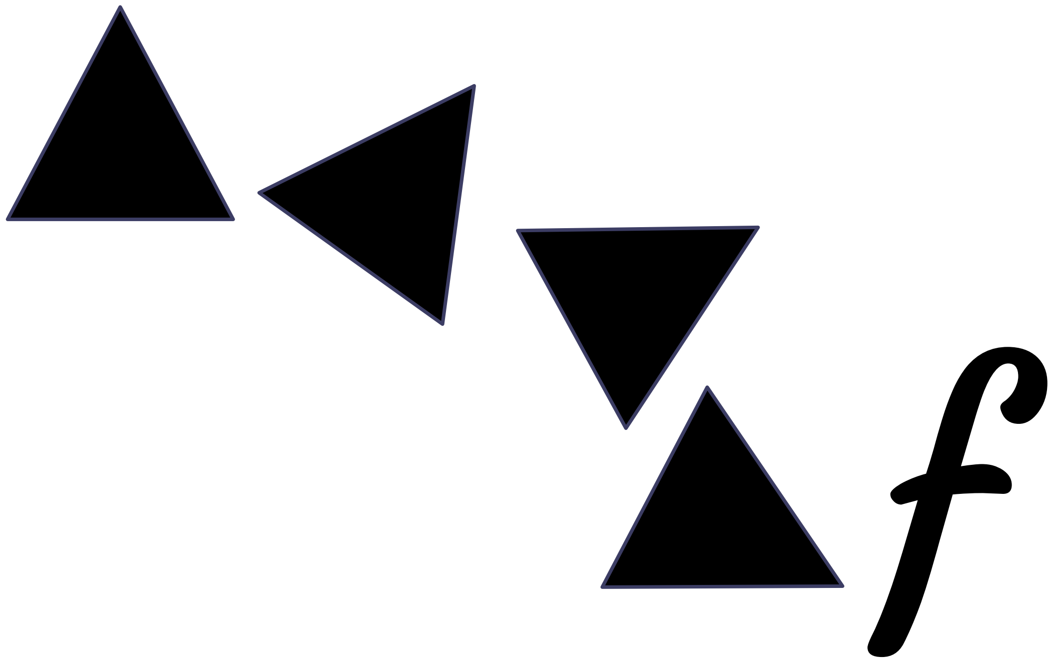
Differential Kinematics: The Jacobian



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EPFL

Agenda

Jacobian

- Definition
- Analytical Jacobian
- Singularities
- Complement - Geometric Jacobian

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The Jacobian

Relationship between joint and tool coordinates

↓

The geometric model (inverse and direct)

Coordinate transformation

Relationship between joint and operational speeds

↓

The Jacobian

3 functions with one Matrix

- Errors
- Speeds
- Forces

- Evaluate the singularities

Consider:

$$q = [q_1 \quad q_2 \quad \cdots \quad q_n]^T \longrightarrow$$ Generalized coordinate vector, Joint positions

$$\dot{q} = [\dot{q}_1 \quad \dot{q}_2 \quad \cdots \quad \dot{q}_n]^T \longrightarrow$$ Generalized velocity vector, Joint speeds

Question: How to represent the vector of operational speed?

The Jacobian: Definition

Mathematicians speak of a Jacobian matrix while roboticists rather use the term « Jacobian »

Matrix connecting the speeds of the end effector (robot wrist) to the joint speeds of the robot.

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = J \cdot \dot{q}$$

v and ω *are the translational and rotational speed vectors of the terminal organ.*

These components are chosen to be able to uniquely represent the speed at the end effector.

The Jacobian: Determination

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = J \cdot \dot{q}$$

J: Jacobian

Geometric construction

Analytical construction

By derivation of the geometric model

!

As for the geometric model!

Analytical Method and Analytical Jacobian

- The robot's output or the coordinates of its terminal organ is described by two types of information:
- Its translational position p(q), and
- The orientation ϕ(q)

$$X(q) = \begin{bmatrix} p(q) \\ \varphi(q) \end{bmatrix}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}$$

m, dimension of the operational space

Example (Euler angles):

$$X(q) = \begin{bmatrix} p_x(q) \\ p_y(q) \\ p_z(q) \\ \alpha(q) \\ \beta(q) \\ \gamma(q) \end{bmatrix}$$

Analytical Jacobian

$$X(q)=\begin{bmatrix} X_1(q) \\ X_2(q) \\ \vdots \\ X_m(q) \end{bmatrix}=\begin{bmatrix} f_1(q_1, & q_2, & \cdots & q_n) \\ f_2(q_1, & q_2, & \cdots & q_n) \\ & & \vdots & \\ f_m(q_1, & q_2, & \cdots & q_n) \end{bmatrix}$$

$$\boldsymbol{dX}=\begin{bmatrix} \boldsymbol{dX_1} \\ \boldsymbol{dX_2} \\ \vdots \\ \boldsymbol{dX_m} \end{bmatrix}=\begin{bmatrix} \frac{\partial f_1}{\partial q_1}\cdot \boldsymbol{dq_1}+\frac{\partial f_1}{\partial q_2}\cdot \boldsymbol{dq_2}+\cdots+\frac{\partial f_1}{\partial q_n}\cdot \boldsymbol{dq_n} \\ \frac{\partial f_2}{\partial q_1}\cdot \boldsymbol{dq_1}+\frac{\partial f_2}{\partial q_2}\cdot \boldsymbol{dq_2}+\cdots+\frac{\partial f_2}{\partial q_n}\cdot \boldsymbol{dq_n} \\ \vdots \\ \frac{\partial f_m}{\partial q_1}\cdot \boldsymbol{dq_1}+\frac{\partial f_m}{\partial q_2}\cdot \boldsymbol{dq_2}+\cdots+\frac{\partial f_m}{\partial q_n}\cdot \boldsymbol{dq_n} \end{bmatrix}$$

$$dX=\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix}\begin{bmatrix} dq_1 \\ dq_2 \\ \vdots \\ dq_n \end{bmatrix}$$

$$\boxed{\boldsymbol{dX}= J \cdot \boldsymbol{dq}}$$

$$\boxed{J=\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdot & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\ & \vdots & & \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix}}$$

Analytical Jacobian

$$\boxed{dX = J \cdot dq}$$

$/dt$ $/dt$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} \cdot \dot{\boldsymbol{q}}_1 + \frac{\partial f_1}{\partial q_2} \cdot \dot{\boldsymbol{q}}_2 + \cdots + \frac{\partial f_1}{\partial q_n} \cdot \dot{\boldsymbol{q}}_n \\ \frac{\partial f_2}{\partial q_1} \cdot \dot{\boldsymbol{q}}_1 + \frac{\partial f_2}{\partial q_2} \cdot \dot{\boldsymbol{q}}_2 + \cdots + \frac{\partial f_2}{\partial q_n} \cdot \dot{\boldsymbol{q}}_n \\ \vdots \\ \frac{\partial f_m}{\partial q_1} \cdot \dot{\boldsymbol{q}}_1 + \frac{\partial f_m}{\partial q_2} \cdot \dot{\boldsymbol{q}}_2 + \cdots + \frac{\partial f_m}{\partial q_n} \cdot \dot{\boldsymbol{q}}_n \end{bmatrix}$$

$$\boxed{\dot{X} = J \ \dot{q}}$$

$$\boxed{\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \cdot & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \cdots & \frac{\partial f_2}{\partial q_n} \\ & \vdots & \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix}}$$

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Differential dispalcements : Joint side versus tool side

$$dX = J \cdot dq$$

$$dq = J^{-1} \cdot dX$$

Analyze how known errors/ resolutions affect the resolution/error on the tool side.

Size sensors: Choose sensors according to the specified resolutions at the tool side.

Specify the control performance according to the specified errors on the tool side.

Velocities: Joint side versus tool side

$$\dot{X} = J \cdot \dot{q}$$

Analyze how known tool velocities varies according to joint velocities.

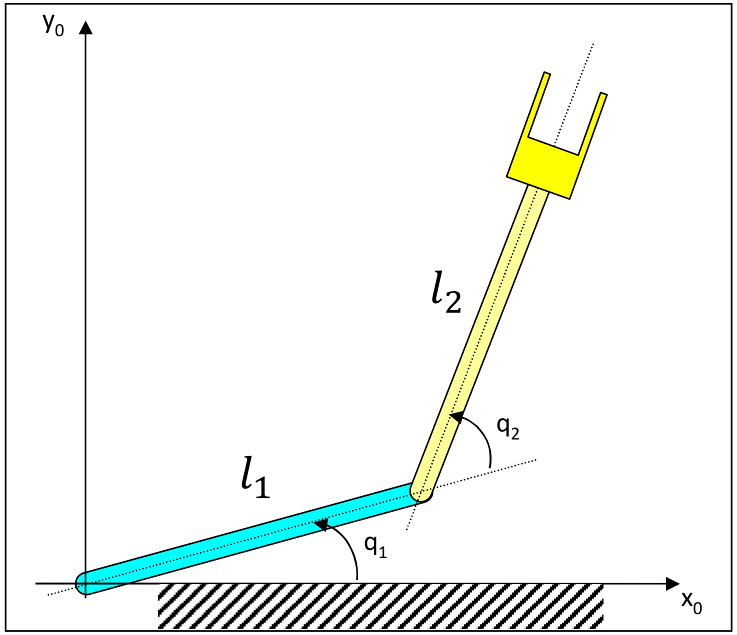
$$\dot{q} = J^{-1} \cdot \dot{X}$$

Size actuators: Choose actuators (motor speed) according to the specified velocity on the tool side.

Example

The direct geometric model

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(q_1,q_2) \\ f_2(q_1,q_2) \end{bmatrix} = \begin{bmatrix} l_1\cos(q_1)+l_2\cos(q_1+q_2) \\ l_1\sin(q_1)+l_2\sin(q_1+q_2) \end{bmatrix}$$



$$J_A = \begin{bmatrix} \dfrac{\partial f_1}{\partial q_1} & \dfrac{\partial f_1}{\partial q_2} \\ \dfrac{\partial f_2}{\partial q_1} & \dfrac{\partial f_2}{\partial q_2} \end{bmatrix} = \begin{bmatrix} -l_1\sin(q_1)-l_2\sin(q_1+q_2) & -l_2\sin(q_1+q_2) \\ l_1\cos(q_1)+l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1\sin(q_1)-l_2\sin(q_1+q_2) & -l_2\sin(q_1+q_2) \\ l_1\cos(q_1)+l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Observations

```
J_11(q1,q2) = -l_1*sin(q1)-l_2*sin(q1+q2);
J_12(q1,q2) = -l_2*sin(q1+q2);

J_21(q1,q2) =  l_1*cos(q1) + l_2*cos(q1+q2);
J_22(q1,q2) =  l_2*cos(q1+q2);
```

```
dx = J_11(q1,q2) * dq1 + J_12(q1,q2) * dq2;

dy = J_21(q1,q2) * dq1 + J_22(q1,q2) * dq2;
```

- The Jacobian elements depend on the position of the robot.

- The precision/resolution at the end effector depends on the sign of the joint errors.

Exercise:

Find the Jacobian for the RR planar following 2 DoF robot.

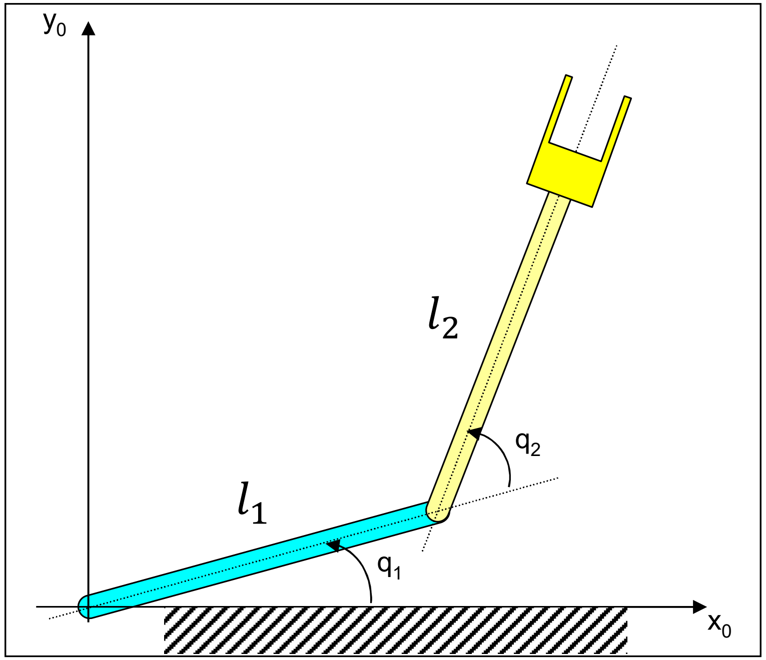
Consider that each link is l_i length.

- assume l_1 = l_2 = 350 mm,
- the resolution of the sensor is 0.001 degrees,
- the backslash is 0.02 degrees,

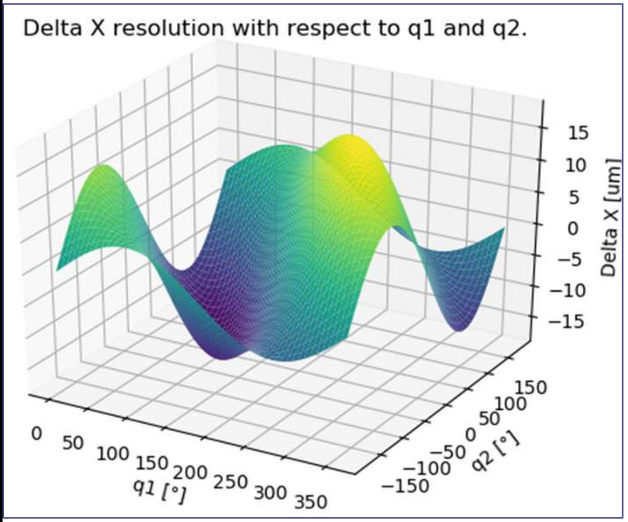
1. To which robot does this remind you?
2. What are the maximum motor speeds if the maximum operational speed is 1m / sec?
3. What are the end effector precisions?

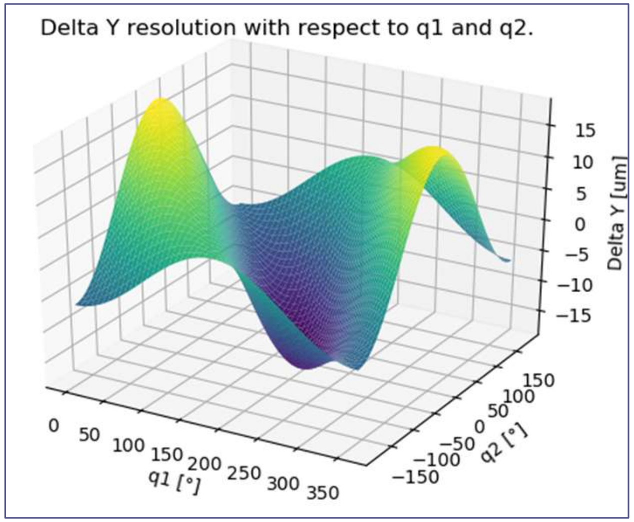
Be careful, distinguish between precision and resolution !!!

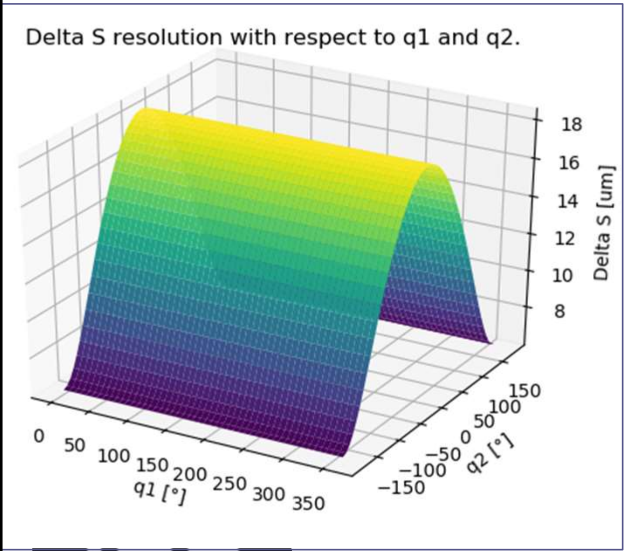
Repeat the calculations for l1 = l2 = 150 mm.



Error Analysis:







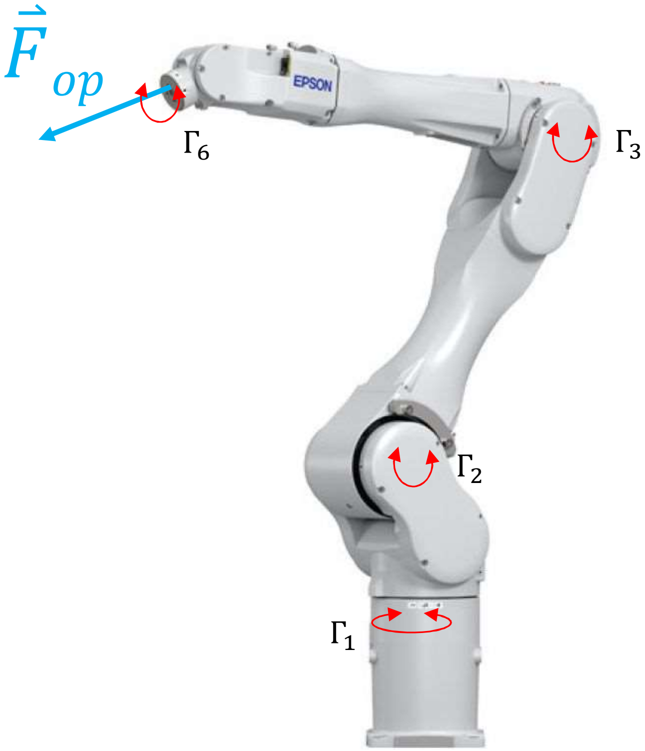
Relationship between Joint and Operational Forces

$$P = F_{op}^T . \dot{X} = \Gamma_{art}^T . \dot{q}$$ (Input / output power equality, assumes efficiency = 1)

$$= F_{op}^T . J . \dot{q} = \Gamma_{art}^T . \dot{q}$$

$$\Rightarrow F_{op}^T . J = \Gamma_{art}^T$$

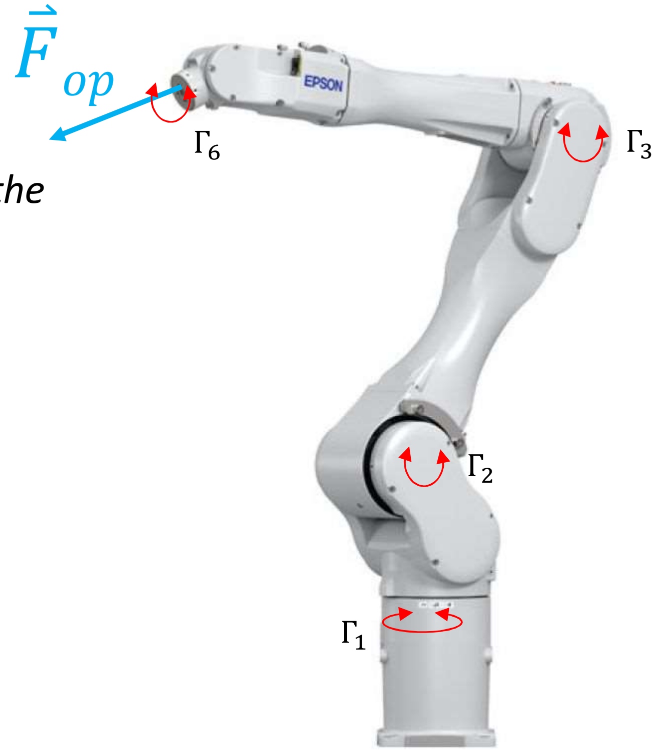
$$\Rightarrow \boxed{J^T . F_{op} = \Gamma_{art}}$$



Relationship between Joint and Operational Forces

$$\Gamma_{art} = J^T \cdot F_{op}$$

This relationship allows the force applied at the tool to be projected onto the various torques required at each joint.



$$\Gamma_{art} = \begin{Bmatrix} \Gamma_1 \\ \Gamma_2 \\ \dots \\ \Gamma_6 \end{Bmatrix} = J^T \cdot \overrightarrow{F}_{op}$$

$\Gamma_i \ (i = 1..6)$ are the joints

Relationship between Joint and Operational Forces

Applications:

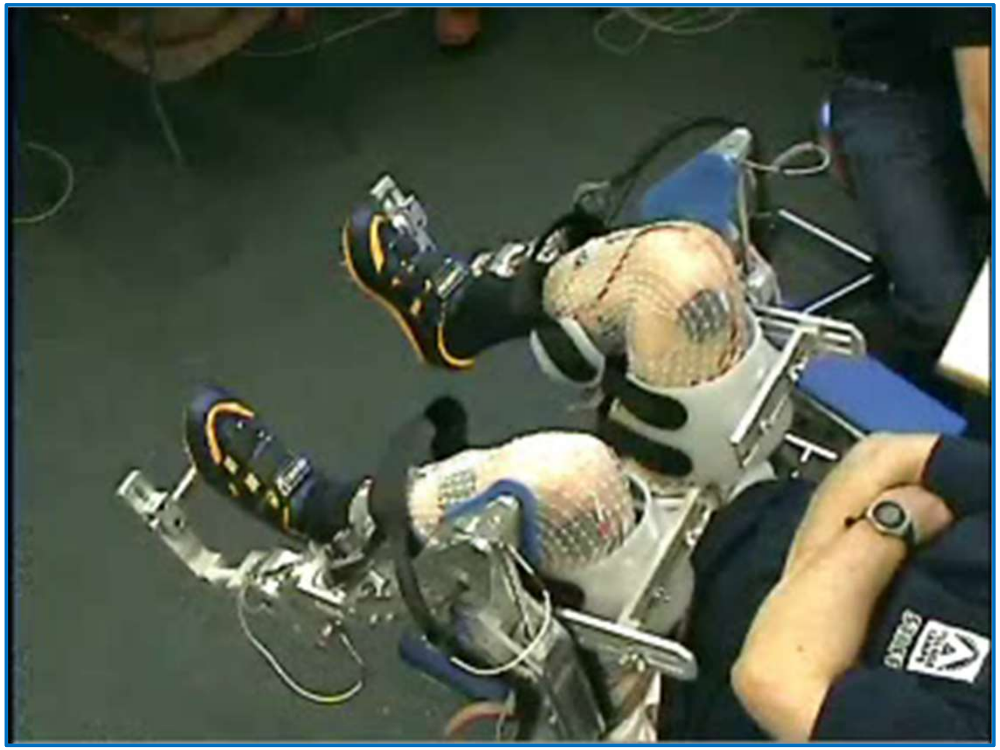
- *Measurement of joint torques if the force sensor is at the side.*
- *Terminal force measurement if sensors are at the joint side.*

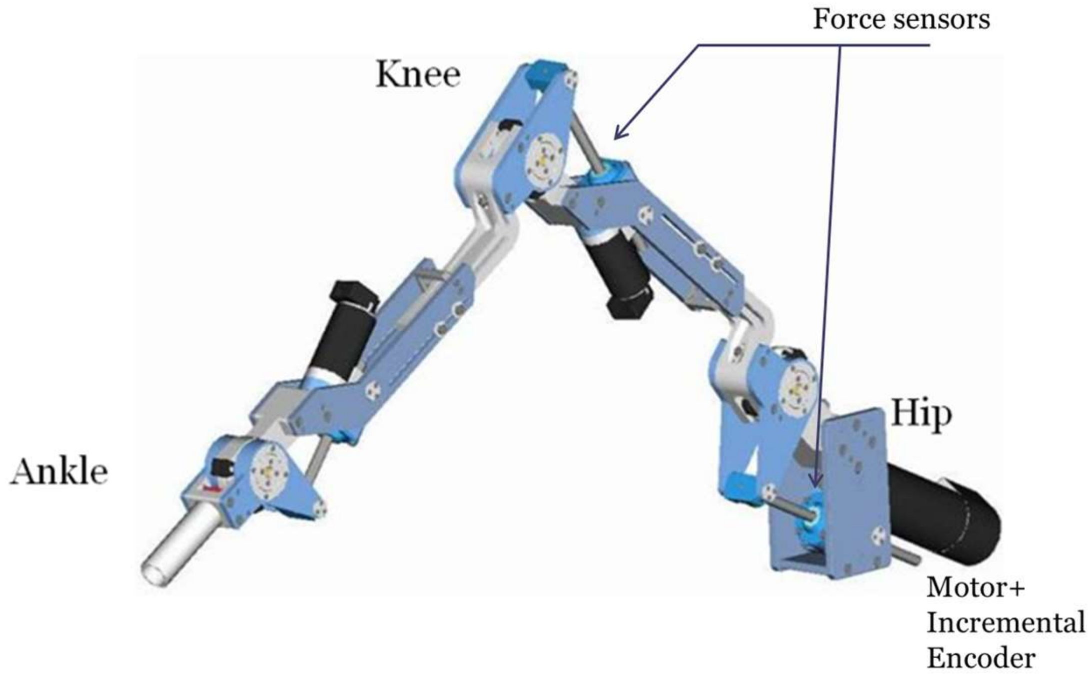
Relationship between Joint and Operational Forces

Example of implementation,

control of the flexion-extension force using muscular electro-stimulation.

The moments are measured at the joint side while being required at the foot side



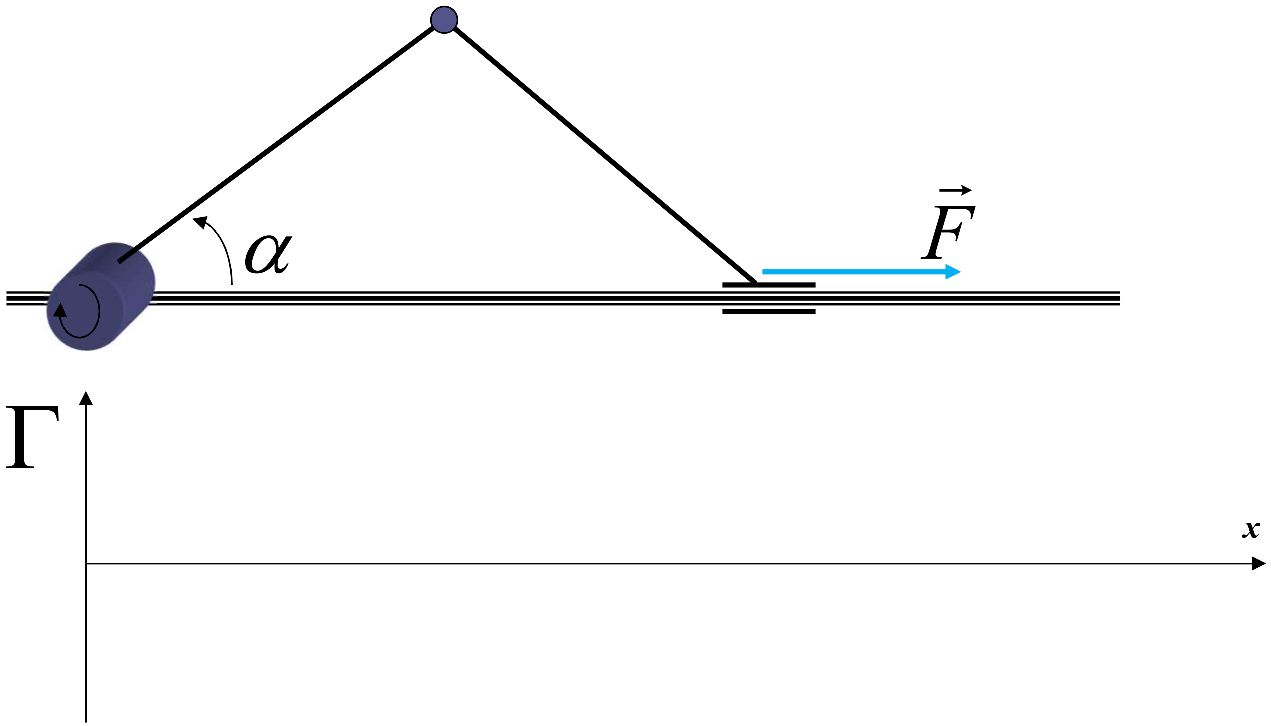


Relationship between Joint and Operational Forces

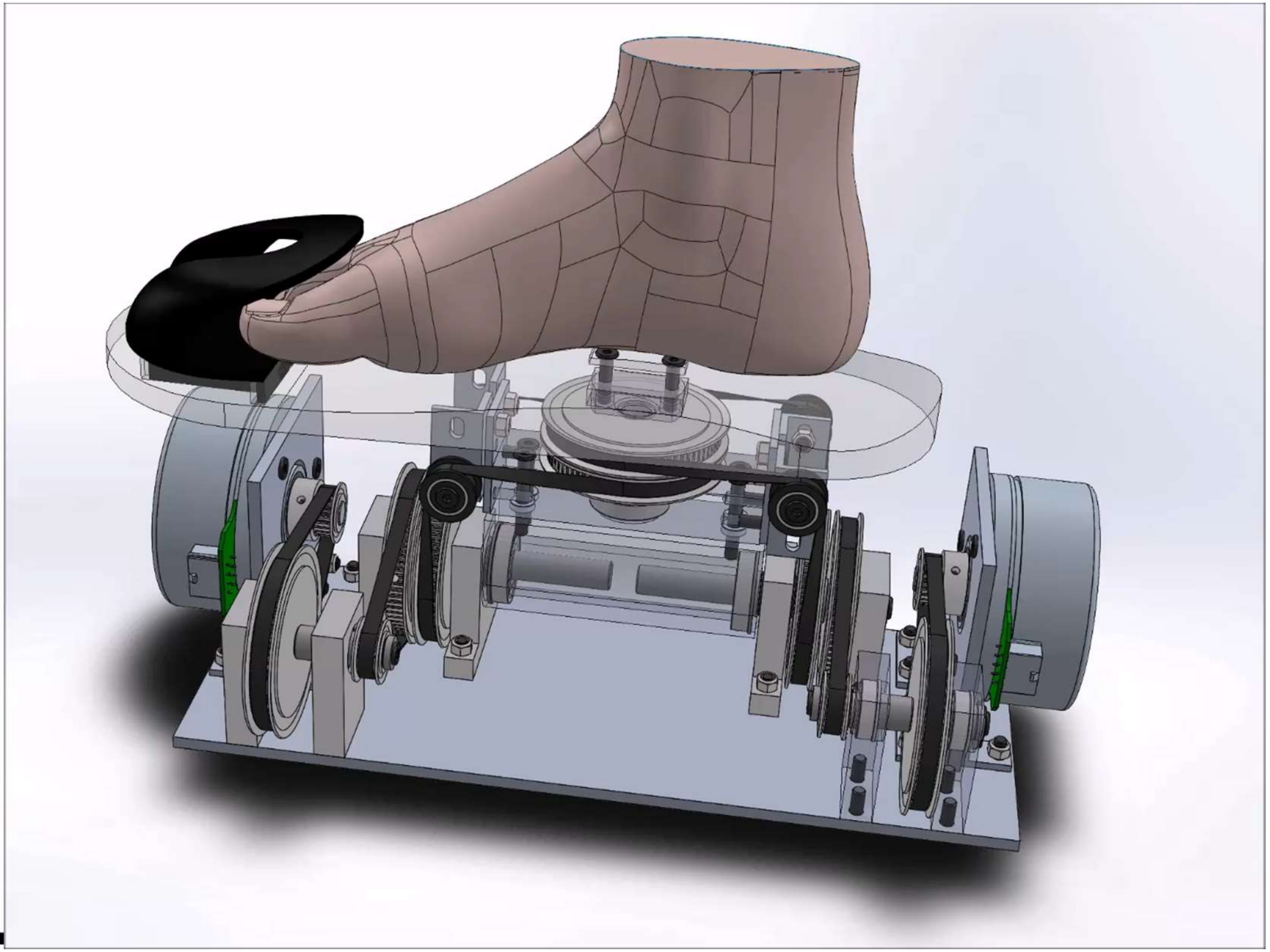
Exercise

Find the torque Γ to be applied by the motor to setup a force F at the output of the following linear axis :

Arm lengths are the same $l_1 = l_2 = l$

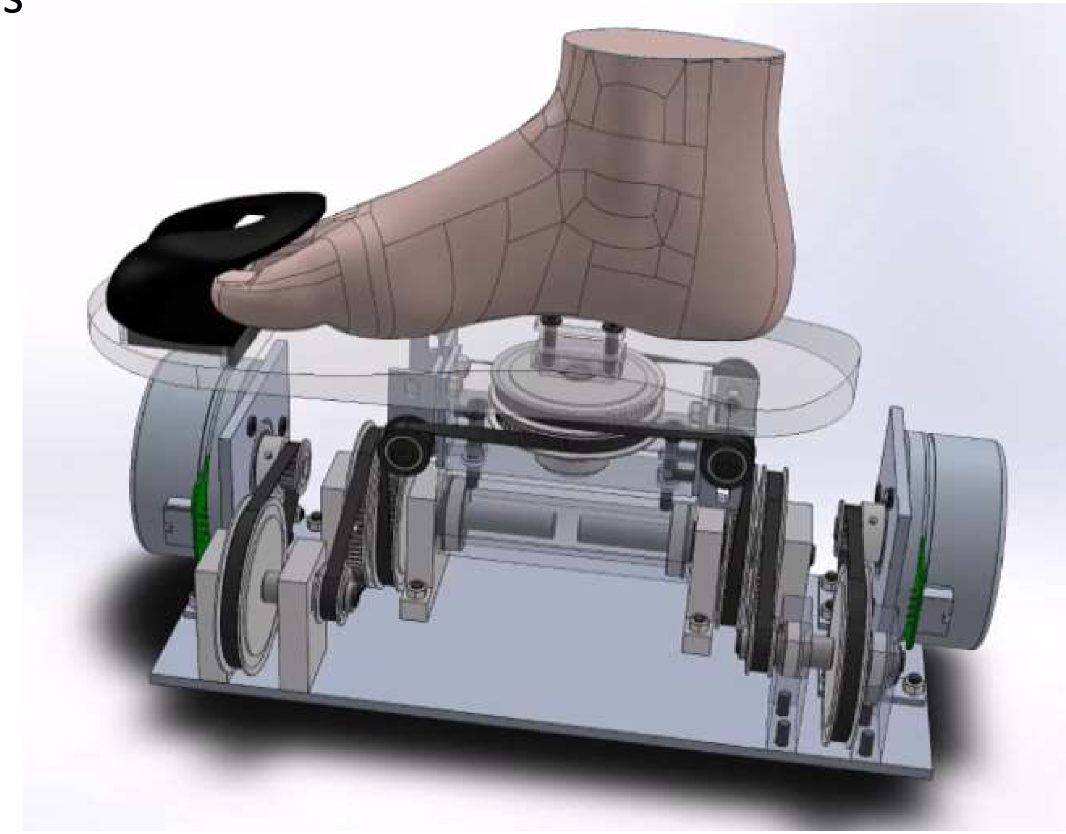


Example 2: 2DOF – Ankle Platform (Design by Jacob Hernandez, REHAssist, EPFL)



Relevant Components

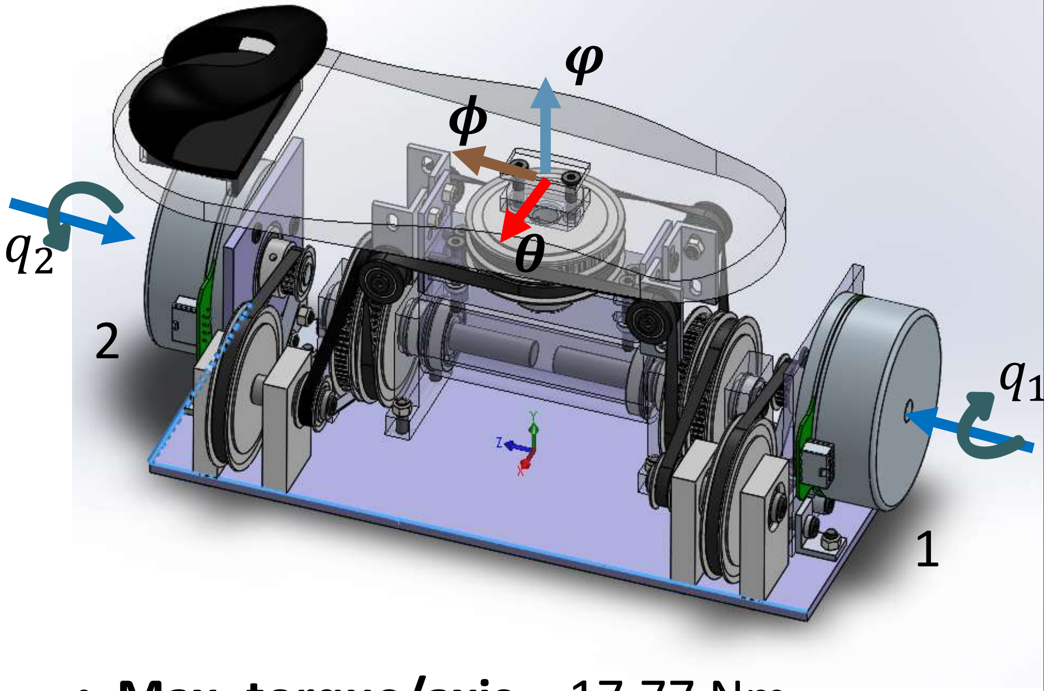
- **Maxon EC-90 (2x):** Flat ø90mm, brushless motor 260W
  - ~1 Nm Nominal Torque
- **S3M 6mm width closed loop belts (6x):**
  - 2x 288mm
  - 2x 291mm
  - 480mm
  - 501mm
- **S3M Pulleys:**
  - 72 (8x) & 22 (4x) Teeth



Mechanism

3-Stages Transmission:

- $\lambda = ({}^{72}/_{22}) \cdot ({}^{72}/_{22}) \cdot (1)$ = **10.71**- total gear ration
- $\eta = (\eta_1) \cdot (\eta_2) \cdot (\eta_3)$ = **0.83** - - total efficiency
- $\eta_{1,2,3} \approx (0.94 \rightarrow 0.96)$
- $\begin{bmatrix} \varphi \\ \phi \end{bmatrix} = -{}^1/_\lambda \begin{bmatrix} q_2 + q_1 \\ q_2 - q_1 \end{bmatrix}$ - Direct Geometric model
- $\begin{bmatrix} \tau_z \\ \tau_y \end{bmatrix} = -\lambda.\eta, \begin{bmatrix} \tau_2 + \tau_1 \\ \tau_2 - \tau_1 \end{bmatrix}$ - Torque transmission



- **Nom. Torque/axis** = 1Nm . 10.71 . 0.83 **= 8.89 Nm**
- **Nom. Speed/axis** =138.81 RPM

- **Max. torque/axis** = 17.77 Nm
- **Max. speed/axis** = 277.6 rpm

- **Max. Rotor's Reflected (λ^2) Inertia** = 2321.83 kg.cm^2

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$$\dot{X} = J \cdot \dot{q} \quad \Rightarrow \quad \dot{q} = J^{-1} \cdot \dot{X}$$

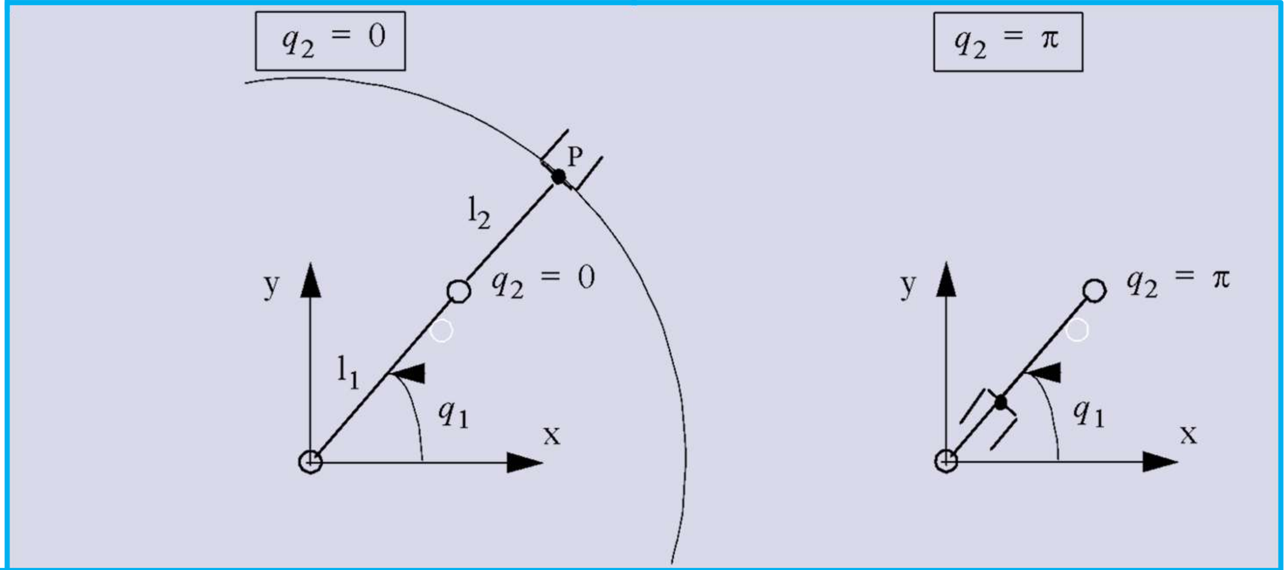
?

det(J) = 0 $\Rightarrow$ *Singularities*

Example, robot RR

$$J^{-1} = \frac{1}{l_1 l_2 \sin q_2} \begin{bmatrix} l_2 \cos(q_1+q_2) & l_2 \sin(q_1+q_2) \\ -l_1 \cos q_1 - l_2 \cos(q_1+q_2) & -l_1 \sin q_1 - l_2 \sin(q_1+q_2) \end{bmatrix}$$

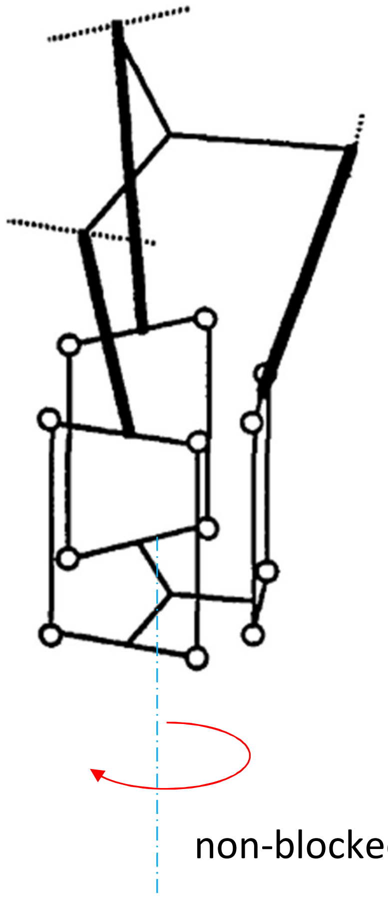
$$l_1 l_2 \sin(q_2) = 0 \Rightarrow q_2 = k \cdot \pi, \quad k = 0, \pm 1, \pm 2, \dots$$

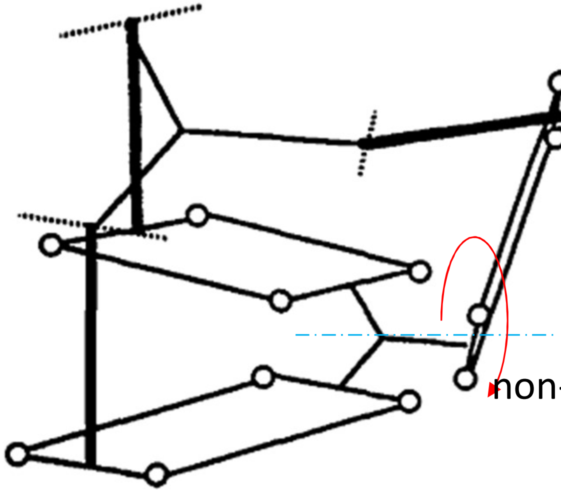


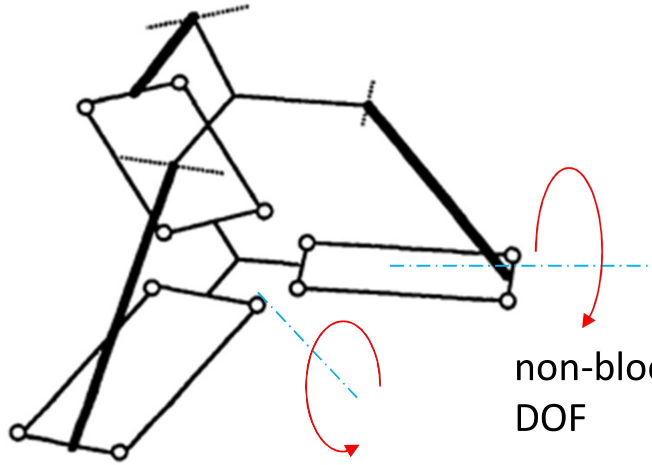
-> They are located at the border (limits) of the robot's working volume.

-> On these singularities, the robot loses one degree of freedom.

Example of Singularities of a parallel robot Delta





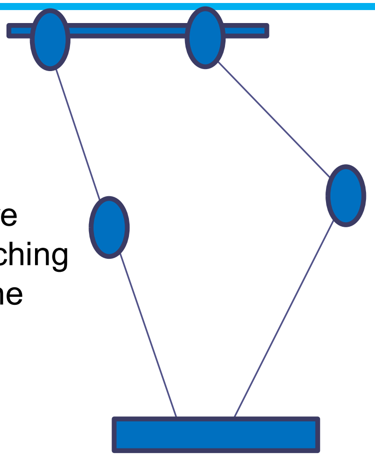


Kinematic singularities

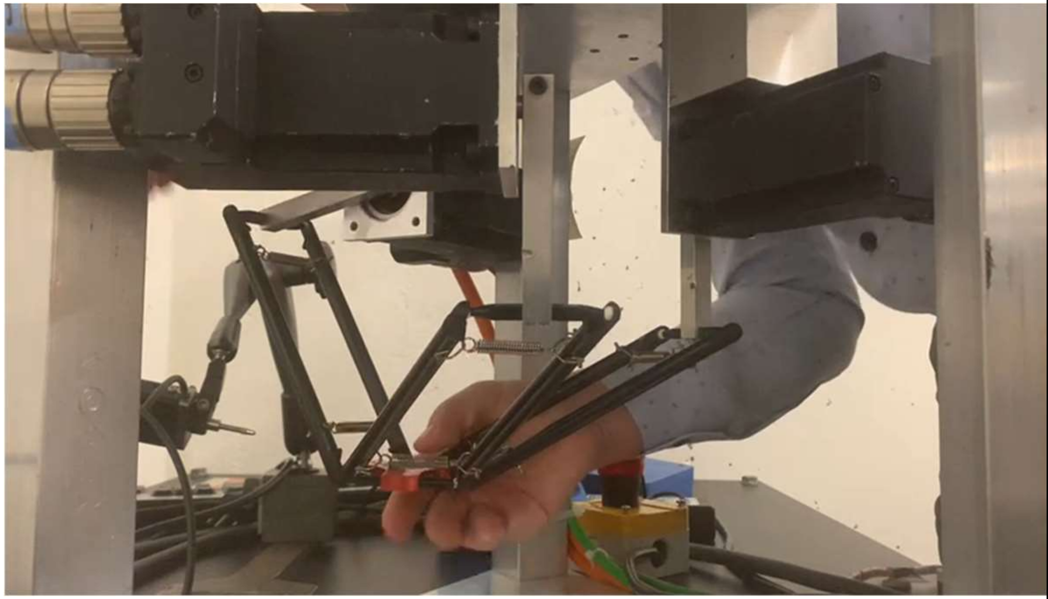
In serial robots, we lose degrees of freedom in singular positions.

In parallel robots, we lose constraints on blocked degrees of freedom (more DOF), in singular positions.

In parallel robotics, border singularities are called (serial-type sigulatries)



zero ion one of the directions



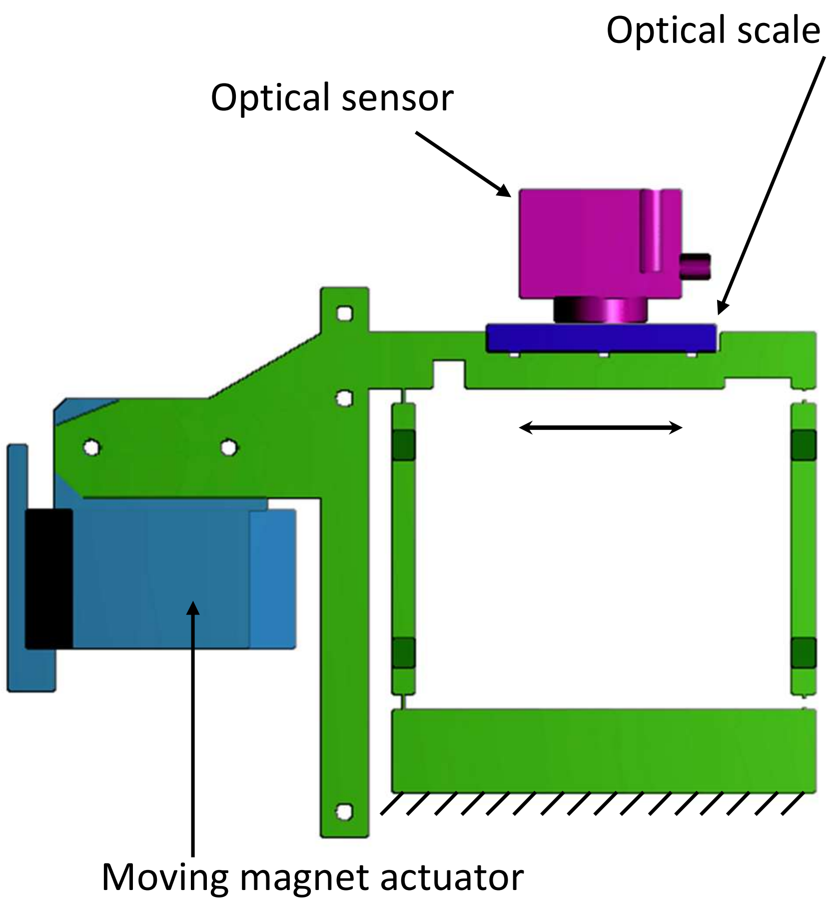
Case study with the microrobot Sigma 6, Numerical inversion of the GM

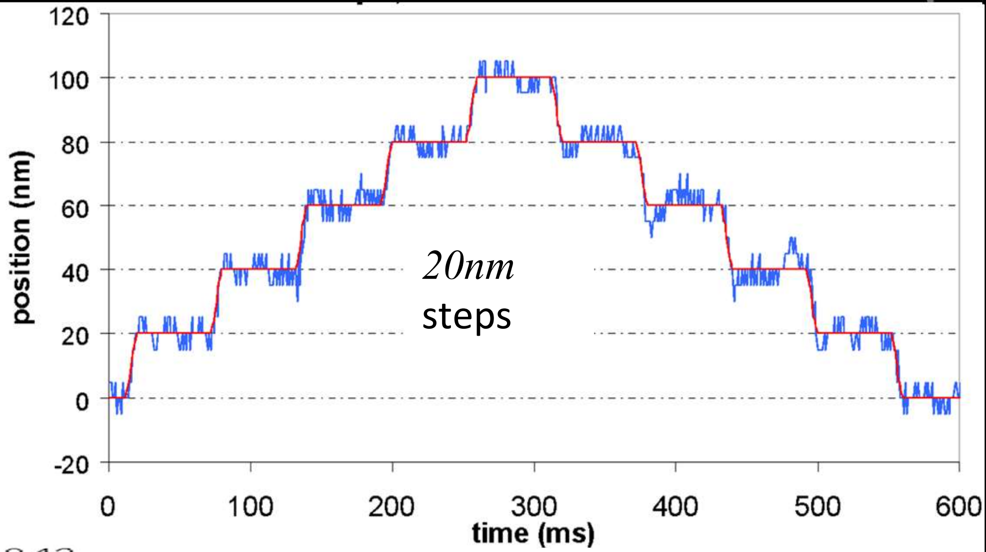
Sigma 6:

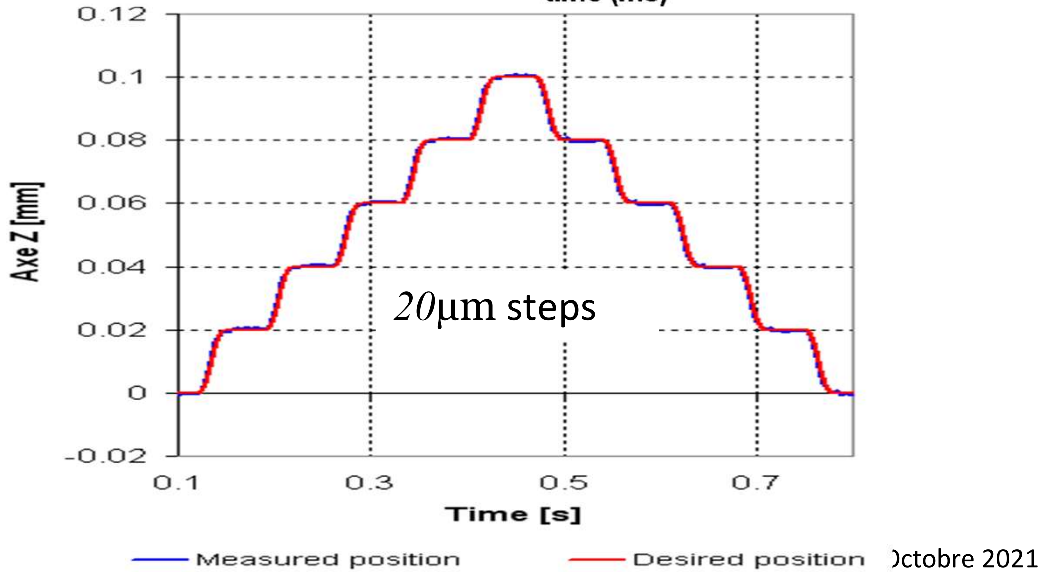
6 DOFs

Dynamic ultra high precision positioning and scanning robot

1 dof linear flexible hinges based guideway







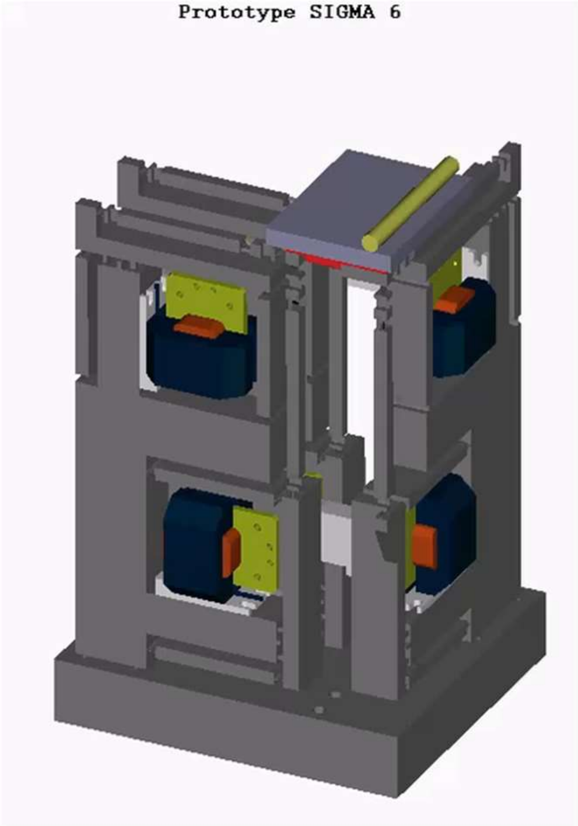
Sigma6

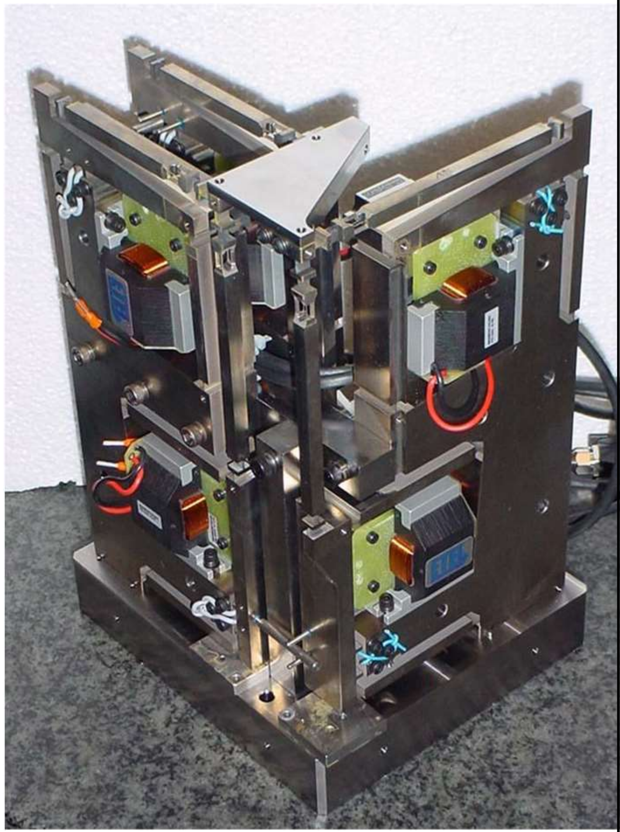
SIGMA 6 characteristics

- Translations x,y,z of 9mm each
- Rotations qx,qy,qz of 10°
- High resolution (5nm and 100 nanoradians)
- High usable bandwidth (>200 Hz)
- Up to 20 N of force in each direction
- Total volume of only ~6dm³

Applications

- Fiber alignment
- Scanning
- Measuring forces and torques
- Ultra high precision positioning and assembly
- …





Prototype

Specifications

SIGMA 6 characteristics

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Applications

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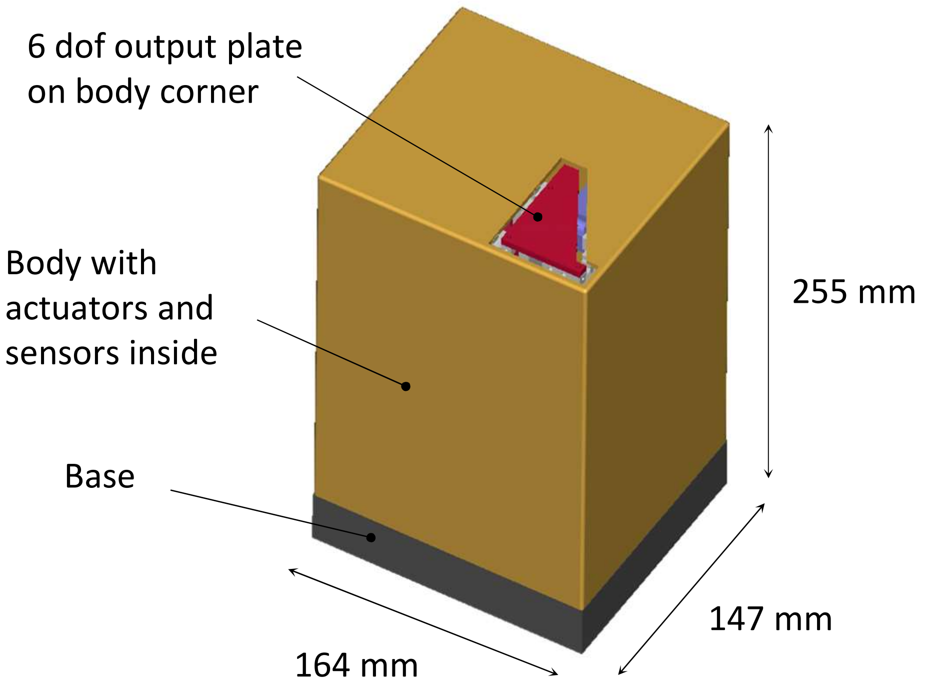


Sigma 6 prototype

6 DOF parallel kinematics

Tool space

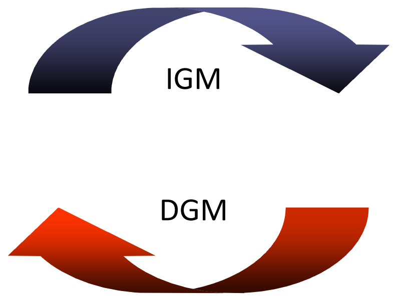
$$\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\vec{\theta} = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}$$

Joint space

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix}$$

analytical is ok



numerical solution by Newton-Raphson method based on the Jacobian

Inverse Geometric Model (IGM)….Done with circles and spheres ☺

- 3 Aij using homogenous mat.

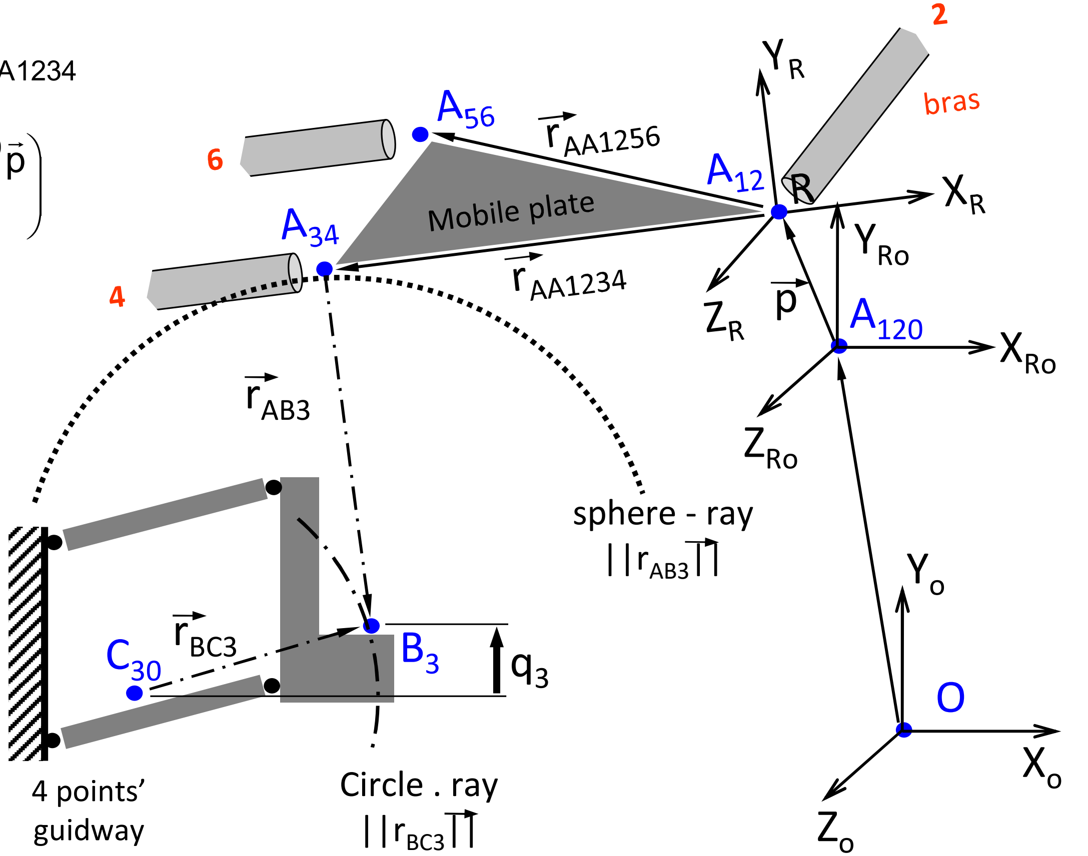
$$ {}^{réf0}\vec{A}_{34} = {}^{réf0}_{réfR0}T \cdot \underbrace{{}^{réfR0}_{réfR}T}_{} \cdot \vec{r}_{AA1234} $$

$$ \begin{pmatrix} {}^{réfR0}_{réfR}R(\vec{\theta}) & {}^{réfR0}\vec{p} \\ \vec{0}^T & 1 \end{pmatrix} $$

- 6 Bi resol. eq. sphere/circle

$$ \begin{cases} \left\| \vec{A}_{34} - \vec{B}_3 \right\|^2 = \left\| \vec{r}_{AB3} \right\|^2 \\ \left\| \vec{B}_3 - \vec{C}_{30} \right\|^2 = \left\| \vec{r}_{BC3} \right\|^2 \end{cases} $$

- 6 qi / function of Bi



The direct Geometric Model (DGM) is obtained using a numerical approach.

Iterative Newton-Raphson method

$\hat{\vec{q}}_k$ is **the computed value** of $\vec{q}$ at the k-th iteration

$\vec{q}$ is the (***input***) and **J is the Jacobian Matrix** (here, **constant**)

$$\boxed{\begin{pmatrix}\hat{\vec{p}}\\ \hat{\vec{\theta}}\end{pmatrix}_{k+1} = \begin{pmatrix}\hat{\vec{p}}\\ \hat{\vec{\theta}}\end{pmatrix}_{k} + J\cdot(\vec{q}-\hat{\vec{q}}_k)}$$

Such that: $\hat{\vec{q}}_k = MGI\begin{pmatrix}\hat{\vec{p}}\\ \hat{\vec{\theta}}\end{pmatrix}_{k}$

k=k+1 till reaching $\left\|\vec{q}-\hat{\vec{q}}_k\right\| \leq \varepsilon$

$\begin{pmatrix}\hat{\vec{p}}\\ \hat{\vec{\theta}}\end{pmatrix}_{k=0}$ for **k=0 is the last calculated position** (i.e. at the previous $t_{sampling}$)

Initial error of $\vec{p}_{estimé}$	5 mm	1 mm	100 µm	10 µm	1 µm	100 nm	10 nm
number of itérations for final error $\Delta\vec{q}$ < 1nm	8	7	6	5	4	3	2

Jacobian Utilities - Important Summary

1. It links the joint speeds to the operational ones, it gives important information on the nominal and maximum speeds of the motors to be selected. It is a **reduction or transmission matrix**.

2. It is also called the **sensitivity matrix** because it is allowing to know the sensitivity at the output level knowing those of the joints (motors + local transmission).

3. It allows the projection of forces between the joint and the tool spaces,

4. It allows to know and control the singularities of the robot. It is also called a **stability matrix**.

5. It allows to **numerically invert the** direct **geometric model** (ie. By using the Newton-Raphson method) to obtain the inverse geometric model (or vice versa).

6. It can be used in several **control approaches**.

7. Useful for dynamic modeling.