Recap Kinematics

KINEMATICS -

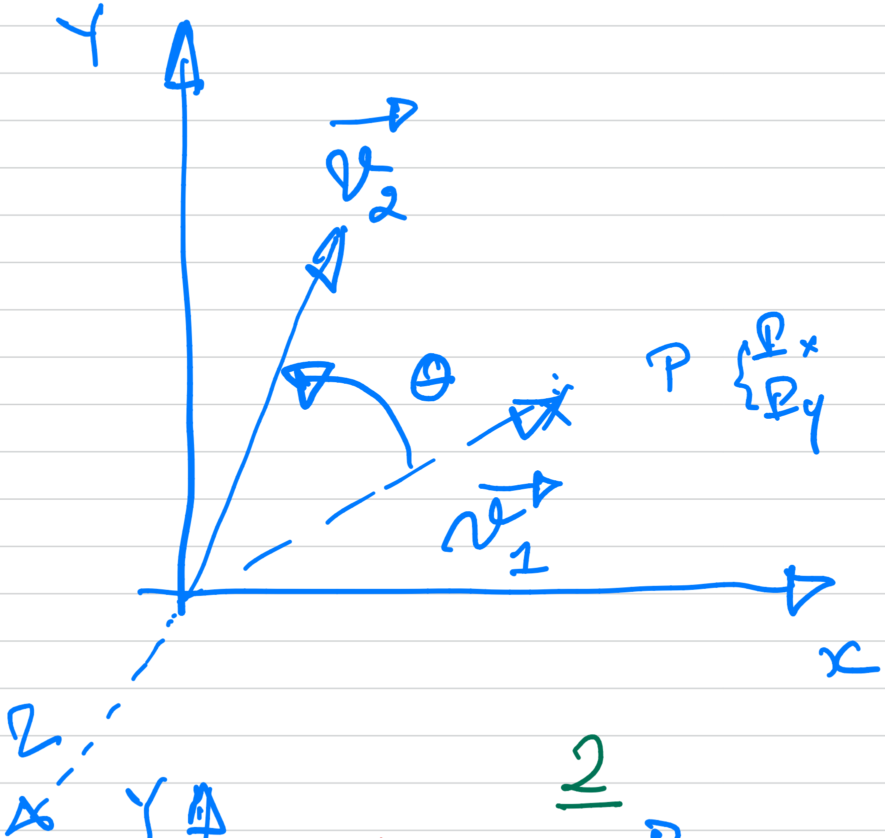
Part 1 Matrices of Rotation, Homogenous Matrices, in 2D - in 3D.

Part 2 Geometric modeling of serial robots

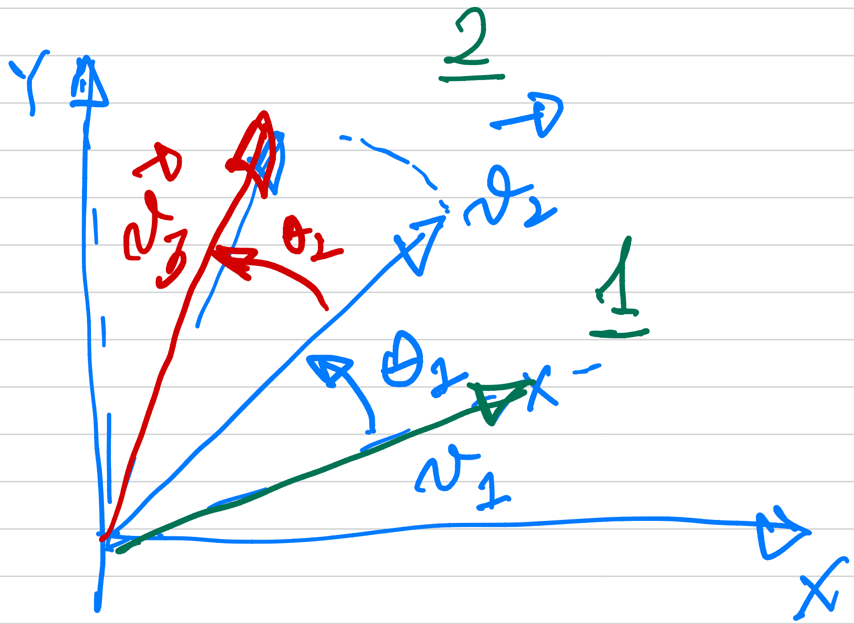
Part 1

2D - Matrices of rotation
translation
→ Homogenous Matrices (2D)

Rotation around an arbitrary point



$$\vec{v}_2 = R_z(\theta)\cdot\vec{v}_1$$



$$\vec{v}_3 = R_z(\theta_2)\,\vec{v}_2$$

$$\vec{v}_2 = R_z(\theta_1)\cdot\vec{v}_1$$

$$\vec{v}_3 = \underbrace{R_z(\theta_2)\cdot R_z(\theta_1)}_{R_z(\theta_1,\theta_2)}\,\vec{v}_1$$

$$R_z(\theta_1, \theta_2) = \underline{R_z(\theta_2)} \cdot R_z(\theta_1)$$

$$R_1 \rightsquigarrow R_2 \rightsquigarrow R_3 \quad \dots \quad R_n$$

$$\boxed{R = R_n \cdot R_{n-1} \cdot R_{n-2} \cdots R_1}$$

Valid in 2D as well as in 3D

$$\xrightarrow{R_1} \quad \xrightarrow{R_2}$$

$$R = R_2 \cdot R_1 \qquad \text{around axis } z$$

$$R_2 = \begin{vmatrix} \cos\theta_2 & -\sin(\theta_2) \\ \sin\theta_2 & \cos(\theta_2) \end{vmatrix} = \begin{vmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{vmatrix}$$

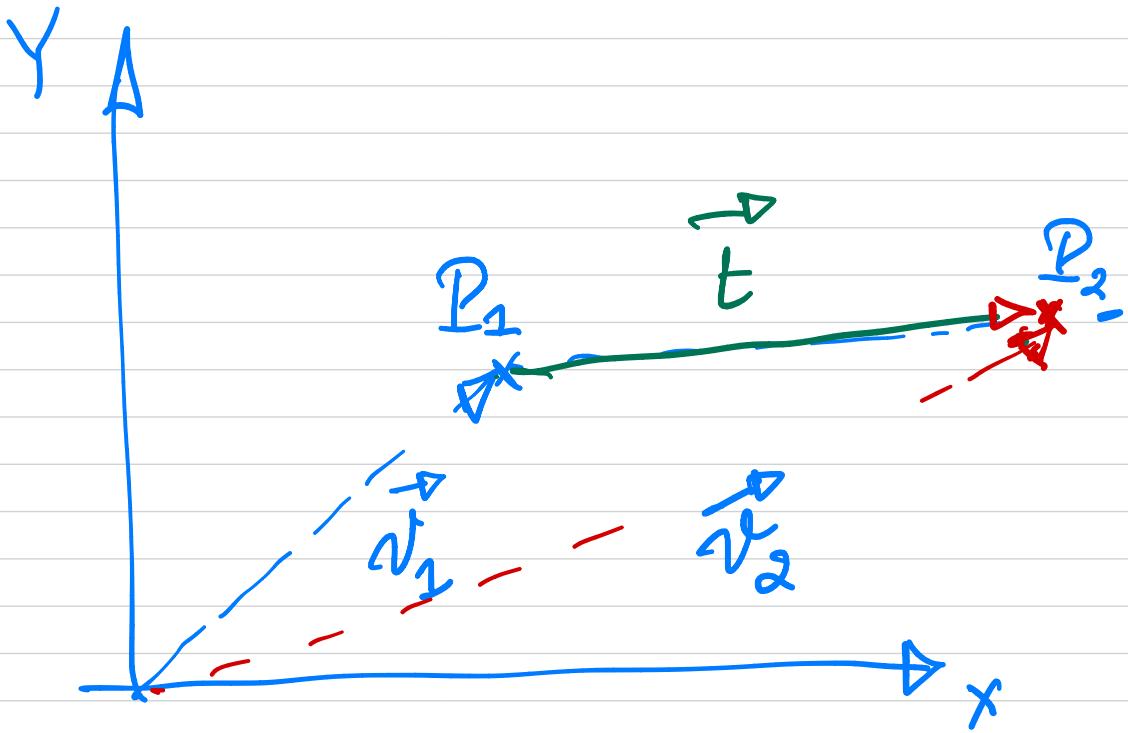
$$R_1 = \begin{vmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{vmatrix}$$

$$R = R_2 \cdot R_1 = \begin{vmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{vmatrix} \begin{vmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{vmatrix} = \begin{vmatrix} c_1 c_2 - s_1 s_2 & -s_1 c_2 - s_2 c_1 \\ c_1 s_2 + c_2 s_1 & -s_1 s_2 + c_1 c_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{vmatrix}$$

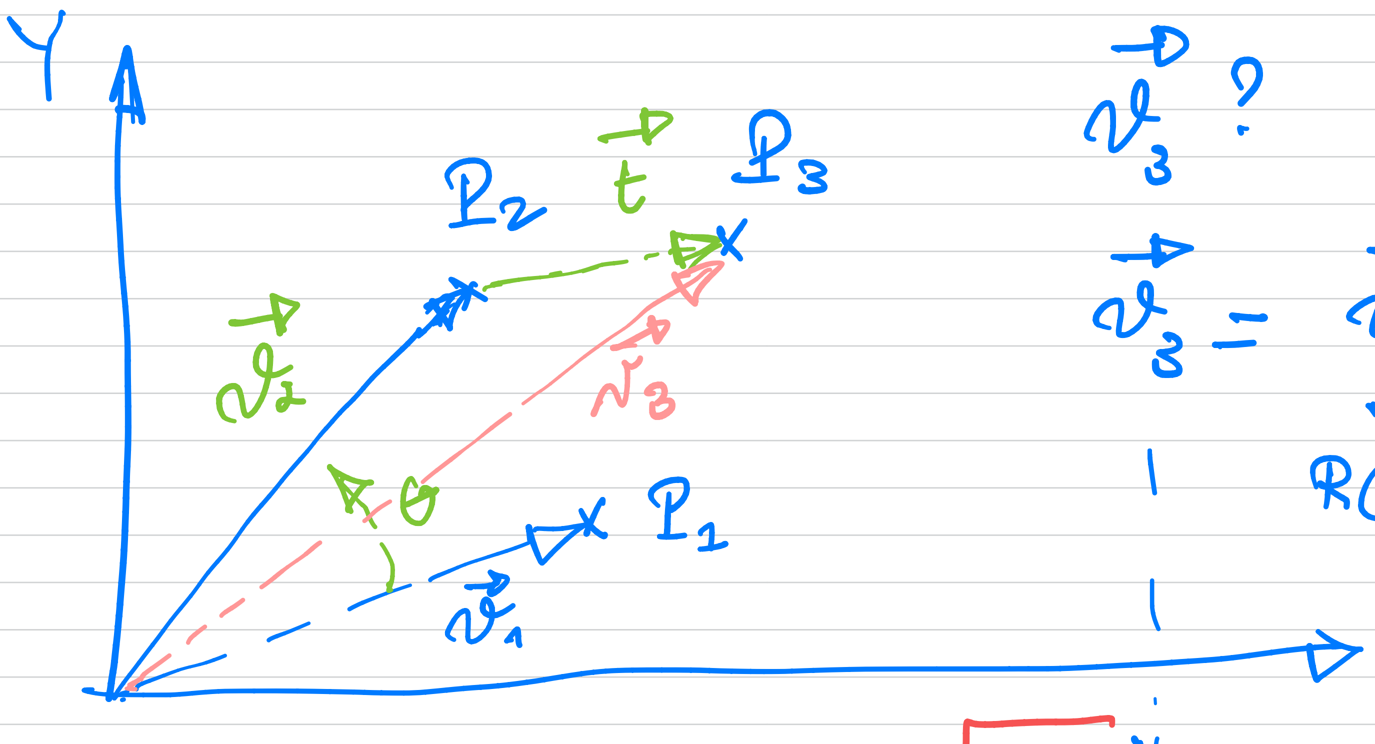
$$= R_z(\theta_1 + \theta_2)$$

2D : translation



$$\vec{v}_2 = \vec{v}_1 + \vec{t}$$

$\vec{t}$ combined with a Rotation



$\vec{v_3}$?

$$\vec{v_3} = \underbrace{\vec{v_2}}_{R(\theta)\cdot\vec{v_1}} + \vec{t}$$

$$\boxed{\vec{v_3} = \underbrace{R(\theta)\cdot\vec{v_1}}_{\text{vector}} + \underbrace{\vec{t}}_{\text{vector}}}$$

? use a matricial representation to combine - translation operation & - Rotation operation.

yes - Using Homogenous Matrices.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} \leftarrow v_3 = R(\theta)\,\vec{v}_1 + \vec{t}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix}}_{M_H(\theta, t)} \cdot \begin{Bmatrix} x_1 \\ y_1 \\ 1 \end{Bmatrix} = \begin{pmatrix} R\vec{v}_1 + \vec{t} \\ 1 \end{pmatrix}$$

$$P = \begin{bmatrix} x \\ y \\ \boxed{1} \end{bmatrix}$$

to avoid

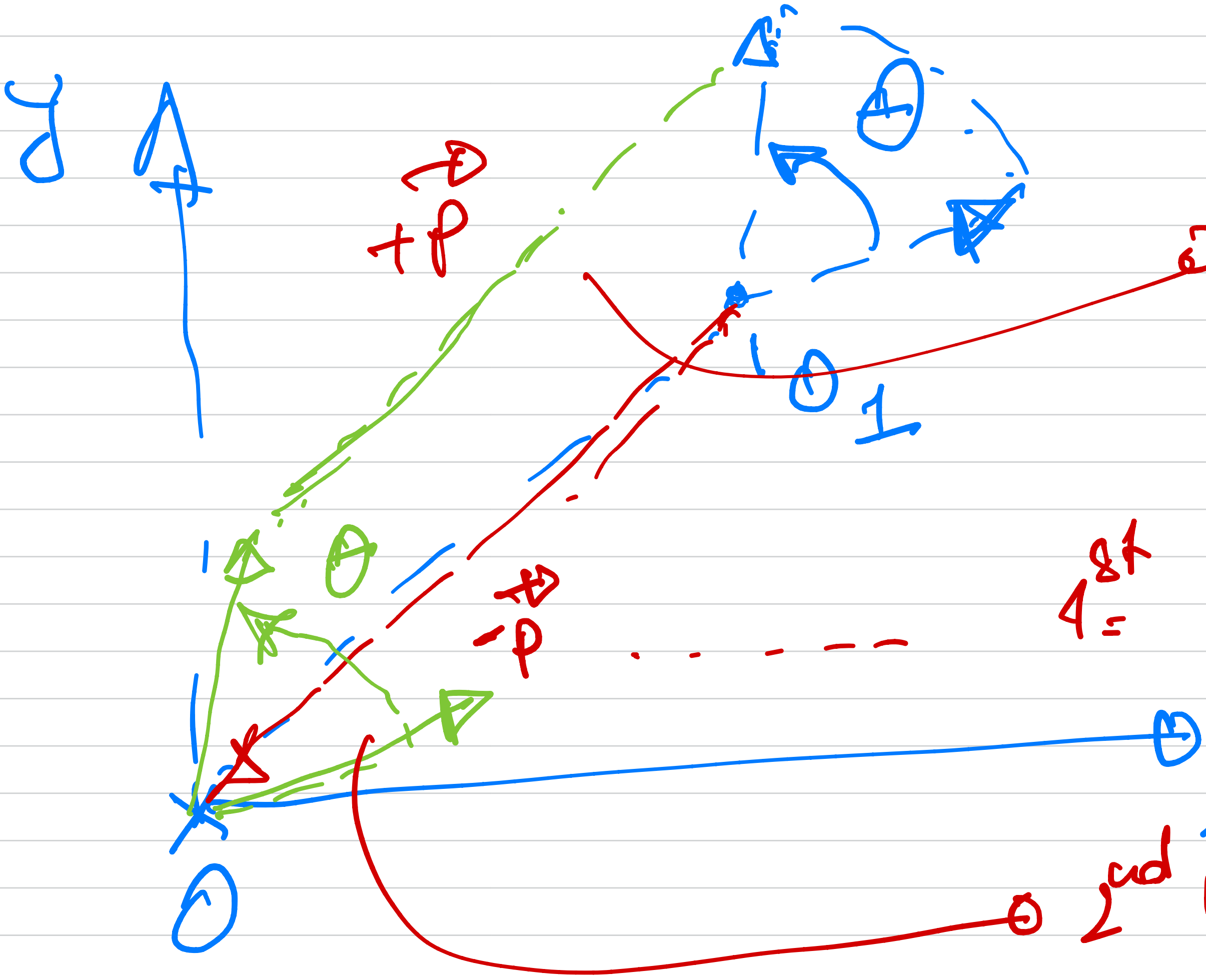
* When using Homog Matrices

$$P = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} [H_{2D}] : 3 \times 3 \\ [H_{3D}] : 4 \times 4 \end{bmatrix}$$

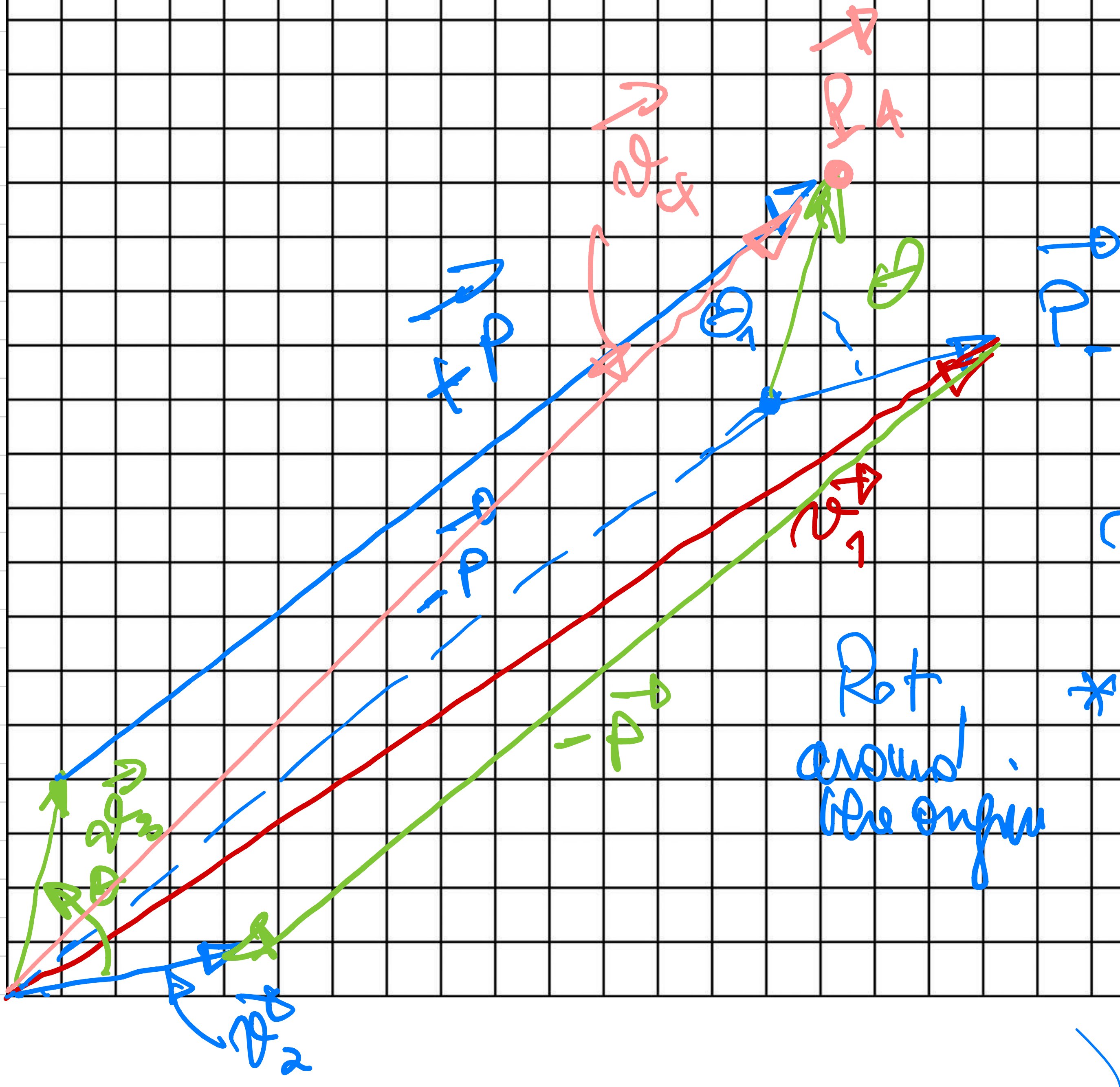
2D : How to do a Rotation around an arbitrary point



1^{st} transf : $-\vec{p}$

2^{nd} transf : $R(\theta)$

3^{rd} transf : $+\vec{p}$



$$\vec{v}_2 = \vec{v}_1 - \vec{P}$$

Rot around the origin

$$*\ \vec{v}_3 = R(\theta) \cdot \vec{v}_2$$

$$\vec{v}_4 = \vec{v}_3 + \vec{P}$$

$P_4 ? : \vec{v}_4$

$\vec{v}_2 = \vec{v}_1 - \vec{p}$

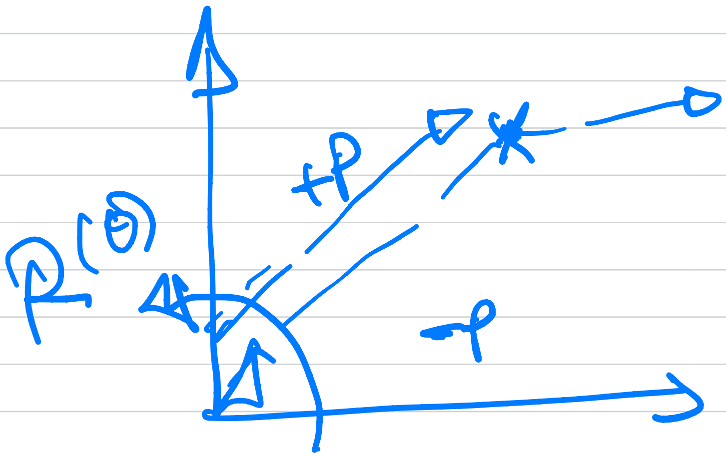
$$\vec{v}_4 = \vec{v}_3 + \vec{p}$$

$$\vec{v}_3 = R \cdot \vec{v}_2$$

$$\vec{v}_4 = \vec{v}_3 + \vec{p} = R\vec{v}_2 + \vec{p} = R\,\vec{v}_1 - R\vec{p} + \vec{p}$$

$$\boxed{\vec{v}_4 = R\,\vec{v}_1 + (\vec{p} - R\vec{p})}$$

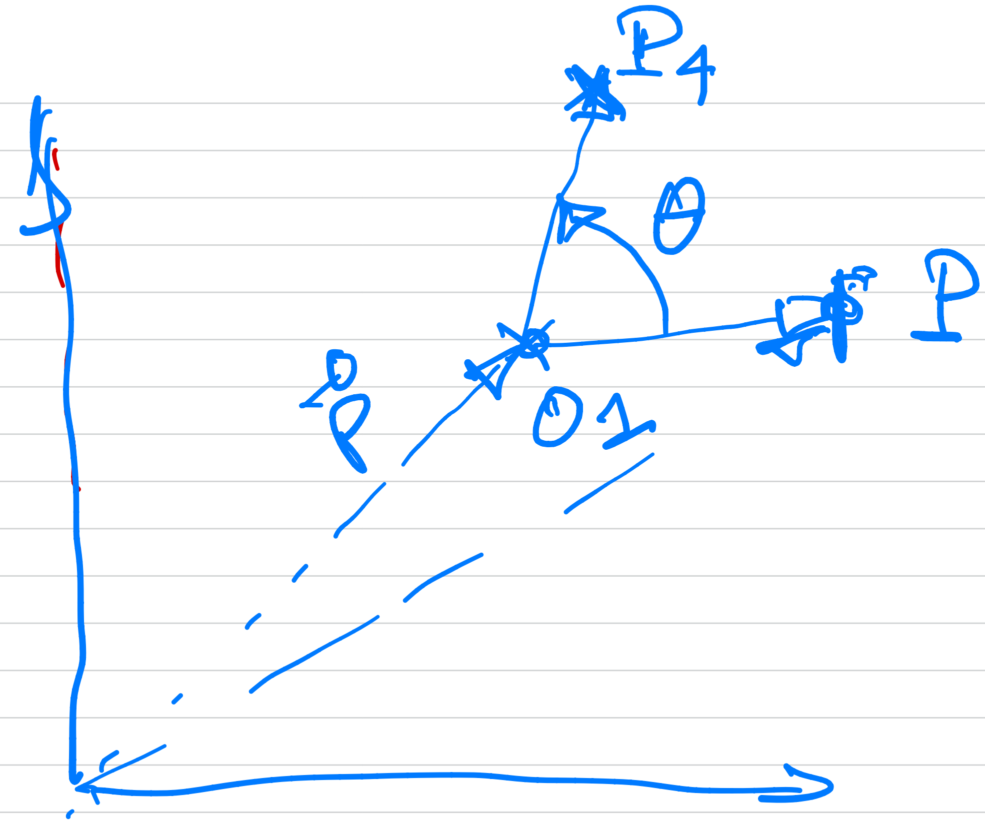
$$\underset{\smile}{v_4} = \begin{vmatrix} R & p - Rp \\ 0 & 1 \end{vmatrix} \underset{\smile}{v_1} \leftarrow \text{extended vectors}$$



$$H(p,\theta) = H_p \cdot H_R \cdot H_{-p}$$

$$= \begin{vmatrix} I & p \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} R & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} I & -p \\ 0 & 1 \end{vmatrix}$$

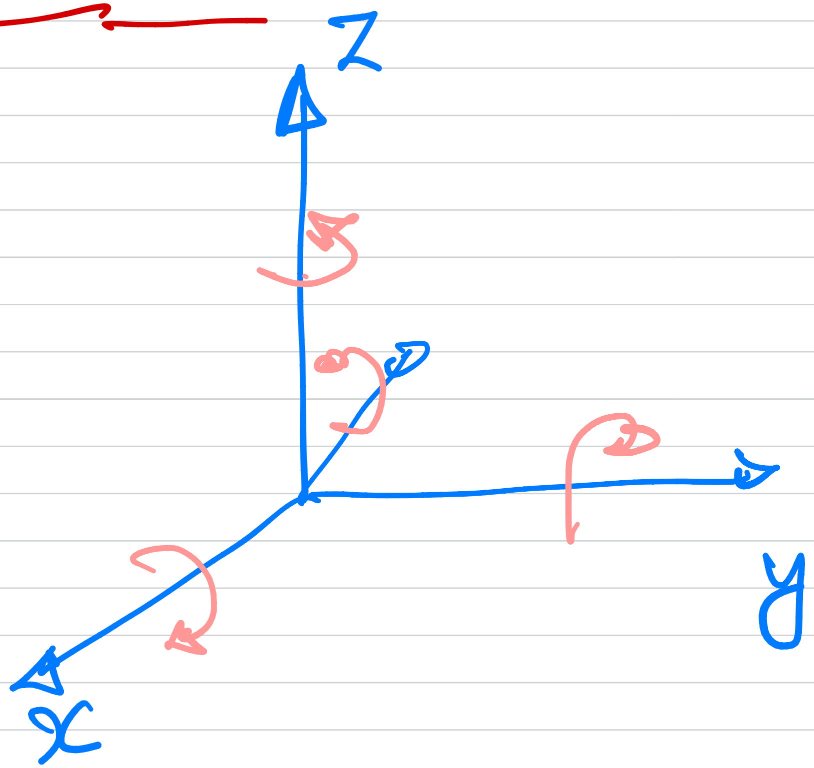
$$= \begin{vmatrix} I & p \\ 0 & 1 \end{vmatrix} \begin{vmatrix} R & -Rp \\ 0 & 1 \end{vmatrix} = \boxed{\begin{vmatrix} R & p - Rp \\ 0 & 1 \end{vmatrix}}$$



Transformation :

$$H = \begin{bmatrix} R & p - Rp \\ 0 & 1 \end{bmatrix}$$

Rotations in 3D



Basic rotations

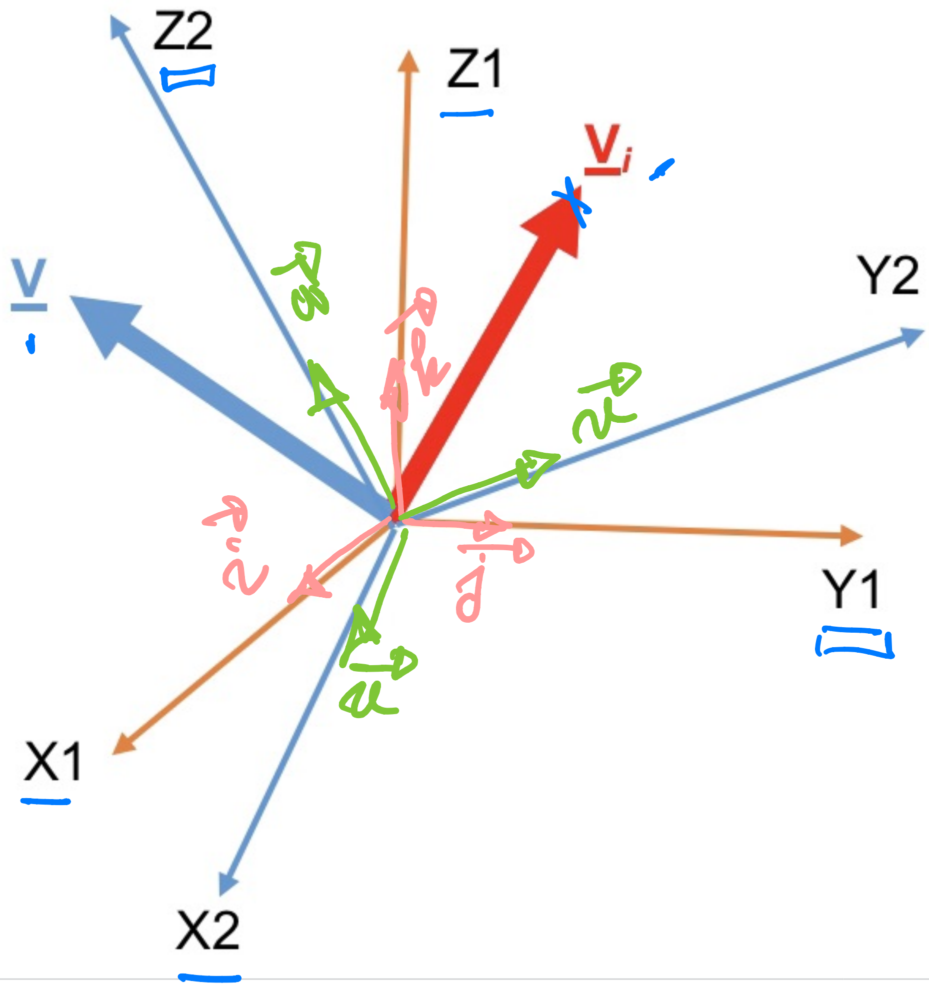
R_x , R_y , R_z

$R = \text{Combi}(R_x, R_y, R_z)$

$\{X_1, Y_1, Z_1\}$ is associated with a base $(\vec{i}, \vec{j}, \vec{k})$

$\{X_2, Y_2, Z_2\}$ is associated with a base $(\vec{u}, \vec{v}, \vec{w})$

$$R = \begin{array}{c} \begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array} \\ \begin{vmatrix} u_i & v_i & w_i \\ u_j & v_j & w_j \\ u_k & v_k & w_k \end{vmatrix} \end{array}$$



Rotations around X, Y, Z

axis_X

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

axis Z

$$\mathbf{R}_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

axis_Y

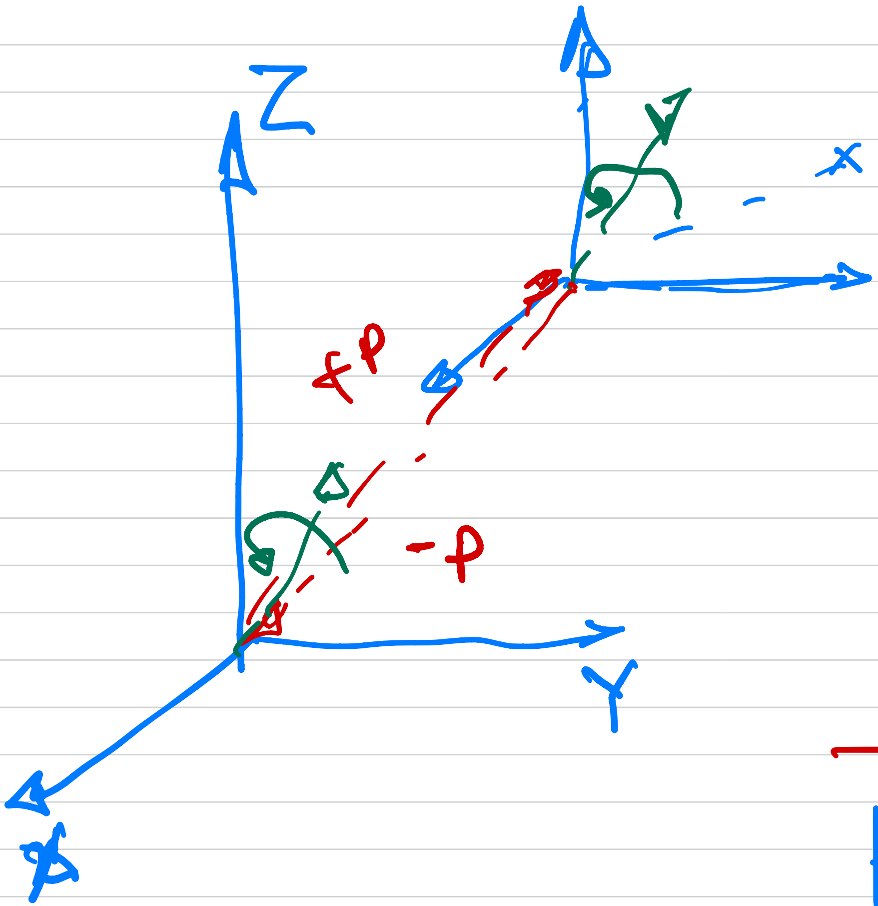
$$\mathbf{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$\det = 1$

$\cos^2 + \sin^2 = 1$

3D

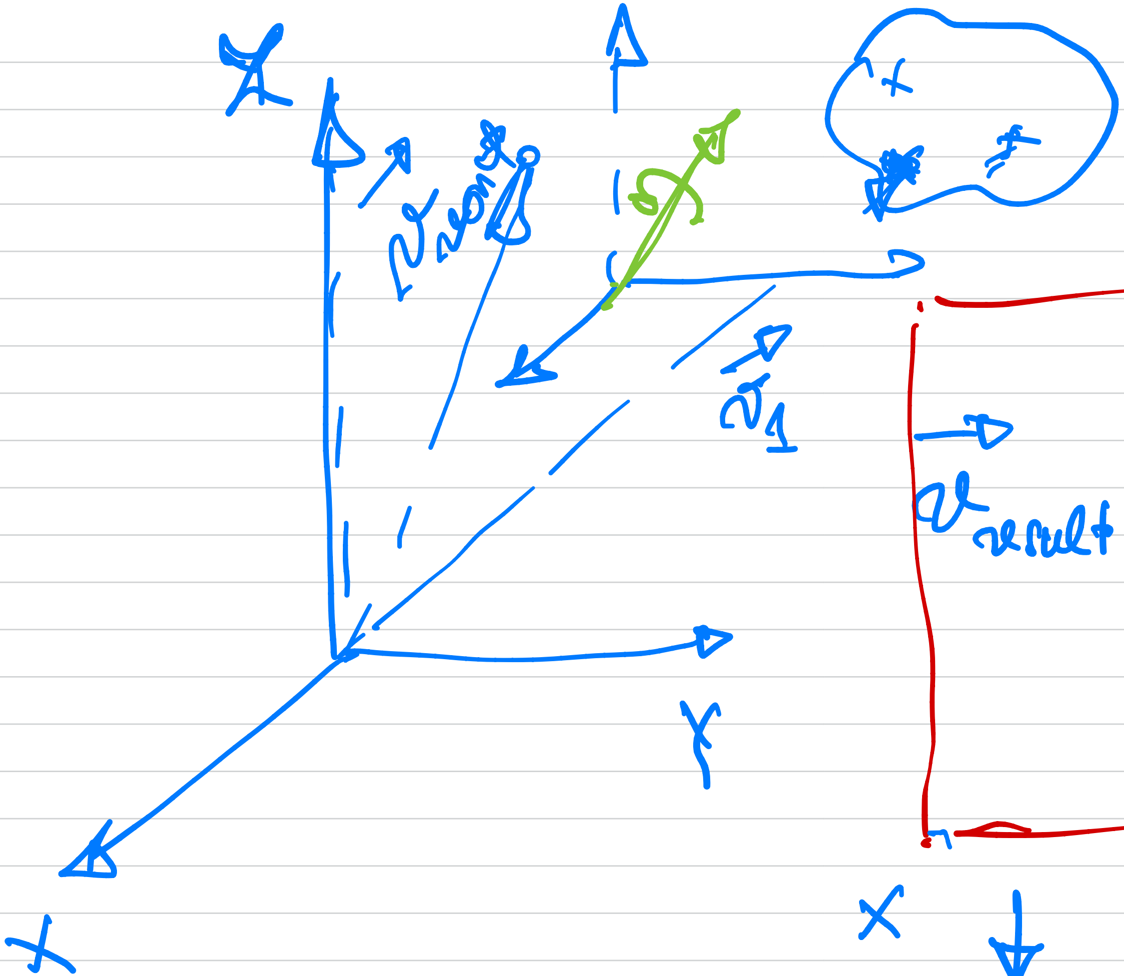
What to do, if the rotation is not realized through (wrt) an axis which passes by the origin?



$$H = H_{p} \, H_{R} \, H_{-p}$$

$$= \begin{vmatrix} I & p \\ 0 & 1 \end{vmatrix} \begin{vmatrix} R & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} I & -p \\ 0 & 1 \end{vmatrix}$$

$$\boxed{H = \begin{vmatrix} R & p - Rp \\ 0 & 1 \end{vmatrix}}$$



$$\vec{v}_{result} = \begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \\ 1 \end{bmatrix} = \begin{bmatrix} R & p-Rp \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \\ 1 \end{bmatrix}$$

X

$\downarrow$

D.G.M.

DIRECT GEOMETRIC MODEL

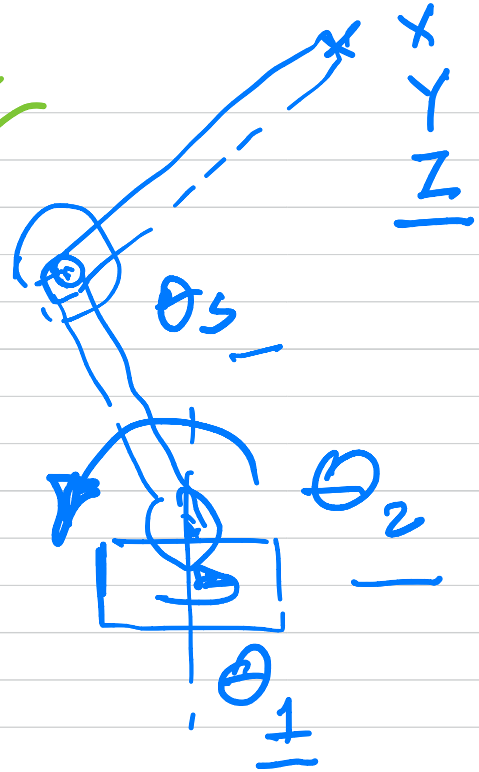
actuator $\rightarrow$ tool

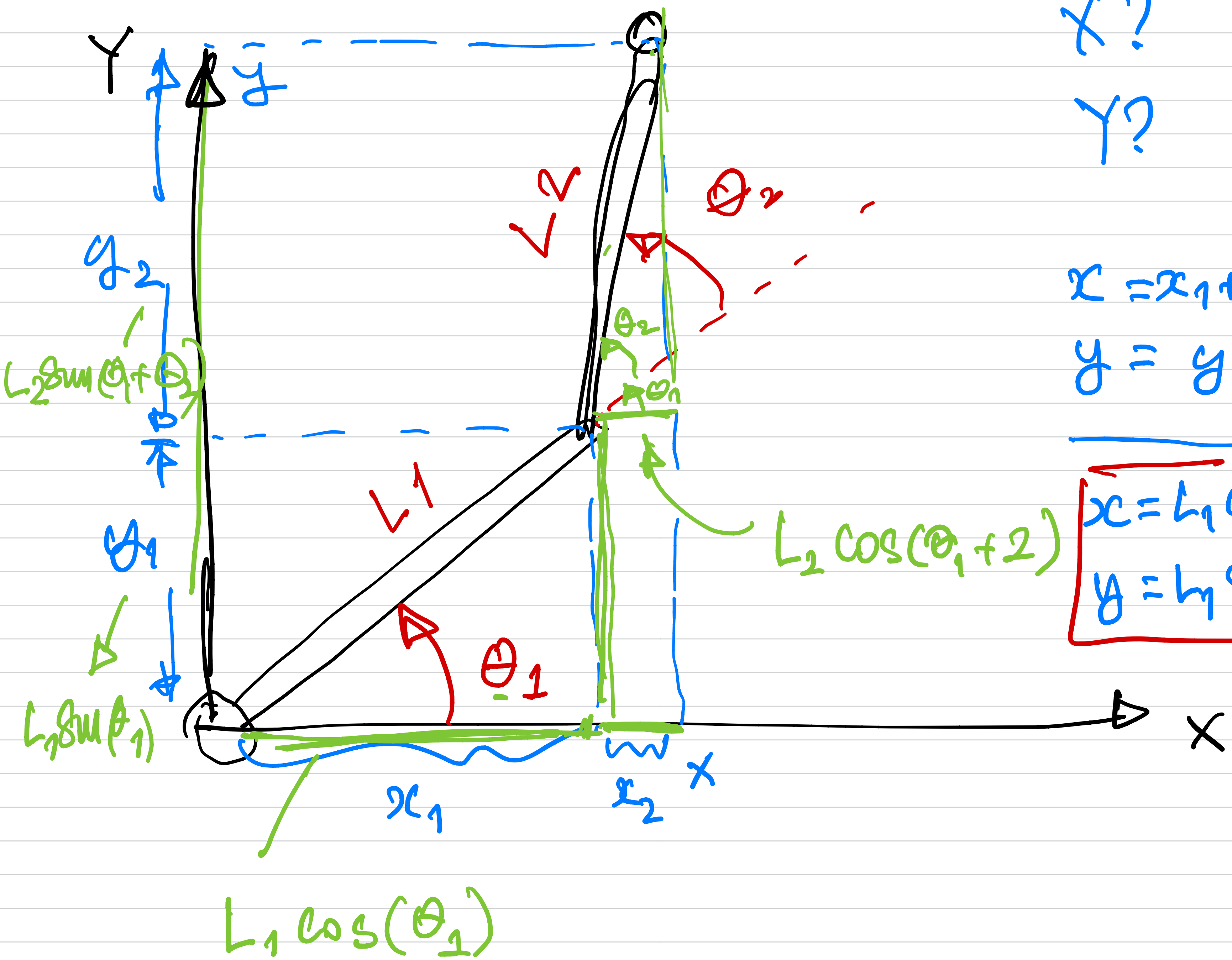
$$\theta_1 \rightarrow \boxed{\begin{matrix} \underline{D} \\ G \\ M \end{matrix}} \rightarrow X$$

$$\theta_2 \rightarrow \qquad \rightarrow Y$$

$$\theta_3 \rightarrow \qquad \rightarrow Z$$

Motor/actuator values





X? as functions

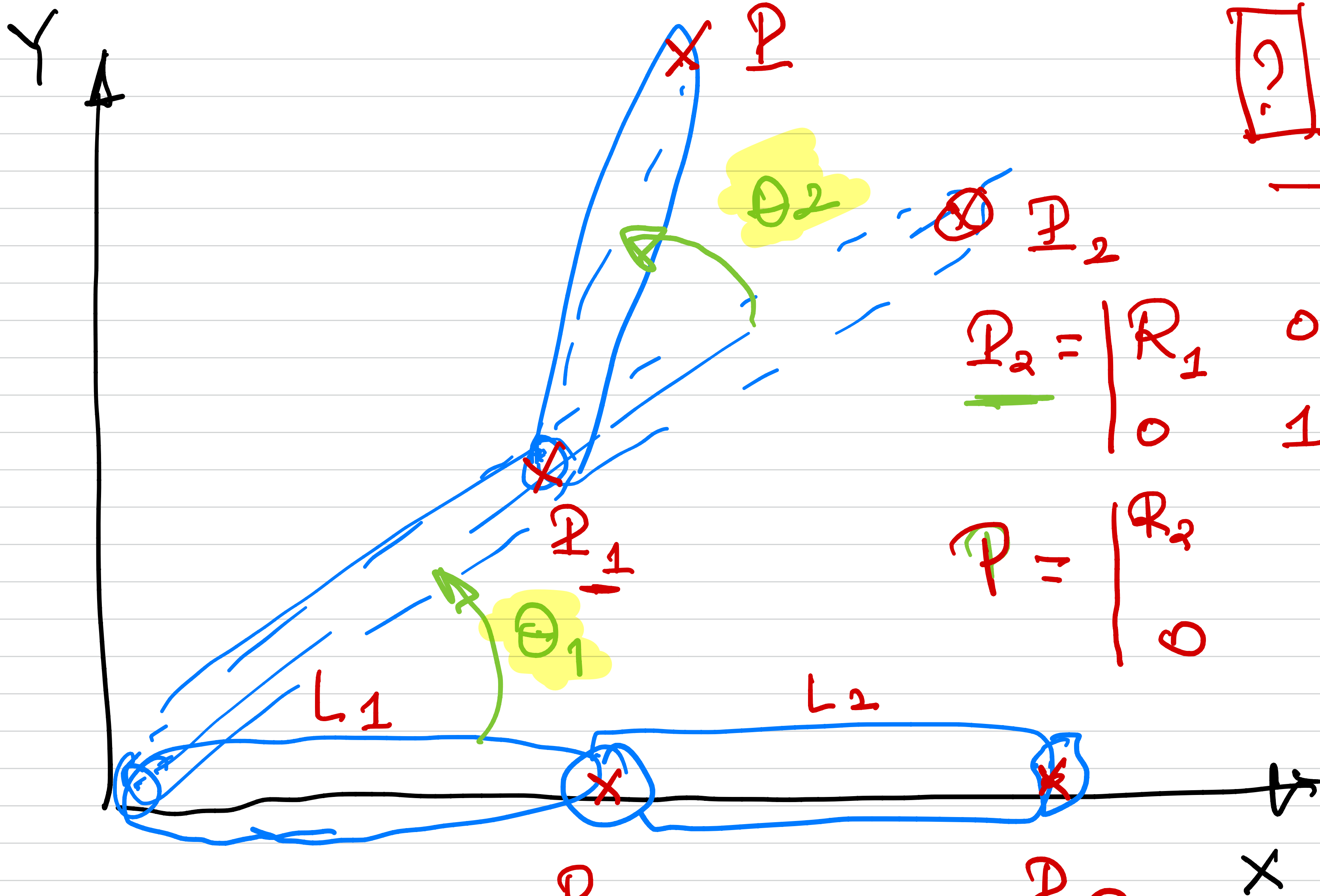
Y? of θ_1 & θ_2

$$x = x_1 + x_2$$

$$y = y_1 + y_2$$

$$\boxed{\begin{aligned} x &= L_1 \cos(\theta_1) + L_2 \cos(\theta_1+\theta_2) \\ y &= L_1 \sin(\theta_1) + L_2 \sin(\theta_1+\theta_2) \end{aligned}}$$

2) GM - Using Homogeneous Matrices that we will name H



$\boxed{?}\ P(\theta_1, \theta_2)\ ?$

$$P_2 = \begin{vmatrix} R_1 & 0 \\ 0 & 1 \end{vmatrix} \cdot P_{20}$$

$$P = \begin{vmatrix} R_2 & P_1 - R_2 P_1 \\ 0 & 1 \end{vmatrix} \cdot P_2$$

$\theta_1 = \theta_2 = 0$

2D

$$P_{10} = \begin{vmatrix} L_1 \\ 0 \\ 1 \end{vmatrix} \qquad P_{20} = \begin{vmatrix} L_1 + L_2 = L_{12} \\ 0 \\ 1 \end{vmatrix} \qquad P_1 = \begin{vmatrix} L_1 \cos(\theta_1) \\ L_2 \sin(\theta_1) \\ 1 \end{vmatrix}$$

$$P_2 = {}^{2}\begin{vmatrix} R_1 & 0 \\ 0 & 1 \end{vmatrix} \cdot P_{20}$$

$$P = \begin{vmatrix} R_2 & P_1 - R_2 P_1 \\ 0 & 1 \end{vmatrix} \cdot P_2$$

$$P = \begin{vmatrix} R_2 & P_1 - R_2 P_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} R_1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} L_{12} \\ 0 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} C_2 & -S_2 & L_1 C_1 - L_1 C_{12} \\ S_2 & C_2 & L_1 S_1 - L_1 C_{12} \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} L_{12} \\ 0 \\ 1 \end{vmatrix}$$