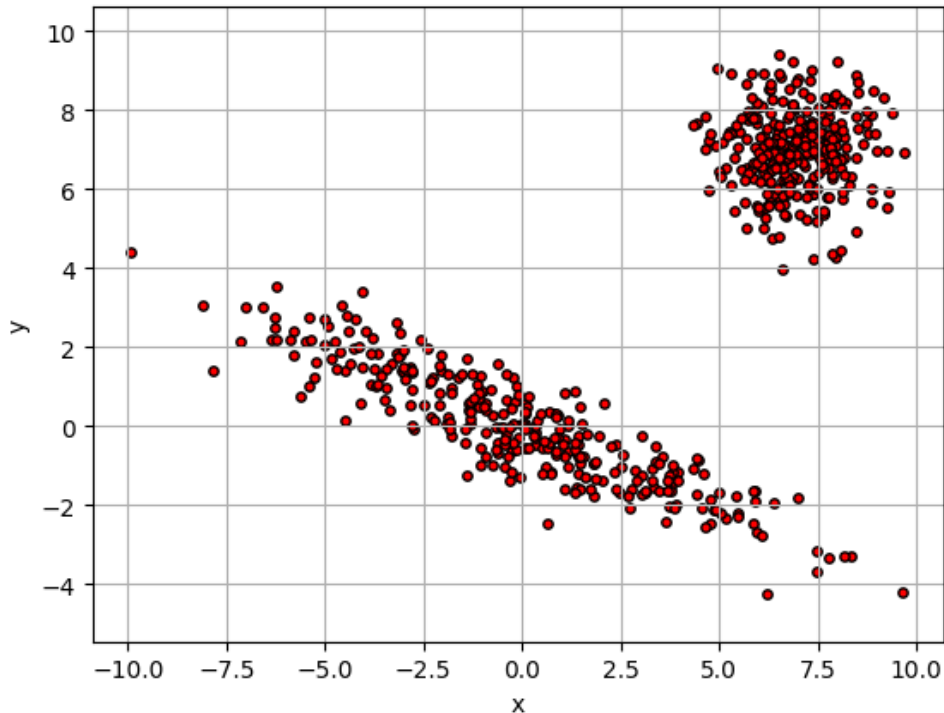


Figure 2: Data consisting of two clusters

1. Let us assume the data set consists of two distinct Gaussian Mixture Model
 - (a) Draw the two distinct Gaussians which would maximize the likelihood. What would the optimal weights be for each Gaussian?
 - (b) What does the conditional distribution look like for $x = 8$
 - (c) What does the conditional distribution look like for $y = 4$
2. Let us assume that we did not see the data before hand, and we only fit one single Gaussian Mixture Model. Does this model have a higher or lower likelihood?
 - (a) Draw the single distinct Gaussians which would maximize the likelihood.
 - (b) What does the conditional distribution look like for $x = 3.5$
 - (c) What does the conditional distribution look like for $y = 0$
3. From the previous model, we realized that one Gaussian did not work well. We try instead with three Gaussians Mixture Models
 - (a) Draw the three distinct Gaussians which would maximize the likelihood. What would the optimal weights be for each Gaussian?
 - (b) What does the conditional distribution look like for $x = 8$
 - (c) What does the conditional distribution look like for $y = 4$

3 Mixture of Gaussians - Likelihood

A) Consider that we have three mixtures of Gaussians. The first one has a single Gaussian. The second and the third have each two Gaussians (see Figure 3).

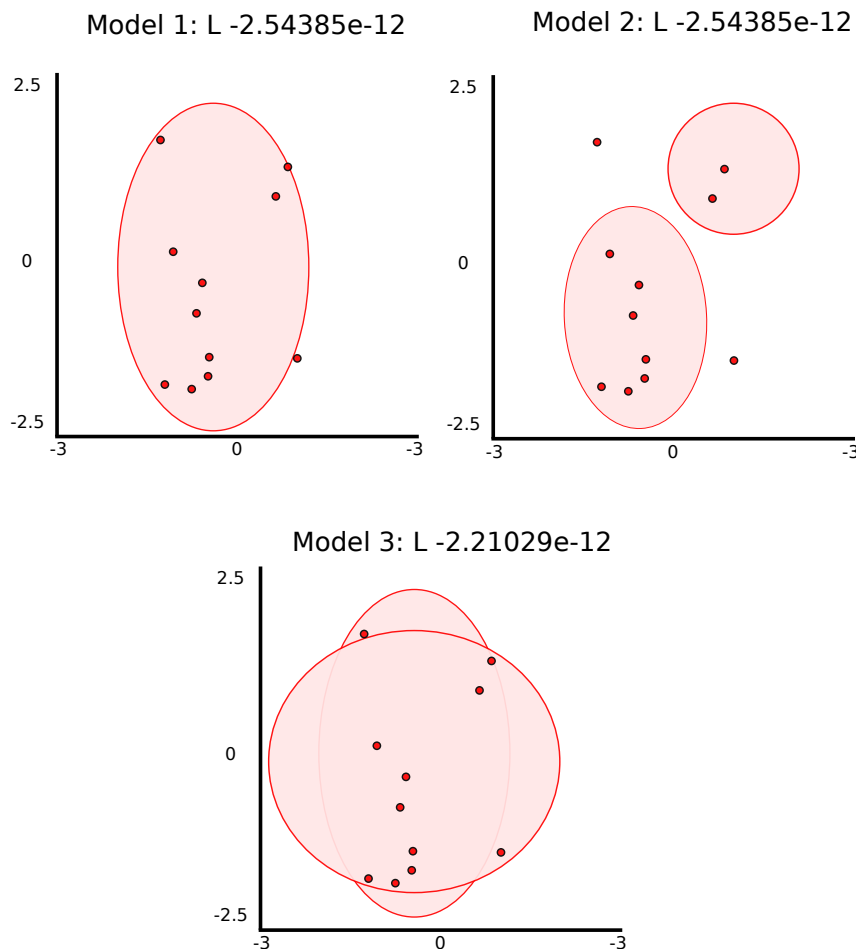


Figure 3: Three different models of “mixture of Gaussians”.

Explain why the likelihood is almost equal for the three mixtures even though they yield very different fits.

Note: If needed, you can help yourself by writing the formal definition of the pdf and the likelihood of each of the two mixtures.

B) Consider a multi-dimensional Gaussian function. What form should the Covariance matrix have to ensure that the marginal distributions are independent?

C) In the case of the PDF of a single normal distribution, the integral of the density computed over the interval $[\mu - \sigma, \mu + \sigma]$ covers 67% of the data-points. This is not longer true for the PDFs of all distributions. Draw a 1D example of a PDF for which this is not true.

D) Compute the mean and the variance for a one-dimensional uniform distribution with boundaries $[-a, a]$.

E) What are the values of μ and σ of a Gaussian Normal Distribution which has the optimal likelihood fit on a specific dataset.

4 (Exercise to be done at home)

A) Show that the solution to fitting a group of data points X with a single Gaussian function is the mean and variance of the data. We assume the data to be one dimensional.

B) If x_1 and x_2 are two uncorrelated variables, show that if g and h are two linear functions, then $y_1 = g(x_1)$ and $y_2 = h(x_2)$ are still uncorrelated. Show that if g and h are nonlinear integrable functions, y_1 , and y_2 are uncorrelated if x_1 and x_2 are statistically independent.