H. Bleuler, M. Bouri, 2018

Contents

2.1 Kinematics

2.1.1 Aim of the Chapter; Active and passive transformations

The task of the robot being to move objects, it is important to precisely describe the geometry of such movements. It is the subject of kinematics, that is to say of the science of **the geometry of movements**, without considering inertia and dynamic forces. “Movement” is defined in our context as a **change of position**. The study of the effect of forces on motions, described by differential equations (**Newton-Euler or Lagrange**), is the field of dynamics. These are the equations of motion (Bewegungs-Differentialgleichungen, equations of motion). The knowledge of the solutions of these equations will allow to develop the appropriate controller of the robot.

Robot joint Variables - Operational variables

Any robot is controlled by angular or linear setpoints sent to actuators (motors). These variables are called “**robot variables**” or “**joint variables**” in English (“joint” meaning “articulation”). These variables define the “**joint space**”. The number *n* of these variables corresponds to the **number of actuators**, and in most cases, it corresponds to the **degrees of freedom**[1] of the robot. We will use for these variables the notations

$\{q_1, q_2, \ldots q_i, \ldots q_n\}$ or $\{\theta_1, \theta_2, \ldots \theta_i, \ldots \theta_n\}$

The robot's task is described in other variables, independent of the robot used. For example, in the case of an assembly, it may be Cartesian coordinates linked to a plate to be assembled, giving the target position of a component to be assembled.

These variables therefore describe the operation to be performed by the robot, and are called “**operational variables**”. We also speak of "**operational space**" or, sometimes also, of "**world coordinates**"

To define the position and the orientation of a (rigid) object, we generally need its six coordinates (three for the position of a point, e.g. of the center of mass, and three to define the orientation in space).

The task of the kinematics will be, to establish the links between the robot variables and the operational variables. For example, the expression of the operational coordinates as a function of the robot variables is called a “**Direct Geometric Model**” (“**Forward Kinematics**” in English). It is therefore the position of the “hand” or of the robot's tool (end-effector) according to its joint positions. The reverse function is the “**Indirect Geometric Model” (Backward Kinematics**), therefore the joint positions according to the desired operational posture. The main goal of this chapter is to establish and study these two essential multivariate functions in robotics.

Kinematics is also necessary to establish the **dynamic model (equations of motion)** used for the sizing of the mechanics, the design of the drive and finally the control of the robot.

Among all these approaches, the kinematics are often the most difficult. It will therefore occupy us a substantial amount of time. Describing the kinematics of a multi-body system (eg of a robot) can represent 80% of the effort in establishing the dynamic model.

[1] If the robot has more actuators than the number of degrees of freedom, it is called redundant

One of the sources of difficulty in kinematics is the fact that you often have to work with multiple frames of reference (coordinate systems), e.g. a **fixed frame references** and **frame references attached to the moving body**, therefore mobile. This results in the need to study changes in the frames of reference of a fixed object or, on the contrary, changes in the position of an object with respect to a fixed frame of reference.

The first case (**change of frame of reference**) is called **passive transformation**, the second case (**movement of the object**) is called **active transformation**, the two types of transformations being obviously closely linked mathematically. (Gruber p.89)

In the first three sections, we limit ourselves to **active transformations**. This facilitates processing considerably, if only by omitting the multiple indexes that characterize so many books on robotics. Once the basic understanding has been acquired, multiple references adapted to the concrete problem will also be introduced and used.

In these same first sections we deal only with the geometry of motion, **without worrying about the time variable, therefore neither speed, nor acceleration.**

2.1.2 2D translations and rotations around the origin

There are **two types of change of position** of a **solid object** (characterized by invariable distances between the points constituting it): Translation and rotation.

The **translation** is defined by a single **vector *t*** displacement valid for all points of the solid.

The **rotation** is defined by a **fixed axis** (not necessarily linked to the object) as well as an **angle of rotation** around this axis. (Reminder of the course "Statics and Dynamics": Where is the fixed axis of a wheel having rolled by a distance *d*?)

The mathematical description of these changes of position is as follows: The points of the object are described by position vectors in a fixed frame of reference. We most often use a Cartesian frame of reference with the three axes *x, y, z* and the origin O. The change in position of a point described by a vector ***v***1 will have the effect of modifying the position vector. The vector giving the new position is called ***v***2 .

The translation is trivial. The vectors of all the points of the solid body C are transformed by the same formula:

$\underline{\boldsymbol{v}}2 = \underline{\boldsymbol{v}}1 + \underline{\boldsymbol{t}}$ (1)

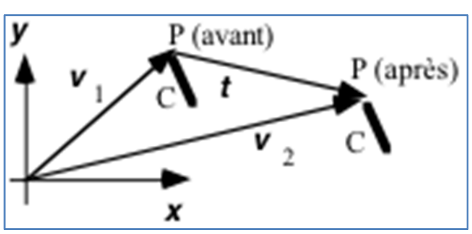
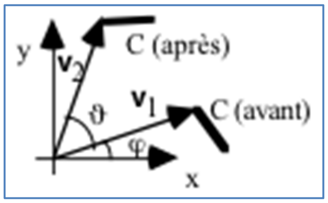


Fig. 1 Translation of point P from position $\underline{\mathbf{v}}_1$ to position $\underline{\mathbf{v}}_2$. This translation is valid for all the points of the object C.

In the case of movements in the plane (2D), **the axis** of rotation is always in direction *z*. The coordinates x_c and y_c of this axis define the **center of rotation** in the plane. We also need to define a third parameter, the angle of rotation θ

A rotation of an object around the origin of an angle θ *will* modify the position vectors of any point fixed to the object in the same way next:



$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$
$$\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} r\cos(\varphi+\vartheta) \\ r\sin(\varphi+\vartheta) \end{bmatrix} \tag{2}$$

Fig. 2 Pure rotation

Using the polar coordinates (r, ϕ) this rotation is trivial. To go directly from [$x1$, $y1$] to [$x2$, $y2$] in Cartesian coordinates, without having to calculate (r, ϕ) from [$x1$,$y1$], the following trigonometric identities will be useful:

cos (*a* + *b*) = cos (*a*) cos (*b*) - sin (*a*) sin (*b*)
sin (*a* + *b*) = sin (*a*) cos (*b*) + cos (*a*) sin (*b*) (3)

By applying them to (2), we obtain the coordinates of ***v***2 expressed in components of ***v***1.

$$\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} r\cos(\varphi+\vartheta) \\ r\sin(\varphi+\vartheta) \end{bmatrix} = \begin{bmatrix} r\cos\varphi\cos\vartheta - r\sin\varphi\sin\vartheta \\ r\cos\varphi\sin\vartheta + r\sin\varphi\cos\vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix} \tag{4}$$

We have therefore found a representation of this rotation by matrix multiplication of the vector ***v***1 with a rotation matrix **R** depending only on the angle of rotation ϑ:

$$\mathbf{v}_2 = \mathbf{R}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{avec} \quad \mathbf{R} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \tag{5}$$

Exercise:

1a) **R**(ϑ= 0) =?

1b) **R**(-ϑ) =?

1c) **R (ϑ)$^{-1}$=? R^{T}** (6)

1d) ***v***3 = **R**(θ2)***v***2 = **R**(θ2)**R**(θ1)***v***1 = **R**(?)***V***1 ? (7)

1e) **R**(θ2)**R**(θ1) = **R**(θ1)**R**(θ2)?

By transforming the two vectors $[1,0]^T$ and $[0,1]^T$, we see that the columns of **R themselves** form an orthogonal coordinate system rotated by θ with respect to the original Cartesian coordinate system. The condition that the volume (or, in the 2D case, the area) of the transformed object remains constant is linked to the value of the determinant ||**R**|| = 1, the geometrical orthogonality of the new coordinate system is related to the orthogonality of the columns (***v***1$\cdot$ ***v***2 = 0). These conditions imply $\mathbf{R}^T=\mathbf{R}^{-1}$ (orthogonal matrices).

The exercise illustrates the concepts of identity, inversion, orthogonality, chaining and commutativity for the case of rotations around the origin. These concepts are necessary to introduce rotations around arbitrary centers. For this purpose, combined movements composed of rotations and translations are introduced.

2.1.3 Combination of 2D rotations and translations; homogeneous representation

A **general motion in the plane** can be described by a combination of translations and rotations around the origin. A sequence of translations and rotations comes up against the fact that the translation is a vector addition while the rotation a matrix multiplication.

It would be very desirable to be able to integrate rotation and translation in a single operation in order to be able to link them together. The **homogeneous matrices** allow this integration of the translation into the transformation matrix. The price to pay is to increase the order of the matrix by one. The translation vector ***t*** is added to the right and a line [0 0 1] at the bottom:

$$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

This is the “**homogeneous** matrix” of transformation in a two-dimensional space (a plane), the third line has no spatial significance (no z axis for the moment !!). This matrix acts on a **homogeneous vector** in a two-dimensional space which consists of its two coordinates plus a "**scale factor**" equal to one:

$\boldsymbol{v} = [x, y, 1]^T$

To find the familiar vectors, just delete the last element. Matrices and homogeneous vectors for **three dimensions** contain **four elements**.

Example of use: Let us assume a movement of the point ***v***1 in two phases composed of

1.) rotation (always around the origin O for all this section 2.1.2) by an angle $\theta 1$

2.) translation of ***t***1

to get $\boldsymbol{v}2 = \mathbf{R}(\theta 1)\, \boldsymbol{v}1 + \boldsymbol{t}1$

Let us apply again a motion composed of two phases ($\mathbf{R}(\theta 2)$, ***t***2) to ***v***2 :

3.) rotation around the origin O by an angle $\theta 2$

4.) translation of ***t***2

to finally get ***v***3 :

$$\boldsymbol{v}3 = \mathbf{R}(\theta 2)\, \boldsymbol{v}2 + \boldsymbol{t}2 = \mathbf{R}(\theta 2)\, \mathbf{R}(\theta 1)\, \boldsymbol{v}1 + \mathbf{R}(\theta 2)\, \boldsymbol{t}1 + \boldsymbol{t}2 \tag{9}$$

These four consecutive movements ($\mathbf{R}(\theta 1)$, ***t***1, $\mathbf{R}(\theta 2)$, ***t***2) can be expressed in a single homogeneous matrix which is calculated simply by matrix products.

Exercise: Perform this operation with the help of two homogeneous matrices. Calculate the homogeneous matrix of the complete operation by matrix product. This result will contain the total translation and the total rotation equivalent to the four movements.

We find that the **rotation** $\mathbf{R}(\theta 2)\, \mathbf{R}(\theta 1)$ is equivalent to $\mathbf{R}(\theta 1+\theta 2)$ and the **translation** totalis equal to $\mathbf{R}(\theta 2)\, \boldsymbol{t}1 + \boldsymbol{t}2$:

$\mathbf{R}(\text{tot}) = \mathbf{R}(\theta 1+\theta 2)$ and $\boldsymbol{t}(\text{tot}) = \mathbf{R}(\theta 2)\, \boldsymbol{t}1 + \boldsymbol{t}2$

Exercise: Give the homogeneous matrices of the following movements:

3a) Pure translation? Pure rotation? Identity?

3b) Translation of **t** then rotation of ϑ?

3c) Operation inverse of 3b), ie rotation of -ϑ then translation of -**t**?

2.1.4 2D rotations around an arbitrary point

These preparations allow us to approach the general movement in the plane, the rotation around a point ***p*** arbitrary. This is simply done by carrying out

1) a translation from ***p*** to the origin ***t***1 = -***p***

2) a rotation around the origin O as before

3) a translation bringing the center of rotation back to ***p*** ***t***2 = ***p*** .

$$\begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I} & -\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p}-\mathbf{Rp} \\ \mathbf{0} & 1 \end{bmatrix} \tag{9}$$

This expression shows us that a rotation around the point ***p*** is equivalent to a rotation around the origin O followed by a translation ***p – Rp*** . Conversely, any combination of translation and rotation can be expressed as pure rotation around a center of rotation ***p***.

Exercise:

4a) Is the previous paragraph completely correct?

4b) Find the center of rotation 1.) by a drawing, 2.) using the previous formula 3.) by looking for an eigenvector of the homogeneous matrix.

4c) Find the homogeneous matrix which describes a rotation of 60 ° around O

4d) Find the homogeneous matrix which describes a translation of un indirection *x* , then a rotation of 60 ° around O

4e) Find the homogeneous matrix which describes a 60 ° rotation around [1,1] T.

4.f) An object with two points ***v***1, ***w***1 is moved so that these points are found at locations ***v***2, ***w***2 .

v1 = $[1.0]^{T}$, ***w***1 = $[1.1]^{T}$, ***v***2 = 0.5 $[1 - \sqrt{3}.1 - \sqrt{3}]^{T}$, ***w***2 = 0.5 $[2 - \sqrt{3}.1]^{T}$

Find the homogeneous matrix, *θ*, ***p*** (graphical solution) which describes this displacement.

2.1.5 3D rotations around the origin, Axis of rotation and Quaternions

2.1.5.1 Matrices of rotation: general case

In three dimensions, it is no longer enough to indicate an angle and a center of rotation. We also need to define the **direction of the axis** around which the rotation is to be performed. The step to be taken between two and three dimensions is considerable, especially at the level of spatial vision and geometric sense.

The number of parameters of the orientation of an object in space is three: The direction of the axis is defined by only two parameters, for example longitude and latitude in spherical coordinates. The third parameter is the angle of rotation around the axis. If the axis of rotation is given by a vector (three parameters), the length of this vector is redundant. We therefore have four parameters, the vector of the axis and the angle of rotation, and a constraint, for example limiting the length of the vector to 1. We thus find the three free parameters of the spatial orientation.

There are several ways to choose these three parameters of spatial orientation. In general, these are angles of rotation around three axes defined according to the application (mobile axes, pitch, roll, yaw, Euler angles, fixed axes *x, y, z*, etc.). We will see several examples later.

We have already seen a (redundant) method to define rotation in the plane: the rotation matrix **R**(θ) has four elements, but only one free parameter, the angle θ itself, since the axis of rotation is always normal to the plan. The elements of the rotation matrix are called "**Direction cosines"**. A three-dimensional directing cosine matrix contains 9 elements. We will now find this matrix.

To make it easier to understand, we start with simple vectors and three-dimensional matrices and then move on (section 2.1.6) to the homogeneous representation. Homogeneous representations considers vectors of four elements and 4 by 4 matrices. Therefore, in this section (2.1.5), no vector and no matrix is in homogeneous representation, the three elements are now the three coordinates Cartesian spaces x, y and z.

We should already know a special rotation matrix in 3D space: That of rotations around the axis z, **R**(θ, $\boldsymbol{v}_z$). The z coordinate remains insensitive to such a rotation. This allows to find **R**(θ, $\boldsymbol{v}_z$). In this notation, the axis of rotation is given by the vector $\boldsymbol{v}_z = [0\ 0\ 1]^T$ and the angle of rotation around this axis is θ.

Exercise Find the basic rotation matrices around the three axes *x, y, z*;

Solutions:

$$\mathbf{Rx} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix} \quad \mathbf{R}y = \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix} \quad \mathbf{R}z = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

See slides for the calculation approach.

Exercise : Are the following two sequences equivalent?

a) 90 ° rotation around z, then 90 ° rotation around y

b) 90 ° rotation around y, then 90 ° rotation around z

Find the solution:

1) By trial

2) By matrix multiplication. Give the rotation matrices of a) and b)

2.1.5.2 Rotation around an arbitrary axis

Exercise series Find the axes and angles of rotation of a) and b) of Exercise 3.2 using the following drawing :

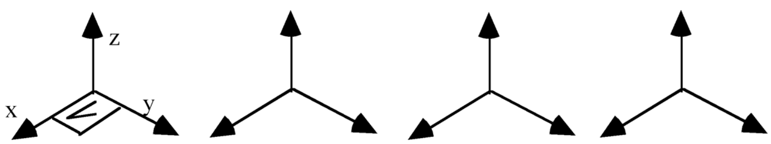


Fig 3 Isometric projection

This example illustrates that any change in spatial orientation can be seen in two equivalent ways:

1.) Composed of several rotations around **several special axes** (e.g. axes of the reference x, y, z or axes defined in relation to the vehicle: pitch roll yaw, cardanic angles, Euler angles etc. In such cases, with axes in motion, processing may become heavy)

2.) A single rotation around a single axis

It is not always easy to guess the direction of this axis. Mathematically, we can find this axis ***v*** for example by the condition that a point on the axis does not change position: **R*v*** = ***v*** (eigenvector).

Which of the three eigenvectors should we choose? Example of result of calculation of MATLAB:

$$A = \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

The command [V, d] = eig (A) returns the eigenvectors in the matrix V and the eigenvalues on the diagonal of the matrix d

$$V = \begin{matrix} -0.05 - 0.57i & -0.05 + 0.57i & 0.57 \\ -0.47 + 0.32i & -0.47 - 0.32i & 0.57 \\ 0.52 + 0.24i & 0.52 - 0.24i & 0.57 \end{matrix} \qquad d = \begin{matrix} -0.50 + 0.86i & 0 & 0 \\ 0 & -0.50 - 0.86i & 0 \\ 0 & 0 & 1.0 \end{matrix}$$

Exercise Find the matrix of directing cosines of a rotation around an axis ***k*** of an angle θ which is in the plane *x* =0 and which is inclined by an angle ϕ with respect to the axis ***z*** (Fig 4 :)

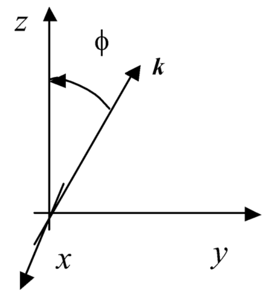


Fig. 4 Find the matrix of a rotation around the axis **w**

An easily understood method of finding the angle of rotation once the axis is found is to 1.) construct a vector ***k*** perpendicular to the axis

2.) transform this vector ***k***2=***Rk***

3.) determine the angle between ***k***2 and ***k*** .

The more complete formulas (10) and (11) for passing from the representation (axis / angle) to the directing cosine matrix and vice versa are given in the literature, eg [J. J Craig, Introduction to Robotics, Addison Wesley 1989, p 52] and derivatives eg. in [Dombre & Khalil: Modeling and control of robots, Hermès, 1988, pp 53]. Here they are:

For a rotation of an angle θ around the axis $[x, y, z]^T$ with $\| [x, y, z]^T \| = 1$ we get the matrix

$$(1-\cos\vartheta)\begin{bmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{bmatrix} + \cos\vartheta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\vartheta \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \quad (10)$$

The following formula allows us the opposite, so to find the axis of rotation and the angle from a matrix ***R*** , and this more directly than by the eigenvector:

with $\mathrm{R} = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{g} \\ \mathrm{b} & \mathrm{e} & \mathrm{h} \\ \mathrm{c} & \mathrm{f} & \mathrm{i} \end{bmatrix}$ one obtains the axis $\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = \dfrac{1}{2\sin(\vartheta)} \begin{bmatrix} \mathrm{f}-\mathrm{h} \\ \mathrm{g}-\mathrm{c} \\ \mathrm{b}-\mathrm{d} \end{bmatrix}$ and the angle θ, *such that*

$$\cos(\vartheta) = \frac{1}{2}(tr(R)-1)$$

$$\sin(\vartheta) = \frac{1}{2}\sqrt{(f-h)^2 + (g-c)^2 + (b-d)^2} \quad (11)$$

These expressions are non-unique: The sign of the root can be positive or negative. Worse, for θ= 0 the axis is not defined. This poses numerical problems in carrying out the control.

These two drawbacks elegantly disappear by using **Euler parameters** or **quaternions** or Olinde-Rodriguez parameters. These three expressions are synonymous in our context.angles **Euler** are other parameters, they will be introduced in Section 2.1.8. Euler parameters or quaternions are used in industrial robotics:

2.1.5.3 Quaternions: Euler parameters, Olinde-Rodriguez parameters

Quaternions are a **generalization of complex numbers**. They were discovered by Hamilton after long and unsuccessful attempts to extend the geometric interpretation of complex numbers in the plane (Argand, 1768-1822, Geneva mathematician) to three spatial dimensions. Hamilton's brilliant intuition consists of two tricks:

1.) There won't be two, but three imaginary parts, in addition to the real part.
2.) We must abandon the commutativity of multiplication.

These new "hypercomplex" numbers, henceforth called quaternions, contain a real scalar part λ_0 and three imaginary parts $[\lambda_1, \lambda_2, \lambda_3]^T$ which are interpreted as vector part $\underline{\lambda}$.

the quaternion Q is therefore the quadruple

$Q = \{ \lambda_0 , \lambda_1, \lambda_2, \lambda_3\} = \{ \lambda_0 , \underline{\lambda}\}$ (11a)

The direction of the axis of rotation $[x, y, z]^T$ is given by the vector $\underline{\lambda} = [\lambda_1, \lambda 2, \lambda 3]$ T.

The angle of rotation ϑ is introduced in the following way in the quaternion Q:

$\lambda_0 = \cos (\vartheta/ 2)$ and $\underline{\lambda} = \sin (\vartheta/ 2) [x, y, z]^T$, $|| x, y, z || = 1$ (11b)

The rotations are therefore represented byquaternions **unit**:

$\lambda_0^2 + \lambda_1^2+ \lambda_2^2+ \lambda_3^2 = 1$ (11c)

The quaternion multiplication rules are a generalization of the complex number multiplication rules:

$Q = \{ \lambda_0 , \lambda_1, \lambda_2, \lambda_3\} = \lambda_0 + i \lambda_1 + j \lambda_2 + k \lambda_3$ (11d)

with $i^2 = j^2 = k^2 = ijk = -1$ (11e)

and $ij = k = -ji$ $jk = i = -kj$ $ki = j = -ik$ (11f)

Note the **multiplication no -commutative**! (William Rowan Hamilton, Dublin, 1843)

These rules lead to the product

$Q_M Q_L = \{ \mu_0 , \underline{\mu}\} \{ \lambda_0 , \underline{\lambda}\} = \{ \mu_0\lambda_0 - \underline{\mu}^T\underline{\lambda} , \quad \mu_0\underline{\lambda} + \lambda_0\underline{\mu} + \underline{\mu} \times \lambda\}$ (11g)

This product defines the sequence of rotations Q_L and Q M.

Exercise Find the solutions of Exercise 7 with quaternions.

Passage from quaternions to rotation matrices

The passage from quaternion to directing cosines and vice versa is: (11h)and

$$R=\begin{bmatrix} 2(\lambda_0^2+\lambda_1^2)-1 & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_1\lambda_3+\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_0^2+\lambda_2^2)-1 & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) & 2(\lambda_0^2+\lambda_3^2)-1 \end{bmatrix}$$

$$\lambda_0=\frac{1}{2}\sqrt{r_{11}+r_{22}+r_{33}+1} \qquad \underline{\lambda}=\frac{1}{2}\begin{bmatrix} \mathrm{sgn}(r_{32}-r_{23})\sqrt{r_{11}-r_{22}-r_{33}+1} \\ \mathrm{sgn}(r_{13}-r_{31})\sqrt{r_{22}-r_{11}-r_{33}+1} \\ \mathrm{sgn}(r_{21}-r_{12})\sqrt{r_{33}-r_{22}-r_{11}+1} \end{bmatrix} \tag{12}$$

2.1.6 Translations and rotations 3D in homogeneous representation

This step is perfectly analogous to the 2D case. The vector in homogeneous representation is composed simply by adding a scale factor "1" which has no meaning of "fourth dimension" in the geometric sense. The complete transformation matrix consists of the rotation matrix **R** and the translation vector **t̲** plus a line [0 0 0 1]:

$$\begin{bmatrix} \mathbf{R}_{3\times 3} & \mathbf{t}_{3\times 1} \\ \mathbf{0}_{1\times 3} & \mathbf{1} \end{bmatrix} \tag{12}$$

A combined transformation (rotation **then** translation) is calculated by multiplying a homogeneous starting vector by this transformation matrix to obtain the transformed vector, exactly as in the 2D case. The transformation then the rotation is found by multiplying the appropriate matrices between them, always with the **matrix of the first change of position on the right**.

A rotation around an axis not passing through the origin is composed in the same way as in the 2D case:

$$\begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{p}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I} & -\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}=\begin{bmatrix} \mathbf{R} & \mathbf{p}-\mathbf{Rp} \\ \mathbf{0} & 1 \end{bmatrix} \tag{13}$$

with the vector $\boldsymbol{p}$ of the origin O at any point **on the axis of rotation**

A difference important with the 2D case should be noted: The general movement in 3D is equivalent to a rotation around an axis **plus a translation in the direction of this axis**. This combination is called a screw (eine Schraube) for obvious reasons.

Exercise : Find the homogeneous transformation matrix leading from the form {A, B, C} to {A ', B', C '}.

A = $[1\ 0\ 0]^T$, B = $[0\ 0\ 0]^T$ and C = $[0\ 1\ 0]^T$ to A '= $[1\ 0\ 1]^T$, B' = $[1\ –1\ 1]^T$ and C = $[0\ –1\ 1]^T$

What are the axis, the angle of rotation, the translation in the direction of the axis? The solution is easily found with a small drawing. Imagine the three points linked to a solid.

2.1.7 Robot variables, direct geometric model (DGM)

Finally, we can move on to the application of these transformations. Concretely, it is about describing the position and orientation of an object manipulated by a robot. The robot is controlled by **joint variables**, ie by sending it **angular/linear movement instructions** for its motors. For these important variables we will use (as in several robotics books) the symbol "*q*"; the symbol Θ (uppercase theta) is also often used in this context.

The **direct geometric model (DGM)** is none other than the formulas giving the position and orientation of the object being manipulated as a function of these variables Θ. The number of variables Θ corresponds to the number of degrees of freedom of the robot. Some of these variables can be **linear variables** (translational), depending on the construction of the robot.

As an example, we will derive the direct geometric model (DGM) of an extremely popular industrial robot, the **SCARA** (Selective Compliance Assembly Robot Arm) presented in Chapter 1. The geometric model of this robot is particularly simple for two reasons:

- First, it only comprises four degrees of freedom. Carefully chosen, these degrees of freedom allow to cover a very large segment of assembly tasks.
- Second, the three angular degrees of freedom are all around a vertical axis, movements out of the xy plane are limited to vertical translation only.

These conditions make it possible to treat the difficult part, the rotations, of the DGM of the SCARA in the xy plane only.

The first step in establishing any DGM consists in defining

- the joint variables (robot variables) ϑ_i ,
- of their **reference positions** $\vartheta_i = 0$ as well as
- the geometric parameters of the robots.

For the SCARA, we define ϑ_1 as the angle at the base of the robot, ϑ_2 as the angle between the first and the second member, ϑ_3 as the **linear displacement** in the direction of the vertical axis z and ϑ_4 as the angle at the wrist. Note that the joint variables are defined **between the segments** and not relatve to the fixed frame of reference <xoy>. This is very important for the practical use of these variables, as angular setpoints of the motors.

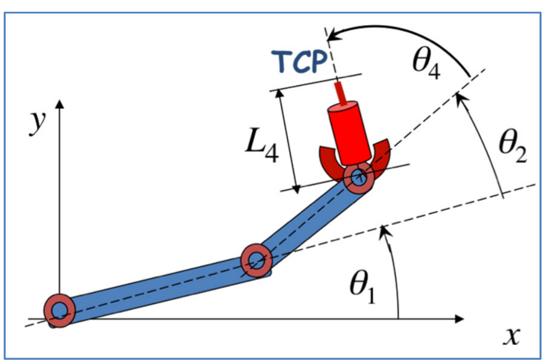


Fig. 8 Definition of the robot variables ϑ_i for the SCARA.

θ_3 is the vertical displacement between the arm L_2 and the gripper. Robot parameters: L_1 and L_2

Reference positions: $\vartheta_i = 0$

TCP = Tool Center Point,

L_4 length of the tool

This way of proceeding from the base (which would therefore be zero member) is also standard, at least in most theoretical works on robotics, although one is in principle free to define these variables differently.

The DGM describes the position and orientation of the object being manipulated, thus making it possible to find the position of any arbitrary point ***p*** of this object as a function of ϑi . It therefore consists of the transformation matrix $\boldsymbol{K}_{DGM}(\vartheta i)$ giving the position of this point as a function of the articular variables ϑi . The point ***P*** on the object (the tool or the gripper) is itself defined in the **reference position** $\vartheta i = 0$.

$$\boldsymbol{P}(\vartheta i) = \boldsymbol{K}_{DGM}(\vartheta i)\, \boldsymbol{P}(\vartheta i = 0) \quad (14)$$

There are two possibilities to define $\boldsymbol{K}DGM(\vartheta i)$: (using c1= cosθ1, s12= sin (θ1+θ2) etc.) (15)

Exercise (on slides):

a.) $\boldsymbol{K}_{DGM}(\vartheta i)$ = (rot ϑ_1 around $[0, 0]^T$) then (rot ϑ_2 around $[L_1c_1 , L_1s_1]^T$) then (rot $\vartheta 4$ around $[L_1c_1+ L_2c_{12},]^T$)

b.) $\boldsymbol{K}_{DGM}(\vartheta i)$ = (rot ϑ_4 around $[\ ?]^T$) then (rot ϑ_2 around $[\ ?]^T$) then (rot ϑ_1 around $[\ ?]^T$)

c.) Calculate $\boldsymbol{K}_{DGM}(\vartheta i)$ of the SCARA. Which method will be simpler? Where does the rotation matrix come from around ϑ_1, left or right in the chain of multiplications?

Exercise (on slides): A working point of the wrist is at a horizontal distance $L4$ from the axis of $\vartheta 4$ with $\boldsymbol{P}(\vartheta i = 0) = [L1 + L2 + L4 , 0, 0, 1]^T$. Give the position of this point as a function of the robot variables $\boldsymbol{P}(\vartheta i)$.

The matrix thus found contains a complete description of the geometry of the SCARA. The equations defined by this matrix are the direct geometric model. They are sometimes called "kinematic equations" of the robot. In the case of the SCARA, they can immediately be obtained by simple trigonometric equations. But the method applied here allows us to find these equations in the same systematic way in less obvious cases such as for the following example.

We now apply this method to find the DGM of a very widespread and classic industrial robot, the **PUMA** (Programmable Universal Manipulator Arm, Stäubli RX) presented in chapter 1.

The **first step** therefore consists in defining the **joint variables,** (θ_i or q_i), then the reference position and the robot parameters.

This should be done on drawings. We directly consider the PUMA with a full 3R wrist, with three degrees of freedom. These three DOFs are added to the three basic DOFs of the PUMA which is therefore a "complete" 6 DOF robot. The joint (or robot) variables 1–3 are found in figure 9, the variables 4-6 are those of the wrist which are found in figure 10 to better see the details.

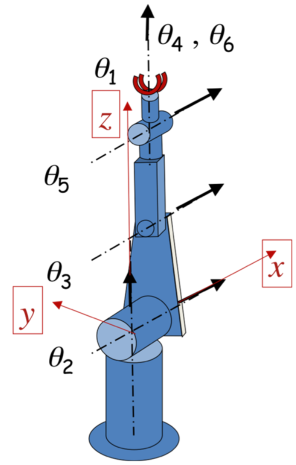


Fig. 9 The PUMA robot and its reference position

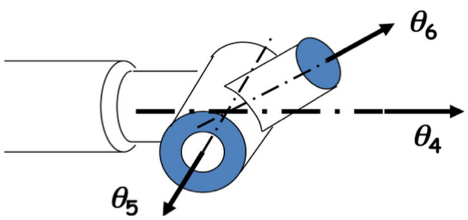


Fig. 10 Wrist with three competing axes

The parameters of the robot are the length L_2 between the axis 2 and the axis 3 as well as the distances D_3 and D_4. The use of different symbols (*L* or *D* in this case) indicates a fundamental difference between these parameters: While the parameters ***Li*** denote a distance **between two axes** consecutive, the parameters ***Di*** indicate a distance **on the axes** themselves. The first are called "Link Length" in English and the second "Joint Offset".

The "Joint Offsets" are each linked to a particular joint while the "Link Lengths" make the transition from one joint to the next.

The parameters “**Link Length**" and "**Joint Offset**" as well as the angle αi between two consecutive axes ("**twist angle**") **fully define the geometry** of a robot. These parameters are called "parameters of **Denavit Hartenberg**" , after a publication which introduced them. They are covered in great detail in many robotics books.

In the case of linear movements (as for example the vertical movement of the SCARA), the offset joint can become one of the robot variables while an articulation angle (**joint angle**) can become a fixed parameter in certain special cases. The definition of the twist angle (*always fixed*) for non-competing axes is done around the shortest line perpendicular to the two axes.

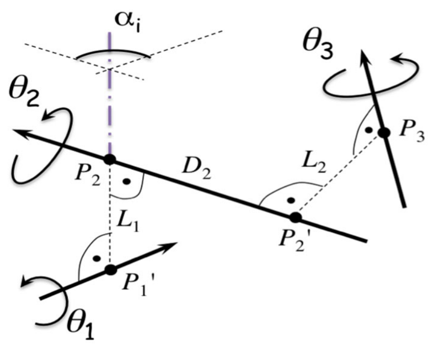


Fig. 11 The four Denavit-Hartenberg parameters (Link length L, Joint angle θ , Joint offset D, Twist angle α) in the general case. An arrow marked θ indicates an articulation of the robot. The variable is either the angle of rotation θ around the axis J is the linear displacement D along the axis.

Back to the PUMA DGM. The drawing shows us the transformation matrices that come into play. As we saw in Exercise 9, it is advisable to start with the rotation $\vartheta 6$ and to continue the operations to end with the rotation ϑ_1. Consequently, the matrices are inscribed in the reverse order in the chain of multiplications. The great advantage of this approach is that the axes of rotation are those of the reference position. We therefore have from left to right:

1.) rotation of ϑ_1 around the z axis.

2.) rotation of ϑ_2 around the x axis

3.) rotation of ϑ_3 around the x axis shifted by L_2 in the direction of z.

4.) rotation of ϑ_4 around the z axis shifted by D_3 in the direction of x.

5.) rotation of ϑ_5 around the x axis shifted by (L_2+ D_4) in the direction of z.

6.) rotation of ϑ_6 around the z axis shifted by D_3 in the direction of x.

Using the abbreviations s_1= sinϑ_1 , c_1= cosϑ_1 and v_1 = versine ($\vartheta 1$) = 1 – cos$\vartheta 1$ we find the following matrices:

$$\begin{bmatrix} & \mathbf{R} & & \mathbf{p}-\mathbf{Rp} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_6 & -s_6 & 0 & D_3 v_6 \\ s_6 & c_6 & 0 & -D_3 s_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{K}_6$$

$$\begin{bmatrix} & \mathbf{R} & & \mathbf{p}-\mathbf{Rp} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_5 & -s_5 & L_{24} s_5 \\ 0 & s_5 & c_5 & L_{24} v_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{K}_5$$

… etc.

The DGM of PUMA is therefore $K_{DGM}(\vartheta\, i) = \boldsymbol{K_1 \cdot K_2 \cdot K_3 \cdot K_4 \cdot K_5 \cdot K_6}$

These expressions can be calculated analytically, to be programmed into the robot controller processor. They are generally too complicated to help a concrete understanding of the robot's behavior. Analyzing a certain part of PUMA's DGM will give us an idea of this complexity while providing us with the mathematical description of an important robotic subsystem: It is the wrist.

2.1.8 Robot Wrist, Euler Angles, Pitch, Roll, Yaw

Figure 10 shows a typical robot wrist (spherical wrist). It is characterized by three competing axes (which have a common intersection). The particularity of this wrist lies in the definition of the reference positions of the angles $\vartheta 4 \vartheta 5 \vartheta 6$. These positions are such that the axes of rotation of $\vartheta 4$ and $\vartheta 6$ coincide for $\vartheta 5 = 0$ (reference position).

In this section, we only discuss rotations without worrying about translations. We therefore move the origin of the frame of reference to the point of intersection of the axes of the wrist. These axes form a mobile frame of reference. The two equivalent ways of describing the orientation defined by the wrist are:

a) Rotation ϑ_4 around the z axis, then rotation ϑ_5 around the x 'axis (forming an angle of ϑ_4 with the x axis) then rotation ϑ_6 around the new z 'axis (which forms an angle of ϑ_5 with the fixed z axis).

b) Rotation ϑ_6 around the z axis, then rotation ϑ_5 around the x axis then rotation ϑ_4 around the z axis (always fixed).

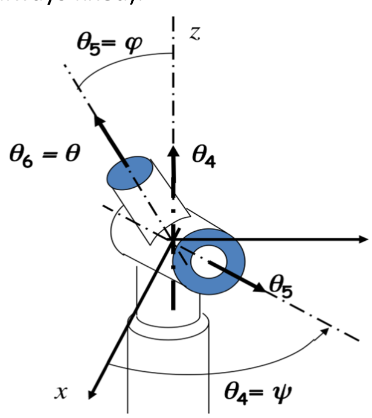


Fig. 12 Euler angles (zxz)

As before, way b) is much simpler, in method a) we would first have to calculate the rotations around the axes with formula (10). We therefore obtain:

$$\boldsymbol{R}(\text{wrist}) = \boldsymbol{R}_4 \,.\, \boldsymbol{R}_5 \,.\, \boldsymbol{R}_6 \tag{15}$$

These angles are called Euler's angle (z, x, z). They are characterized by the coincidence of the axis of the first and the last rotation in the reference position. By choosing a reference position other than π / 2 for the second angle of rotation, we obtain the angles of a cardanic wrist which is characterized by three mutually perpendicular axes in the reference position. These angles are also called roll pitch yaw depending on the context.

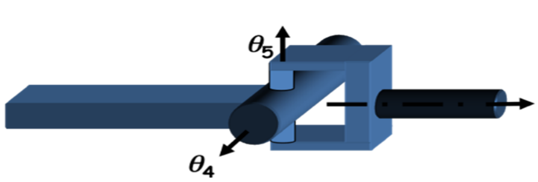


Fig. 13
Spherical wrist in so-called “cardanic” configuration

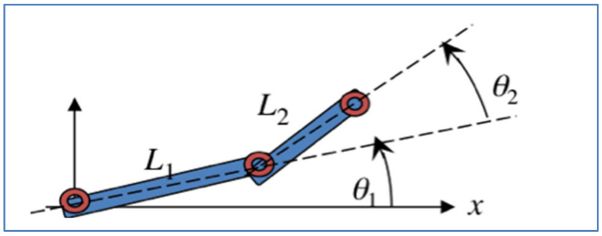
Pitch: Θ_4

Roll (roll) Θ_6

Yaw (yaw)Θ_5

2.1.9 Inverse geometric model, postures

The DGM gives the position and orientation as a function of the robot variables. In practice, we are more often confronted with the opposite problem, ie finding the robot variables necessary to reach a given position and orientation. These equations are called "inverse geometric model" (IGM, inverse kinematics).



As a first example, we treat the simple two-membered planar manipulator. It is therefore a question of finding the angles θ_1 and θ_2 from a given position *x, y* . The DGM is

$x = L_1 \cos \theta_1 + L_2 \cos (\theta_1 + \theta_2)$

$y = L_1 \sin \theta_1 + L2 \sin (\theta_1 + \theta_2)$

To find the solution, we must first apply the trigonometric equations (2). Using the symbols $c_1 = \cos \theta_1$ and $s_1 = \sin \theta_1$, etc. , we get

$$x = L1\, c_1 + L2\, c_1 c_2 - L2\, s_1 s_2 \qquad \textbf{(a)}$$

$$y = L1\, s_1 + L2\, s_1 c_2 + L2\, c_1 s_2 \qquad \textbf{(b)}$$

$$c_1^2 + s_1^2 = 1 \qquad \textbf{(c)}$$

$$c_2^2 + s_2^2 = 1 \qquad \textbf{(d)}$$

As these are four quadratic equations, we can expect multiple solutions. The sum of the squares of (a) and (b) gives us the well-known relation of the cosine of a triangle:

$(x^2 + y^2) = L1^2 + L2^2 + 2\, L1\, L2\, c_2$, which gives us c_2 , and, with (d), two solutions for s_2. The unknowns s_1 and c_1 are found with weighted sums of expressions (a) and (b):

$x\, c_1 + y\, s_1$ and $- x\, s_1 + y\, c_1$.

In the case of the puma robot, the IGM of the first three DOF consists of two chains identical to the previous example, with two solutions for each pair (θ_1 , θ_2) and (θ_2 , θ_3). In combination, this therefore gives four distinct solutions. In practical terms, this means that the first three arms of the kinematic chain from the base to the wrist can reach a given point in four different ways (with different orientations each time). **Such solutions are called the "postures" of the robot**. (Fig. 14)

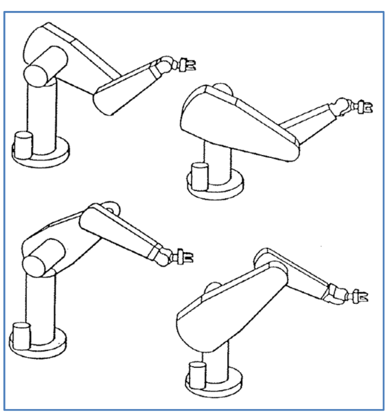


Fig. 14 **The four postures** of the first three dof of the PUMA

To this, we must add the postures of the wrist. (Which ones? How many PUMA IGM solutions?) The number of postures of a general robot is unknown.

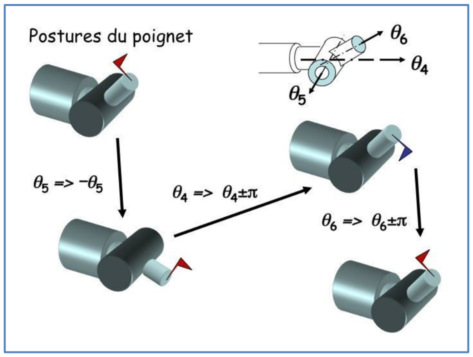


Fig. 15 Postures of a spherical wrist

2.1.10 Summary, parallel robots:

As we have seen with the previous examples (Puma, SCARA), the DGM for serial robots has only one solution: For given joint variables, the position of the robot is well defined. While the IGM may have several solutions (eg Fig. 14, 15).

In the case of **parallel robots**, we find **just the opposite**! The DGM has several solutions (called robot "contortions"), the IGM has only one.

The following simple parallel robot with 2 DOF in the plane highlights this claim:

The IGM of the parallel robot, has only one solution, as we can see with the same parallel structure planar: If the position and the orientation of the mole plae (end effector) are given, the lengths of the actuated cylinders are well determined, without any ambiguity).

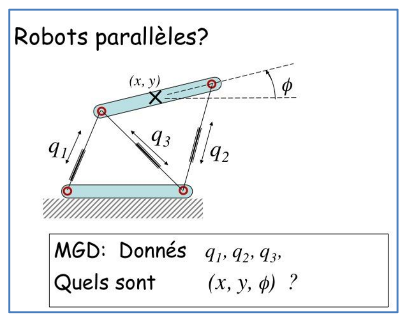


Fig. 16: Planar parallel robot

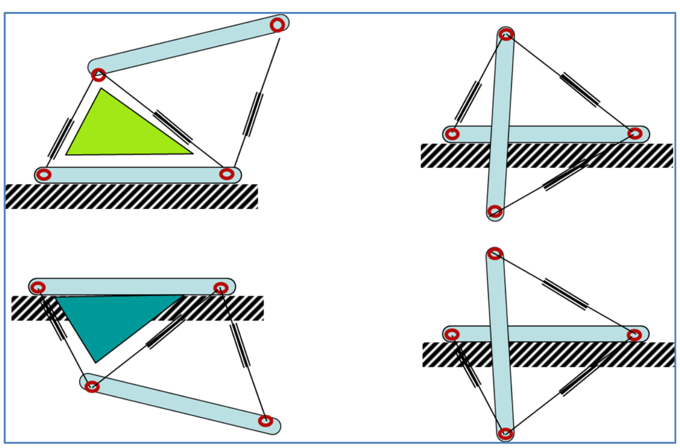


Fig. 17: Different solutions of DGM, called contortions. The same joint variables q, give different robot configurations