7

Gaussian Beams

(MAKING A LASER BEAM DO WHAT YOU WANT IT TO)

The output of a laser is very different than most other light sources. Those sources that have been used in the examples up to this point in the text have been incandescent or fluorescent lights, or in the case of Example 3.C, a light emitting diode. The description of collecting light from such sources and sending it through an optical system was discussed in Chapter 3, Radiometry. Because lasers are used in many optical systems, a chapter is devoted to a description of the propagation of laser beams and their modification by optical systems.

The design of systems incorporating a laser beam should be looked at more as a problem akin to radiometry than as a problem in imaging. Unless the beam is interrupted by a slit, aperture, or transparency, there is no image. The ray optics presented in the first six chapters is insufficient to explain the behavior of laser beams, since the effects of diffraction must be included right from the beginning. Because of this, intuitions based on geometrical optics arguments can be misleading.

After a description of the simplest type of beam, the TEM_{00} mode Gaussian beam and its parameters, we look at means of modifying the beam using equivalent thin lens and paraxial equations for Gaussian beams. The concept of conjugate distances will be introduced and the effect of higher-order modes on the calculations will be described. Finally, the variation in beam irradiance throughout the beam will be described and some methods for reshaping the distribution to give useful non-Gaussian shapes will be given.

7.1. CHARACTERISTICS OF A GAUSSIAN BEAM

The term *Gaussian* describes the variation in the irradiance along a line perpendicular to the direction of propagation and through the center of the beam, as shown in Fig. 7.1. The irradiance is symmetric about the beam axis and varies radially outward from this axis with the form

$$I(r) = I_0 e^{-2r^2/r_1^2} \tag{7.1}$$

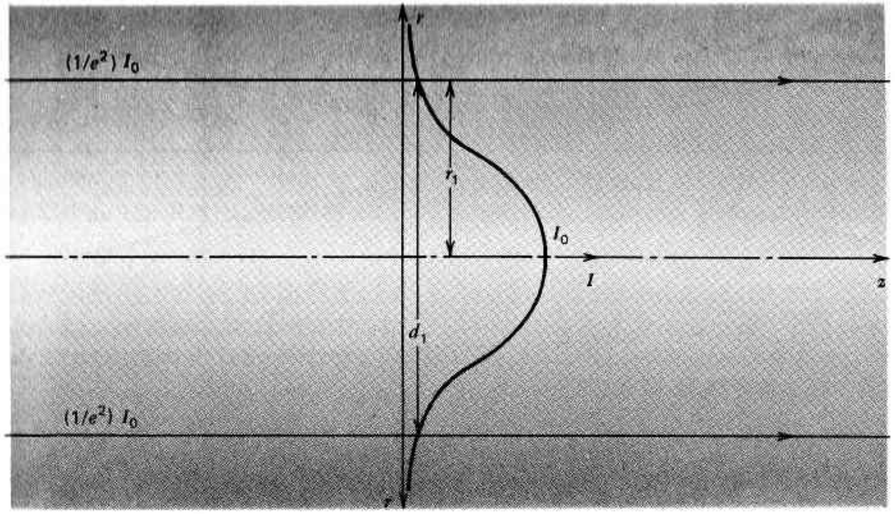


Figure 7.1. Gaussian beam profile. Plot of irradiance versus radial distance from the beam axis.

or in terms of a beam diameter

$$I(d) = I_0 e^{-2d^2/d_1^2}, \tag{7.2}$$

where r_1 and d_1 are the quantities that define the radial extent of the beam. These values are, by definition, the radius and diameter of the beam where the irradiance is $1/e^2$ of the value on the beam axis, I_0.

7.1.1. Beam Waist and Beam Divergence

Figure 7.1 shows a beam of parallel rays. In reality, a Gaussian beam either diverges from a region where the beam is smallest, called the *beam waist*, or converges to one, as shown in Fig. 7.2. The amount of divergence or convergence is measured by the *full angle beam divergence* θ, which is the angle subtended by the $1/e^2$ diameter points for distances far from the beam waist as shown in Fig. 7.2. In some laser texts and articles, the angle is measured from the beam axis to the $1/e^2$ asymptote, a *half* angle. However, it is the full angle divergence, as defined here, that is usually given in the specification sheets for most lasers. Because of symmetry on either side of the beam waist, the convergence angle is equal to the divergence angle. We will refer to the latter in both cases.

Under the laws of geometrical optics a Gaussian beam converging at an angle of θ should collapse to a point. Because of diffraction, this, of course, does not occur. However, at the intersection of the asymptotes that define θ, the

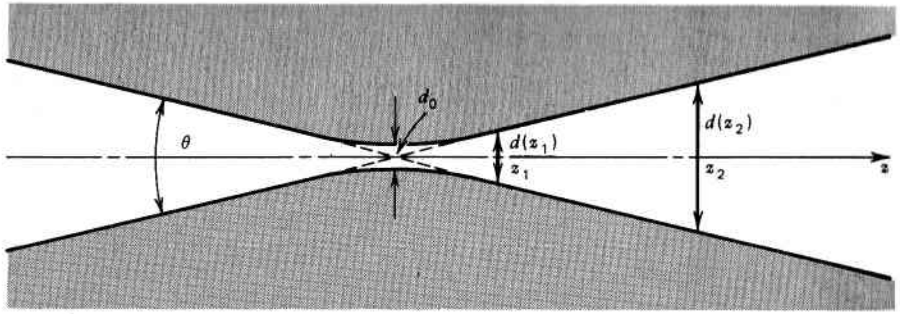


Figure 7.2. Variation of Gaussian beam diameter in the vicinity of the beam waist. The size of the beam at its smallest point is d_0; the full angle beam divergence, defined by the asymptotes for the $1/e^2$ points at a large distance from the waist is θ.

beam does reach a minimum value d_0, the *beam waist diameter*. It can be shown (Ref. 7.1) that for a TEM_{00} mode d_0 depends on the beam divergence angle as:

$$d_0 = \frac{4\lambda}{\pi\theta}, \tag{7.3}$$

where λ is the wavelength of the radiation. Note that for a Gaussian beam of a particular wavelength, the product $d_0\theta$ is a constant. This means that for a very small beam waist the divergence must be large, for a highly collimated beam (small θ), the beam waist must be large.

The variation of the beam diameter in the vicinity of the beam waist is shown in Fig. 7.2 and given as

$$d^2 = d_0^2 + \theta^2 z^2, \tag{7.4}$$

where d is the diameter at a distance $\pm z$ from the waist along the beam axis. Note that far from the beam waist ($\theta z \gg d_0$), $d = \theta z$. Thus, at points z_1 and z_2 satisfying $z_1, z_2 \gg d_0/\theta$, $d_1 = \theta z_1$, and $d_2 = \theta z_2$. Subtracting yields $d_2 - d_1 = \theta(z_2 - z_1)$, and if we set $z_2 - z_1 = t$, an axial distance, we can write the equation as

$$d_2 = d_1 + \theta t. \tag{7.5}$$

This is similar to the transfer equation in geometrical optics. Indeed, well away from the beam waist one can use geometrical optics.

7.1.2. The Rayleigh Range

It is useful to characterize the extent of the beam waist region with a parameter called the *Rayleigh range*. (In other descriptions of Gaussian beams this extent is

sometimes characterized by the *confocal beam parameter* and is equal to twice the Rayleigh range.) Rewriting Eq. 7.4 as

$$d = d_0\sqrt{1 + \left(\frac{\theta z}{d_0}\right)^2}. \tag{7.6}$$

we define the Rayleigh range as the distance from the beam waist where the diameter has increased to $\sqrt{2}\, d_0$. Obviously this occurs when the second term under the radical is unity, that is, when

$$z = z_R \equiv \frac{d_0}{\theta}.$$

Although the definition of z_R might seem rather arbitrary, this particular choice offers more than just convenience. Figure 7.3 shows a plot of the radius of curvature of the wavefronts in a Gaussian beam as a function of z. For large distances from the beam waist the wavefronts are nearly planar, giving values tending toward infinity. At the beam waist the wavefronts are also planar, and, therefore, the absolute value of the radius of curvature of the wavefronts must

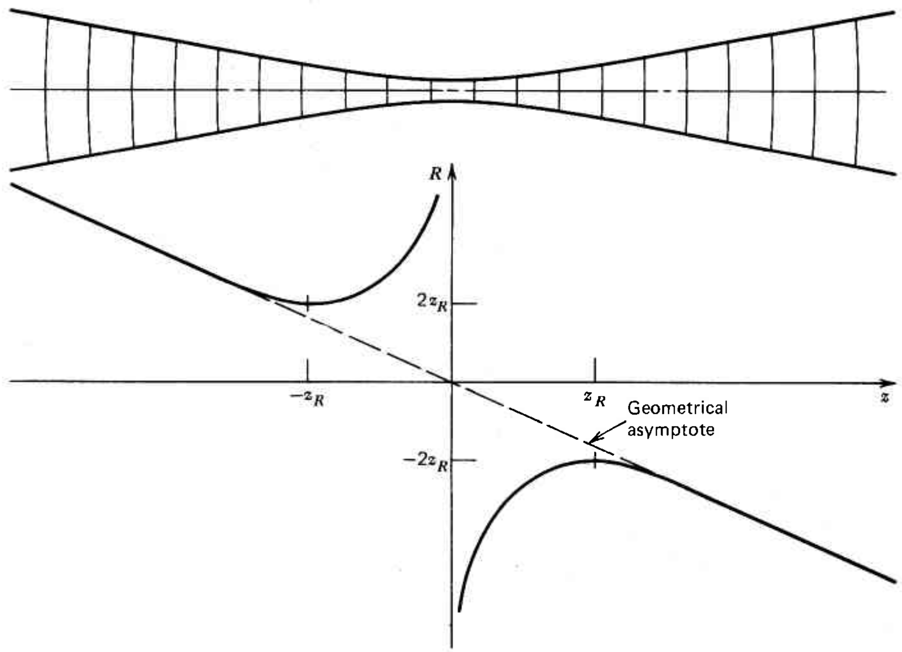


Figure 7.3. Plot of radius of curvature (R) versus distance (z) from the beam waist. The absolute value of the radius is a minimum at the Rayleigh range point, z_R. In the limit of geometrical optics, the radius of curvature of the wavefronts follows the dashed line.

go from infinity at large distances through a minimum and return to infinity at the beam waist. This is also true on the other side of the beam waist but with the opposite sign. It can be shown that the minimum in the absolute value of the radius of curvature occurs at $z = \pm z_R$, that is, at a distance one Rayleigh range either side of the beam waist. From Fig. 7.3, the "collimated" region of Gaussian beam waist can be taken as $2z_R$.

The Rayleigh range can be expressed in a number of ways:

$$z_R = \frac{d_0}{\theta} = \frac{4\lambda}{\pi\theta^2} = \frac{\pi d_0^2}{4\lambda}. \tag{7.7}$$

From this we see that all three characteristics of a Gaussian beam are dependent on each other. Given any of the three quantities, d_0, θ, z_R, and the wavelength of the radiation, the behavior of the beam is completely described [Problem 7.1]. For example, if a helium–neon laser (λ = 633 nm) has a specified TEM_{00} beam diameter of 1 mm, then $\theta = 4\lambda/\pi d_0 = (1.27 \times 6.33 \times 10^{-7}\ \text{m})/(1 \times 10^{-3}\ \text{m}) =$ 0.8 mrad and $z_R = d_0/\theta = (1 \times 10^{-3}\ \text{m})/(0.8 \times 10^{-3}\ \text{rad}) = 1.25\ \text{m}$. The Rayleigh range of a typical helium–neon laser is considerable.

The initial Rayleigh range (and, therefore, beam waist and divergence) is determined by the radii of curvatures of the laser mirrors R_1 and R_2 and by their separation L. If the same sign convention used earlier in the text is retained, the Rayleigh range of a laser is given by

$$z_R^2 = \frac{L(R_2 + L)(R_1 - L)(R_2 - R_1 + L)}{(R_2 - R_1 + 2L)^2} \tag{7.8}$$

(The expression is the same as in Ref. 7.2, except that the sign convention for the radii of curvature is the opposite of standard ray-tracing conventions.) It should be noted that there are some geometries for which the value of z_R^2 would be negative. These mirror arrangements are referred to as *unstable resonators*. If we define $g_1 \equiv (1 - L/R_1)$ and $g_2 \equiv (1 + L/R_2)$, it can be shown that [Problem 7.2] the condition for stable resonators based upon the requirement that z_R^2 be positive is that $0 < g_1 g_2 < 1$. This is known as the *stability condition* for laser resonators (Ref. 7.3).

7.2. MODIFYING A GAUSSIAN BEAM

There are many design situations in which the "raw" beam as it comes out of the laser cannot be used. The beam must be modified in some way—expanded, focused, recollimated. This modification can be performed by lenses, mirrors, prisms, or even slabs of glass. For our example, we will use a lens, but the focal length or surface power of any optical element can be substituted into the equations in the appropriate place.

The thin lens equation was expressed in Chapter 1 in both the Gaussian form

$$\frac{1}{t_1} - \frac{1}{t_0} = \frac{1}{f}. \tag{1.16}$$

where t_0 and t_1 are the object and image distances, respectively, and in the Newtonian form [Problem 1.13]

$$-xx' = f^2 \tag{7.9}$$

where x is the distance from the front focal point to the object and x' is the distance from the back focal point to the image. For Gaussian beams we can use the Newtonian form with some minor redefinitions and the addition of a single term to account for the effects of diffraction.

7.2.1. Thin Lens Equation For Gaussian Beams

In Fig. 7.4 the relevant distances are shown. Note that in contrast to the neatly subscripted variables used to trace a ray through a system, a series of unprimed and primed variables is used. This is done because the Gaussian beam is most easily described by locating the origin of the coordinate system at the beam waist. In contrast, the ray-trace coordinate origins are located at the vertices of each element or in the plane of interest. The beam waist always has a zero subscript. Other beam diameters have other subscripts, usually d_1 is to the left of d_0 and d_2 to the right. If there is more than one lens, each additional region between lenses is given another prime. Also unlike conventional ray tracing, beam waist distances relative to the lens position are positive if on the input side of the lens the beam narrows down to a waist before striking the lens, and negative if the beam is intercepted by the lens before the beam waist occurs.

If the positions of the input beam waist and output beam waist are replaced by the object and image distances in Eq. 7.9, the thin lens equation in Newtonian form becomes $(-x \rightarrow z_2 - f;\ x' \rightarrow z_1' - f)$

$$(z_2 - f)(z_1' - f) = f^2 - f_0^2, \tag{7.10}$$

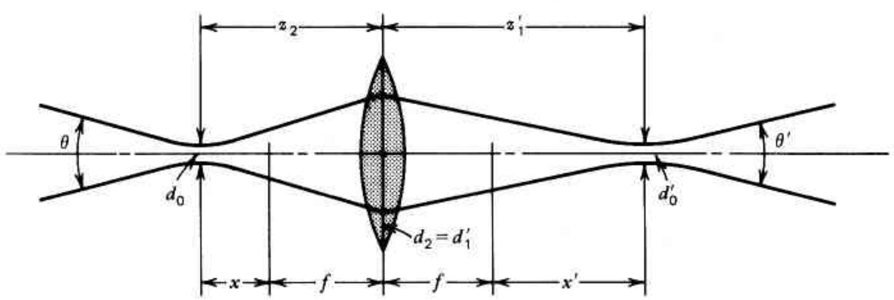


Figure 7.4. Transformation of a Gaussian beam by a thin lens.

where f_0^2 is the term introduced to account for diffraction. There are a number of ways of writing this term since it is an expression of the beam properties on both sides of the lens. The most symmetric form is

$$f_0^2 = z_R z'_R, \tag{7.11}$$

but other useful forms are

$$\begin{aligned} f_0 &= \frac{d'_0}{\theta} \\ &= \frac{d_0}{\theta'} \\ &= \frac{\pi d_0 d'_0}{4\lambda} \\ &= \frac{4\lambda}{\pi\theta'\theta} \end{aligned} \tag{7.12}$$

Another useful relation between the beam waist diameters and their locations, derived in Ref. 7.1, is

$$\frac{(z_2 - f)}{(z'_1 - f)} = \frac{d_0^2}{d_0'^2}. \tag{7.13}$$

Using Eqs. 7.10 and 7.13, a set of simple relations can be derived between the positions of the beam waists and the beam parameters. Solving for $(z'_1 - f)$ in Eq. 7.10, inserting it into Eq. 7.13 and rearranging terms one has

$$\frac{(z_2 - f)^2}{f^2 - d_0'^2/\theta^2} = \frac{d_0^2}{d_0'^2}.$$

Multiplying by the left-side denominator,

$$(z_2 - f)^2 = f^2\frac{d_0^2}{d_0'^2} - \frac{d_0^2}{\theta^2} = f^2\frac{d_0^2}{d_0'^2} - z_R^2,$$

and then by $d_0'^2$

$$d_0'^2[(z_2 - f)^2 + z_R^2] = f^2 d_0^2,$$

One obtains the new beam waist d'_0 in terms of the old beam waist d_0:

$$d_0'^2 = \frac{f^2 d_0^2}{(z_2 - f)^2 + z_R^2} = \alpha^2 d_0^2, \tag{7.14}$$

The parameter α, given by

$$\alpha = \frac{|f|}{\sqrt{(z_2 - f)^2 + z_R^2}}, \tag{7.15}$$

is a function of axial quantities related to the input beam (z_R), the lens (f) and the beam–lens position ($z_2 - f$). Once α is evaluated for the arrangement at hand, the output parameters are quickly determined since

$$d'_0 = \alpha d_0, \tag{7.16}$$

$$\theta' = \frac{4\lambda}{\pi d'_0} = \frac{4\lambda}{\pi\alpha d_0} = \frac{\theta}{\alpha}, \tag{7.17}$$

$$z'_R = \frac{d'_0}{\theta'} = \frac{\alpha d_0}{\theta/\alpha} = \alpha^2 z_R. \tag{7.18}$$

Using Eqs. 7.13 and 7.16, the position of the new beam waist z'_1 can be expressed as

$$(z'_1 - f) = \frac{d_0'^2}{d_0^2}(z_2 - f) = \alpha^2(z_2 - f),$$

$$z'_1 = f + \alpha^2(z_2 - f) = f + \frac{f^2(z_2 - f)}{(z_2 - f)^2 + z_R^2}. \tag{7.19}$$

In the thin lens approximation, all of these equations are quite general. Now we will look at some approximations.

7.2.2. Focusing of a Collimated Beam to a Small Spot

Since α is directly proportional to f, a small spot requires a short focal length lens. Since the beam is collimated z_2 and z_R are large compared to f, $\alpha = f/\sqrt{z_2^2 + z_R^2}$. There are two limiting cases:

1. The lens is well inside z_R ($z_2 \ll z_R$). Then $\alpha = f/z_R = f\theta/d_0$. This gives

$$d'_0 = \alpha d_0 = f\theta, \tag{7.20}$$

a simple equation for finding d_0'. Also

$$\theta' = \frac{\theta}{\alpha} = \frac{\theta d_0}{f\theta} = \frac{d_0}{f}, \tag{7.21}$$

$$z_R' = \alpha^2 z_R = \frac{f^2}{z_R^2} \cdot z_R = \frac{f^2}{z_R}, \tag{7.22}$$

$$z_1' = f + \alpha^2(z_2 - f) = f + \frac{z_2 \cdot f^2}{z_R^2} \cong f. \tag{7.23}$$

2. The lens is well outside z_R ($z_2 \gg z_R$; beware, the lens may be overfilled if it is too far away). Then $\alpha = f/z_2$ and

$$d_0' = \frac{f d_0}{z_2}, \tag{7.24}$$

$$\theta_0' = \frac{z_2\theta}{f} \tag{7.25}$$

(note $z_2\theta = d$ at the lens),

$$z_R' = \frac{f^2 z_2}{z_R^2}, \tag{7.26}$$

$$z_1' = f + \frac{(z_2 - f) \cdot f^2}{z_2^2} = f + \frac{f^2}{z_2} \cong f. \tag{7.27}$$

In Example 7.A three situations are illustrated. Note that the placement of the lens relative to the size of the Rayleigh range has a dramatic effect on the spot size.

It should be understood that we do not always know where the beam waist is in a laser. In many cases the beam waist is at or near the output mirror, and for an initial attempt to design a system this is a reasonable initial assumption. However, later in the design, the sensitivity of the design to the initial beam waist location should be checked. If the sensitivity is high, a more precise calculation can be made. Given the radius of curvature of the two laser mirrors R_1 and R_2 and their separation L, the distance between the beam waist and the first mirror is

$$z_1 = \frac{L(R_2 + L)}{R_1 - R_2 + 2L}. \tag{7.28}$$

EXAMPLE 7.A. A helium–neon laser beam with a divergence of 0.8 mrad is focused with a 100-mm focal length lens. What are the beam diameters in the cases

(a) $z_2 \ll z_R$
(b) $z_2 = z_R$
(c) $z_2 \gg z_R$?

Solution: Since $\theta = 0.8 \times 10^{-3}$ rad, $d_0 = \dfrac{4\lambda}{\pi\theta} = 1$ mm. Therefore $z_R = d_0/\theta = 1 \times 10^{-3}/0.8 \times 10^{-3} = 1.25$ m.

(a) $z_2 \ll z_R$, $d_0' = f \cdot \theta = 10^{-1}\,\text{m} \times 0.8 \times 10^{-3} = 80\,\mu\text{m}$.
(b) $z_2 = z_R$. The complete expression must be used.

$$d_0' = \frac{f d_0}{\sqrt{(z_2 - f)^2 + z_R^2}} = \frac{10^{-1} \cdot 10^{-3}}{\sqrt{(1.25 - 0.1) + (1.25)^2}} = 60\,\mu\text{m}.$$

(c) $z_2 \gg z_R$. Let us assume $z_2 = 12.5$ m. The beam diameter there is ab 10 mm

$$d_0' = \frac{f d_0}{z_2} = \frac{0.1}{12.5} \times 1 \times 10^{-3}\,\text{m} = 8\,\mu\text{m}.$$

Note the distance between the beam waist and the lens is extremely large.

The distance between the beam waist and the second mirror is $z_2 = L + z_1$ ($z_1 < 0$) [Problem 7.3]. If the outside surface of the output mirror is not flat, then the effect of the optical power of this surface must also be calculated using the paraxial method given in Section 7.2.6.

7.2.3. Refocusing a Beam Waist to a New Beam Waist

This is a case of magnification or demagnification of the beam waist. Since $d_0' = \alpha d_0$, α is the magnification constant for the waist. At the same time the beam divergence will be demagnified by θ/α, and the Rayleigh range will be multiplied by α^2. (This parallels the relation between the transverse magnification and the longitudinal magnification for imaging: $M_l = M_t^2$.)

Once the value of α has been decided on, there are still two degrees of freedom in the system: the focal length of the lens (f) and location of the lens with respect to the beam waist (z_2). While there is one independent input parameter and one independent output parameter, there is an infinite combination of the focusing geometries that will provide the required beam waist. There are some restrictions, however, on the choice of the focal length. Assuming the value of α has been chosen, then we can solve (7.15) for z_2:

$$z_2 = f \pm \sqrt{\frac{f^2}{\alpha^2} - z_R^2} = f \pm \frac{1}{\alpha}\sqrt{f^2 - f_0^2} \tag{7.29}$$

Thus, f may be chosen to be any focal length practical (or handy) subject to the limitation that $f > f_0^2 = z_R z_R'$ [Problems 7.4–7.6]. In Example 7.B, the use of Eq. 7.29 is illustrated by a practical problem.

EXAMPLE 7.B. Reduce the beam waist of a helium–neon laser ($d_0 = 1$ mm; $\theta = 0.8$ mrad; $z_R = 1.25$ m) to 184 μm. Assume waist is at the output mirror.

Solution:

$$\frac{d_0'}{d_0} = \frac{184\ \mu\text{m}}{1\ \text{mm}} = 0.184 = \alpha,$$

$$z_R = 1.25\ \text{m} \qquad z_R' = \alpha^2 z_R = 42\ \text{mm},$$

$$f_0^2 = z_R z_R' = 1250 \times 42\ \text{mm}^2,$$

$$f_0 = 230\ \text{mm} \qquad f > 230\ \text{mm}.$$

(a) Choose $f = 300$ mm, then

$$z_2 = 300 + \frac{1}{0.184}\sqrt{(300)^2 - (230)^2} = 1347\ \text{mm}.$$

The positive root was chosen since the laser would get in the way for the negative root. This is an extremely long distance.

(b) Then choose $f = 235$ mm and

$$z_2 = 235 + \frac{1}{0.184}\sqrt{(235)^2 - (230)^2} = 497\ \text{mm},$$

$$z_1' = 235 + (0.184)^2(497 - 235) = 244\ \text{mm}.$$

Total distance from laser to focus: $z_2 + z_1' = 741$ mm.

(c) For the shortest distance, choose $f = f_0 = 230$ mm. Then $z_2 = z_1' = f$. Total distance: $z_2 + z_1' = 2f = 460$ mm. Thus by choosing the correct lens a practical laser-to-new-focus distance can be provided.

A special case of the transfer of focus is that of relaying the waist without magnification ($\alpha = 1$). Starting with the definition of α (7.15),

$$\alpha^2 = \frac{f^2}{(z_2 - f)^2 + z_R^2} = 1,$$

and solving for z_2, one obtains

$$z_2 = f \pm \sqrt{f^2 - z_R^2}, \quad (f \geq z_R) \tag{7.30}$$

or, in terms of f,

$$f = \frac{z_2^2 + z_R^2}{2z_2}.$$

The distance between original beam waist and the lens is found to be equal to the distance between the lens and the new beam waist as would be expected for unit magnification [Problem 7.7].

$$z_1' = f + \alpha^2(z_2 - f) = f + z_2 - f = z_2.$$

Thus the total distance between waists in terms of f and z_R is

$$z_{12} = z_2 + z_1' = 2f \pm 2\sqrt{f^2 - z_R^2}. \tag{7.31}$$

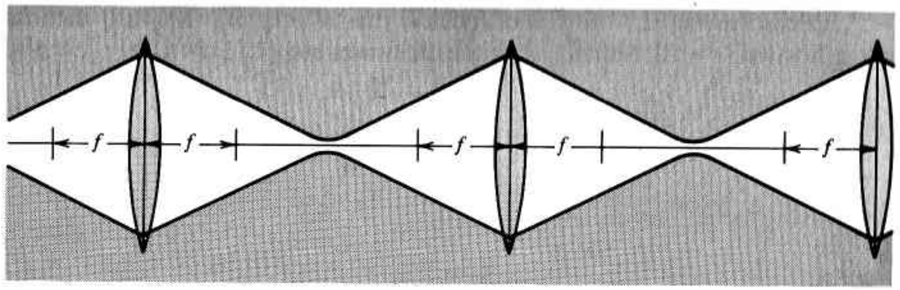
In addition to this general case there are two special cases (Fig. 7.5)

1. $f \gg z_R$. Then $z_{12} = 4f$. This is similar to the relaying of objects and images in geometrical optics, where the unit magnification occurs at $t_1 = -t_0 = 2f$. Note for this to occur for any reasonable focal length, z_R must be $\ll f$ and therefore the beam must be strongly divergent.
2. $f = z_R$. Then $z_2 = z_1' = f$ and $z_{12} = 2f$. This is obviously different than the previous case. The beam waists are separated by only $2f$. Since it is not always possible to match f to z_R, some prior beam conditioning is usually done. This second case is sometimes called a *Gaussian relay*.

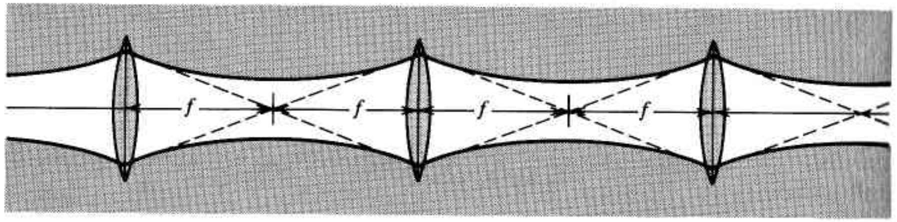
At one time this type of relay was considered for information transmission over some distances. Since the advent of low loss optical fibers this is no longer considered to be a realistic method of transmitting laser radiation.

7.2.4. Collimating a Gaussian Beam

If one cannot achieve truly collimated light, $\theta = 0$ (since d_0 would have to be infinite in extent), then what should be the definition of collimated light? There



(a) $z_R \ll (z_2 - f)$



(b) $z_R = z_2 = f$

Figure 7.5. Gaussian beam relays. (a) Case 1. Highly divergent beams, similar to an image relay. (b) Case 2. Rayleigh range equal to the focal length.

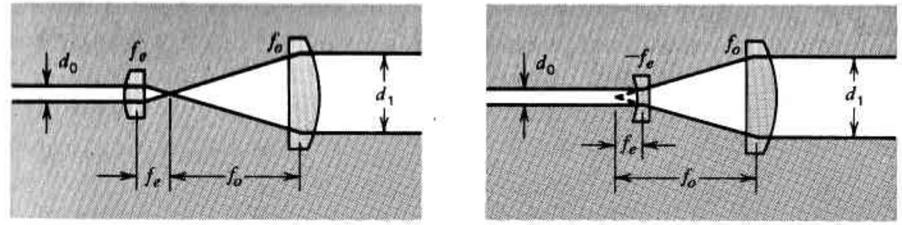
are two possible definitions. One is that the divergence be as small as possible, the other is that the distance to the next beam waist be a maximum. In the first case, one can show [Problem 7.8] that z_R' is a maximum if $z_2 - f = 0$. (Thus, $\alpha = f^2/z_R^2$ and $z_R' = f^2/z_R$.) In the second case z_1' is a maximum when $z_2 = f + z_R$. That is, the beam is considered collimated when a positive focal length lens is positioned such that its focal point is one Rayleigh range beyond the beam waist. Inserting this condition into Eq. 7.15 yields

$$\alpha = \frac{|f|}{\sqrt{(z_2 - f)^2 + z_R^2}} = \frac{|f|}{\sqrt{2z_R^2}}, \tag{7.32}$$

and this result in Eq. 7.19 determines the location of the new beam waist:

$$z_1' = f + \alpha^2(z_2 - f) = f + \frac{f^2}{2z_R^2}(z_R) = f + \frac{f^2}{2z_R}. \tag{7.33}$$

From Eqs. 7.32 and 7.33 we see that z_1' and z_R' will be large if z_R is small. Thus, a long focal length lens accepting a highly diverged beam (small z_R) will give



(a) (b)

Figure 7.6. Gaussian beam collimation. (a) Keplerian telescope. (b) Galilean telescope. Eyepiece focal length, f_e; objective focal length, f_o.

excellent collimation. If the initial beam is well collimated, however, then the beam must be focused with a short focal length lens before being recollimated with the long focal length collimation lens. This arrangement is nothing more than an inverted telescope (light goes in the eyepiece and out the objective (Fig. 7.6a)). It is also possible to reduce z_R with a diverging lens that forms a virtual beam waist in front of the negative lens. Together with the collimating lens this system is an inverted Galilean telescope (Fig. 7.6b).

We can use the information gained so far to calculate the increase in the collimation distance. The input beam is assumed to be well collimated. Therefore, the beam diameter at the eyepiece lens (focal length, f_e) is approximately equal to d_0 and the lens is well inside the Rayleigh range. This is the same as Case 1 of Section 7.2.2. so that

$$\alpha = \frac{f_e}{z_R},$$

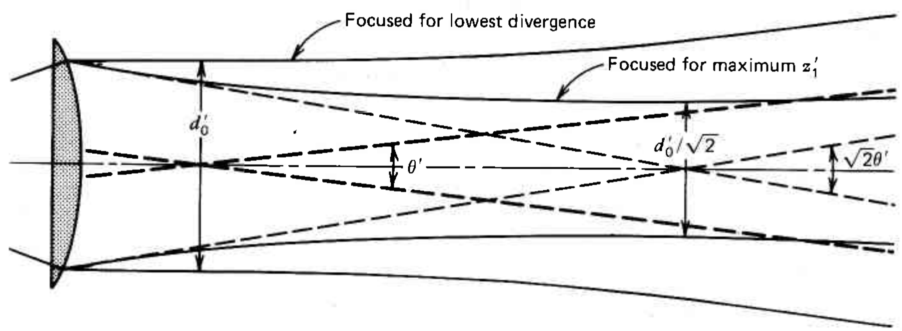


Figure 7.7. Comparison of two definitions of a collimated Gaussian beam. The bold lines show the asymptotes and $1/e^2$ points for a beam focused for smallest divergence, θ'. The fine lines indicate the asymptotes and $1/e^2$ points for a beam focused for maximum lens to beam waist distance. For a distance of about $2z_R$ from the lens this second definition maintains a smaller beam.

and

$$d_0' = \alpha d_0 = f_e \theta,$$

$$\theta' = \frac{d_0}{f_e},$$

$$z_R' = \frac{f_e^2}{z_R}, \qquad \text{so that} \quad z_R' \ll z_R$$

$$z_1' = f_e.$$

For maximum collimation we want $z_2' = f_o + z_R'$, where f_o is the focal length of the telescope objective. It is not to be confused with f_0, the diffraction term in Eq. 7.10. Using Eq. (7.32), $\alpha' = f_o/\sqrt{2z_R'}$. The output beam waist is then

$$d_0'' = \alpha' d_0' = \alpha' \alpha d_0 = \alpha'' d_0$$

and the overall magnification is

$$\alpha'' = \alpha' \alpha = \frac{f_e}{z_R} \times \frac{f_o}{\sqrt{2z_R'}} = \frac{f_e}{z_R} \cdot \frac{f_o}{\sqrt{2(f_e^2/z_R)}},$$

$$\alpha'' = \frac{f_o}{\sqrt{2f_e}} = \frac{M}{\sqrt{2}} \tag{7.34}$$

where $M = f_o/f_e$ is the magnification of the telescope. The collimation has increased by a factor of $M/\sqrt{2}$, since the divergence has decreased to $\theta'' = \theta/\alpha''$ and the Rayleigh range has increased by

$$z_R'' = \alpha''^2 z_R = \frac{M^2}{2} z_R. \tag{7.35}$$

Finally, it can be shown [Problem 7.9], that the beam waist is located at

$$z_1'' = f_o + \frac{M^2}{2} z_R = f_o + z_R'' \tag{7.36}$$

In Example 7.C, the collimation of a helium–neon laser by a 40 power telescope is illustrated [Problem 7.10].

If instead of a maximum lens to beam-waist distance (z_1'), a minimum beam divergence is required, the above results are changed by replacing $\alpha'' = M/\sqrt{2}$ by $\alpha'' = M$ and $z_1'' = f_o + M^2/2z_R$ by $z_1'' = f_o$. Thus the beam waist is at the back focal point of the telescope objective lens and the divergence is a factor of $\sqrt{2}$ smaller. In Fig. 7.7, the two cases are compared. As one can see from the figure, if collimation is needed close to the optical system, the beam waist can be smaller by a factor of $1/\sqrt{2}$ and collimated over a distance some $2z_R'$ from the lens. If collimation is needed in the far field (i.e., smallest beam divergence) then the focal point of the lens should be located at the original beam waist.

Of the two telescopes pictured in Fig. 7.6, the Keplerian type should be used if there is a need to spatially filter the beam (Ref. 7.10), a method for removing any spatial irregularities introduced by the optics before the collimator by putting a pinhole slightly larger than d_0' at the focus of f_e. For high power lasers focusing the beam to a small diameter could result in breakdown of the air by the high electrical field strength at the focus. In such cases, a Galilean type telescope should be employed (Fig. 7.6b).

EXAMPLE 7.C. Collimate a helium–neon laser beam ($d_0 = 1$ mm, $\theta = 0.8$ mrad, $z_R = 1.25$ m) using a 40-power telescope ($f_o = 160$ mm, $f_e = 4$ mm (40× microscope objective)).

Solution: First,

$$\alpha'' = \frac{f_o}{\sqrt{2f_e}} = 28.28 \quad \text{and} \quad (\alpha'')^2 = \frac{40^2}{2} = 800.$$

The new beam characteristics are:

$$d_0'' = \alpha'' d_0 = 28.28 \text{ mm},$$

$$\theta'' = \theta/\alpha'' = 28.28\ \mu\text{rad},$$

$$z_R'' = (\alpha'')^2 z_R = 1 \text{ km (!)}.$$

Unless extreme care is taken with collimation optics, this diffraction limited performance cannot be achieved. Even with simple lenses, however, the collimation can be quite good.

If minimum divergence is required, then $\alpha'' = 40$ and $(\alpha'')^2 = 1600$. In this case $d_0'' = 40$ mm, $\theta'' = 20\ \mu$rad, and $z_R'' = 2$ km.

7.2.5. Combinations of Lenses

It is possible using the above equations to calculate beam waist positions for a combination of lenses in a manner similar to the special case of beam-collimating telescopes. Instead of the ray-trace technique of the preceding chapters, the approach is similar to the thin lens calculations used in introductory physics. The equations are applied once, an intermediate waist is found, then the equations are applied again (Example 7.D). In the example note that the final beam characteristics are the same as those in Example 7.B. In that example the lens was nearly 500 mm from the laser and the focal length of the lens had to be greater than 230 mm. By using two lenses, a similar result was achieved with shorter focal length lenses placed closer to the laser. In addition, the ability to move one lens relative to the other allows one to adjust the output beam parameters easily.

EXAMPLE 7.D. Using two lenses, obtain a "long" focus from a helium–neon laser. Assume beam waist at output mirror.

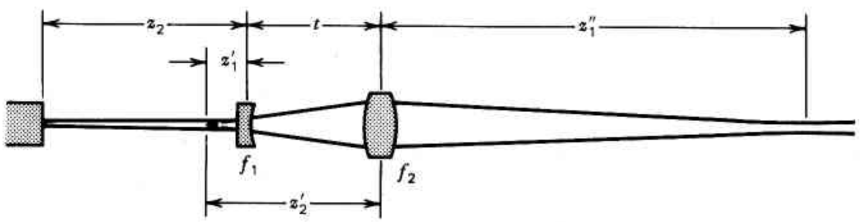
Helium–neon laser output:

$$d_0 = 1\,\text{mm},\ \theta = 0.8\,\text{mrad},\ z_R = 1.25\,\text{m}.$$

System dimensions:

$$z_2 = 135\,\text{mm}, \qquad t = 58.42\,\text{mm},$$

$$f_1 = -50\,\text{mm}, \qquad f_2 = 88.9\,\text{mm}.$$



Solution: For the first lens:

$$z_2 - f_1 = 185\,\text{mm}, \qquad \alpha = \frac{|-50|}{\sqrt{(185)^2 + (1250)^2}} = 4 \times 10^{-2}$$

$$d_0' = 4 \times 10^{-2} \times 1\,\text{mm} = 40\,\mu\text{m}; \qquad \theta' = 20\,\text{mrad}; \qquad z_R' = 2\,\text{mm}.$$

$$z_1' = f_1 + \alpha^2(z_2 - f_1) = -50 + 16 \times 10^{-4}(185) = -50 + 0.3 = -49.7\,\text{mm}$$

z_1' is negative (i.e., virtual), since the beam is diverging.

$$z_2' = t - z_1' = 58.42 - (-49.7) = 108.1\,\text{mm}.$$

For the second lens:

$$f_2 = 88.9 \quad \text{and} \quad z_2' - f_2 = 108.1 - 88.9 = 19.2\,\text{mm}.$$

$$\alpha' = \frac{88.9}{\sqrt{(19.2)^2 + 2^2}} = 4.605.$$

$$d_0'' = \alpha' d_0' = 4.605 \times 40\,\mu\text{m} = 184.2\,\mu\text{m}.$$

$$\theta'' = 20\,\text{mrad}/4.605 = 4.34\,\text{mrad}.$$

$$z_R'' = (\alpha')^2 z_R' = 42.4\,\text{mm}.$$

$$z_1'' = f_2 + (\alpha')^2(z_2' - f) = 88.9 + (4.605)^2 19.2 = 496\,\text{mm}.$$

Total distance:

$$\text{laser to beam waist} = 135 + 58.4 + 496 = 689\,\text{mm}.$$

The method given above is only one of several methods for calculating beam waist parameters. There are matrix based approaches (see Refs. 7.1 and 7.2) and chart based approaches (Ref. 7.4). In the case of periodic lens structures, such as a series of relay lenses, the matrix based approach may be more compact, whereas the chart-based approach may yield some insight as to poor choices of lenses and placements. The method given here is attractive because it is straightforward with certain useful approximations.

7.2.6. Paraxial Equations for Gaussian Beams

Following much the same mathematics used for the thin lens equations, it is possible to derive similar equations for the propagation of Gaussian beams at the interface of two media having optical power. Just as the paraxial equations for ray tracing bear a resemblance to the thin lens equations, so also are these paraxial equations similar to Eqs. 7.10 and 7.13. They are

$$(n' - z_1'\phi)(n - z_2\phi) = nn' - z_R z_R' \phi^2 \tag{7.37}$$

and

$$\frac{n - z_2\phi}{n' - z_1'\phi} = \frac{z_R}{z_R'}, \tag{7.38}$$

where $\phi = (n' - n)/r$ is the optical power of the interface as it is in paraxial optics (Section 4.1) and n' and n are the refractive indices of the refracting and

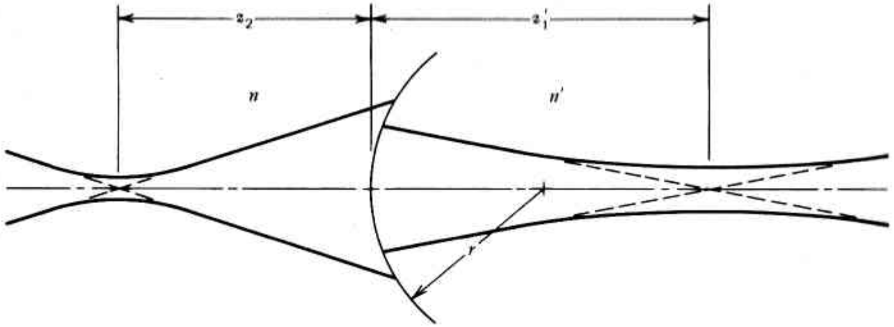


Figure 7.8. Paraxial ray trace of a Gaussian beam. The radius of curvature of the interface is r; the refractive indices of the incident and refracting media are n and n', respectively. The optical power of the surface is $\phi = (n' - n)/r$.

incident media, respectively as shown in Fig. 7.8. The distances z_2 and z_1' are defined as before. Multiplying these equations by ϕ/nn', one sees that $1/\phi$ replaces f and z/n replaces z in the respective media. Using these two equations, we may solve for $\alpha^2 = z_R'/z_R$ and find that

$$\alpha = \frac{\sqrt{nn'}}{\sqrt{(n - z_2\phi)^2 + z_R^2\phi^2}} \tag{7.39a}$$

and that

$$z_1' = \frac{z_R z_R' \phi - z_2 n'}{n - z_2\phi}. \tag{7.39b}$$

These are the paraxial counterparts to Eqs. 7.15 and 7.19. In Example 7.E, a Gaussian beam focused by a biconvex lens is analyzed two ways, a paraxial treatment and the thin lens method. The differences are small but significant [Problems 7.11–7.13].

For the special case of focusing a beam through a flat surface ($\phi = 0$), $\alpha = \sqrt{n'/n}$, where n is the incident medium refractive index and n' is that of the transmitting medium. If the incident medium is air ($n = 1$) the values of the beam parameters in the second medium are $d_0' = \sqrt{n'}\,d_0$; $\theta' = \theta/\sqrt{n'}$; $z_R' = n'z_R$. The quantities preserved in all media are $d_0/\sqrt{n}$, $\sqrt{n}\theta$, and z_R/n, where d_0, θ and z_R are the values for the beam in a vacuum.

If a Gaussian beam is focused inside of a material through a plane surface, where is the beam waist located? Setting $\phi = 0$ in Eq. 7.39b, $z_1' = -n'z_2$. In terms of ray tracing the refraction of a ray of light along the asymptote at an angle of β with respect to the surface normal, as shown in Fig. 7.9, is refracted to

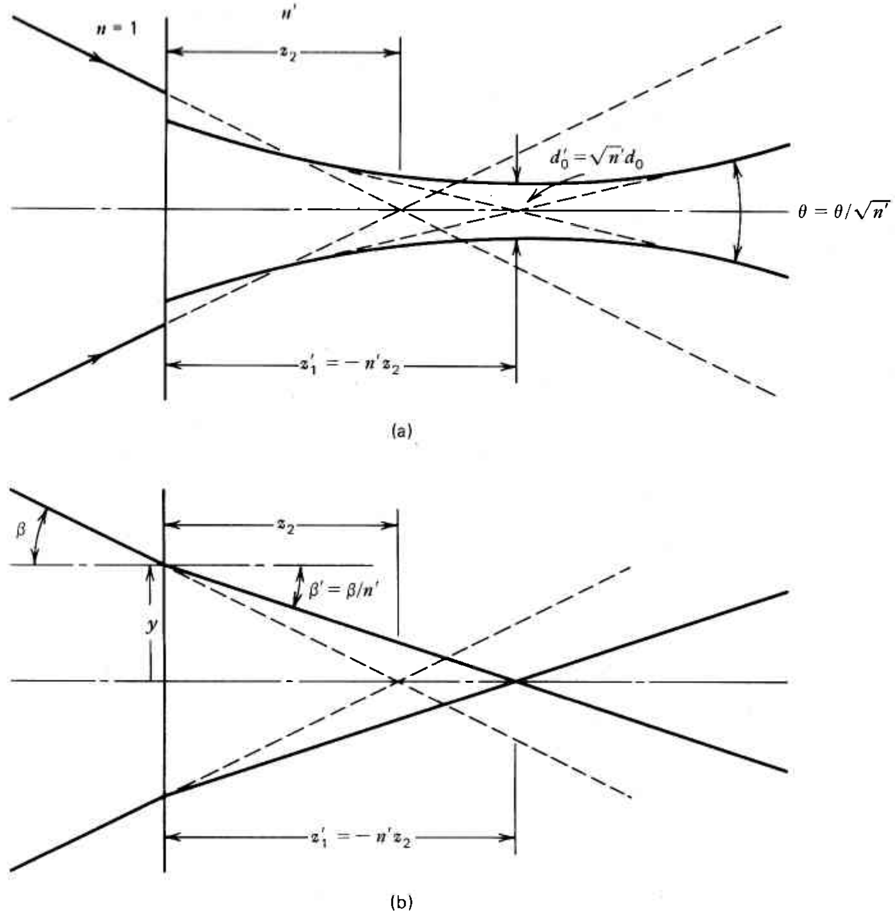


Figure 7.9. Focusing of a Gaussian beam into a plane interface. (a) Gaussian beam asymptotes, (b) ray trace of incident asymptotes. Both locate the beam waist at the same point.

an angle β', where $\sin\beta' = \sin\beta/n'$ ($n = 1$). If d is the diameter of the beam at the surface then

$$\tan\beta = \frac{d}{2z_2} \quad \text{and} \quad \tan\beta' = \frac{d}{-2z_1'}.$$

For small angles, setting the sines and tangents equal to their arguments, $\beta = n'\beta'$, we can write

$$\frac{d}{2z_2} = \frac{n'd}{-2z_1'},$$

$$z_1' = -n'z_2, \tag{7.40}$$

EXAMPLE 7.E. A biconvex lens ($n = 1.5$, $R = -R' = 100$ mm, $t = 20$ mm (thick!)) is placed 100 mm from the output mirror of a laser ($d_0 = 1$ mm, $\theta = 0.8$ mrad, $z_R = 1.25$ m). Find the position and size of the beam waist after it passes through the lens. Compare it to the thin lens values.

Solution:

$$z_2 = 100 \text{ mm}, \qquad \phi_1 = (1.5 - 1)/100 = 1/200 \text{ mm}.$$

$$z_2\phi = \tfrac{1}{2}, \qquad z_R\phi_1 = 1250/200 = 6.25.$$

$$\alpha = \frac{\sqrt{1.5}}{\sqrt{(1-\frac{1}{2})^2 + (6.25)^2}} = \frac{\sqrt{1.5}}{\sqrt{\frac{1}{4} + (6.25)^2}} = \frac{\sqrt{1.5}}{6.27} = 0.195,$$

$$\alpha^2 = 0.0382.$$

$$d_0' = 195\ \mu\text{m}, \qquad \theta' = 4.10 \text{ mrad}, \qquad z_R' = 47.7 \text{ mm}, \qquad z_1' = 296 \text{ mm}.$$

The distance from the beam waist at the next surface is:

$$z_2' = t - z_1' = 20 - 296 \text{ mm} = -276 \text{ mm}.$$

$$\phi_2 = (1 - 1.5)/-100 = 1/200 \text{ mm} = \phi_1.$$

$$z_2'\phi_2 = -1.381, \qquad n - z_2'\phi_2 = 1.5 - (-1.381) = 2.881.$$

$$z_R'\phi_2 = 47.7/200 = 0.2385.$$

$$\alpha' = \frac{\sqrt{1.5}}{\sqrt{(2.88)^2 + (0.239)^2}} = \frac{\sqrt{1.5}}{\sqrt{8.36}} = 0.424, \qquad \alpha'^2 = 0.179.$$

$$d_0'' = 82.7\ \mu\text{m}, \qquad \theta'' = 9.67 \text{ mrad}, \qquad z_R'' = 8.56 \text{ mm}, \qquad z_1'' = 96.56 \text{ mm}.$$

Also

$$\alpha_{\text{total}} = \alpha\alpha' = 0.195 \times 0.424 = 0.0827.$$

In the thin lens treatment:

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{100} - \frac{1}{-100}\right) = \frac{1}{2} \times \frac{2}{100} = \frac{1}{100}, \qquad f = 100 \text{ mm}.$$

$$d_0'' = f \cdot \theta = 100 \text{ mm} \times 0.8 \text{ mrad} = 80\ \mu\text{m} \rightarrow \alpha = 0.08.$$

$$\theta'' = \theta/\alpha = 0.8/0.08 = 10 \text{ mrad}.$$

$$z_R'' = 1250 \text{ mm} \times (0.08)^2 = 8 \text{ mm}.$$

$$z_1'' = f + \alpha^2(z_2 - f) = 100 + \alpha^2(0) = 100 \text{ mm}.$$

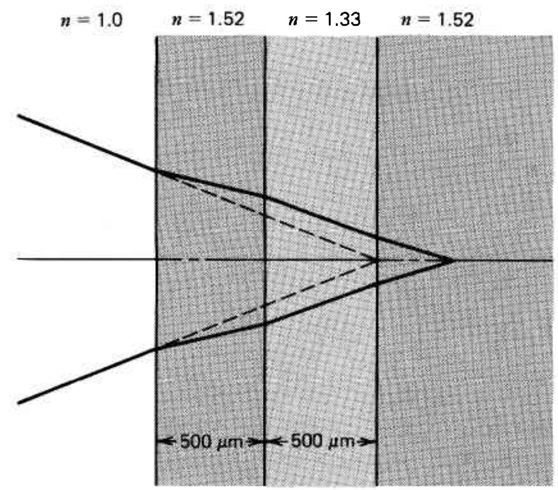
Thus, the thin lens values in this case give a good approximation to the paraxial figures.

the same relation just derived using Eq. 7.39b with $\phi = 0$ and $n = 1$. Thus, the position of a Gaussian beam waist after passage through a plane interface can be computed as if the asymptotes were refracted as ordinary light rays with their intersection locating the waist. As was shown earlier in the case of displacement of an image (Section 5.1.4), the beam waist is displaced by a parallel plate of thickness t and refractive index n by an amount

$$\Delta t = \frac{n-1}{n} t.$$

An example of a calculation for focusing through parallel slabs of material is given in Example 7.F.

EXAMPLE 7.F. A helium–neon laser beam ($z_R = 1.25$ m) is focused by a short-focal-length lens ($f = 8$ mm) located well inside the Rayleigh range. The beam waist is located at the surface of a microscope slide. If a drop of water ($n = 1.33$) is put on the surface and covered with a 500 μm thick cover glass ($n = 1.52$) and the water entrapped between the glass and slide also has a 500 μm thickness, where is the new beam waist located?



Solution: In this case $\alpha = \sqrt{n'/n}$ for each interface. Since the power of the surfaces multiply the Rayleigh range in the expression for α, Gaussian beams can be treated very simply. All we need to do is follow the asymptotes of the beam in each region to their crossing at the beam waist.

1. Since $z_R, z_2 > f$, we may use Case 1 in Section 7.2.2. The original laser beam has (as we have seen earlier) the parameters $d_0 = 1$ mm, $\theta = 0.08$ mrad, $z_R = 1.25$ m. If $f = 8$ mm, then $d_0' = \theta \cdot f = 0.8 \times 10^{-3} \times 8 \times 10^{-3} = 6.4\ \mu$m. $\theta' = 0.125$ rad; $z_R' = 51.2\ \mu$m, and $z_1' = f = 8$ mm.

2. In the paraxial approximation, a parallel plate displaces an image a distance $[(n-1)/n] \times t$, where t is the thickness of the plate (Example 5.1.4). Thus, to find the position of the beam waist we need only add the additional contributions to each layer.

(a) The first glass surface is 1 mm from the original waist. The cover slip adds a distance of $[(1.52-1)/1.52] \times 500\ \mu m = 171\ \mu m$.

(b) The water layer adds an additional $[(1.33-1)/1.33] \times 500\ \mu m = 124\ \mu m$.

(c) Therefore, if there were air beyond the water, the beam waist would be 295 μm beyond the water–air surface. However, since the glass substrate is beyond this, the increase in distance into the surface will be nd, where d is distance in air, 295 μm. Therefore the final beam position is $1.52 \times 295\ \mu m$ or 448 μm beyond the original substrate surface.

3. An alternate calculation:

$$z'_1 = -1.52(-1000\ \mu m) = 1520\ \mu m.$$

$$z'_2 = t - z'_1 = 500 - 1520 = -1020\ \mu m.$$

$$z''_1 = -1.33(-1020)/1.52 = 892.5\ \mu m.$$

$$z''_2 = t - z''_1 = 500 - 892.5 = -392.5\ \mu m.$$

$$z'''_1 = -1.52(-392.5)/1.33 = 448\ \mu m.$$

7.3. CONJUGATE DISTANCES

As was pointed out in Example 7.D, the replacement of a single lens by two lenses provides the flexibility to change the position and size of the new beam waist. By moving the positive lens toward the negative lens, the beam waist is moved away from the two lenses. For example, by changing z'_2 in Example 7.D by 19.2 mm, from 108.1 to 92.9 mm ($z'_2 - f = 3.97$ mm there), the beam waist moves out to 1.69 m (see Example 7.G(a)). Since the distance between lenses was chosen to give $\alpha' = 20$ the beam waist is 0.800 mm. ($d''_0 = 20 \times d'_0 = 20 \times 40\ \mu m = 0.8$ mm). If we erect a target at the beam waist ($z''_1 = 1.69$ m), the spot will be as small as possible with this lens configuration. But it is *not* the smallest achievable spot at this target distance! If the positive lens is now moved away from the first lens by 0.8 mm so that ($z'_2 - f$) is changed to 4.77 mm, α' is reduced to 16.8 and the beam waist is reduced to $d''_0 = 0.672$ mm. ($\theta = 1.12$ mrad) and is now located at 1.44 m from the positive lens, between the lens and the target. What is most interesting is the size of the spot on the target. The target is 1.69 m − 1.44 m, or 0.25 m beyond the new beam waist, so that

using Eq. 7.4, $d''_2 = 0.734$ mm (see Example 7.G(b)). The spot beyond the beam waist is *smaller* than the beam waist when it was located at the target! It might seem that if focusing inside the target produced a smaller spot on the target, then an even shorter focus would produce even better results. But this is not so. Moving the second lens to a point where $\alpha' = 15$ ($z'_2 - f = 5.58$ mm) results in a smaller $d''_0 = 0.600$ mm, but $d''_2 = 0.759$ mm [Problem 7.8]. Thus, there is something special about the second configuration that was used. This effect is explained by the concept of conjugate distances.

EXAMPLE 7.G. (a) If the second lens of the combination in Example 7.D is moved from a point where $z'_2 - f = 19.2$ mm to a point where $z'_2 - f = 3.97$ mm, what is the beam diameter at the waist (d''_0) and where is it located (z''_1)?

Solution:

$$\alpha' = \frac{88.9}{\sqrt{(3.97)^2 + (2)^2}} = 20.$$

$$d''_0 = 20 \times 40 \times 10^{-6}\ \text{m} = 0.800\ \text{mm}.$$

$$z''_1 = f + \alpha'^2(z'_2 - f) = 88.9 + 400(4) = 1.69\ \text{m}.$$

Thus the spot is 0.8 mm at ~1.7 m from lens.

(b) How does the spot size change on a target located at the beam waist location in part (a) when the lens is moved to a point where $z'_2 - f = 4.77$ mm?

Solution:

$$\alpha' = \frac{88.9}{\sqrt{(4.77)^2 + 2^2}} = 16.8.$$

$$d''_0 = 16.8 \times 40 \times 10^{-6}\ \text{m} = 0.672\ \text{mm}, \qquad \theta_0 = 1.19\ \text{mrad}.$$

$$z''_1 = 88.9 + 283(4.77) = 1.44\ \text{m}.$$

At the target

$$z''_2 = 1.69 - 1.44\ \text{m} = 250\ \text{mm}$$

and the spot size is

$$d''_2 = \sqrt{(0.672)^2 + (1.19 \times 10^{-3} \times 250)^2} = 0.734\ \text{mm},$$

smaller than when the beam waist was located at the target.

7.3.1. Conjugate Distance Equations

To illustrate this, some simple differentiation is necessary. However, if you are not familiar with calculus, the end result of this short derivation is still of use. In Fig. 7.10 we have set up the geometry for this derivation. Let d_2 be the diameter of the beam at a fixed distance r from the exit pupil of an optical system. The angle subtended by the spot at that distance is $\beta = d_2/r$. While r is fixed, z_1 and z_2 are not (i.e., the beam waist can be anywhere between the exit pupil and the target). The basic relations are (all of the primes on the variables have been dropped):

$$r = z_1 + z_2,$$

$$d_2 = \sqrt{d_0^2 + \theta^2 z_2^2},$$

$$\beta = \frac{d_2}{r} = \frac{\sqrt{d_0^2 + \theta^2 z_2^2}}{z_1 + z_2}. \tag{7.41}$$

Since we wish to minimize β, let us take the derivative of β with respect to z_1 or z_2 and set it equal to zero:

$$\frac{d\beta}{dz_2} = \frac{1}{2} \cdot \frac{2\theta^2 z_2}{\sqrt{d_0^2 + \theta^2 z_2^2}} \cdot \left(\frac{1}{z_1 + z_2}\right) - \frac{\sqrt{d_0^2 + \theta^2 z_2^2}}{(z_1 + z_2)^2} = 0.$$

Rationalizing, we obtain

$$\theta^2 z_2(z_1 + z_2) = d_0^2 + \theta^2 z_2^2$$

$$\theta^2 z_1 z_2 = d_0^2$$

and

$$z_1 z_2 = \frac{d_0^2}{\theta^2} = z_R^2. \tag{7.42}$$

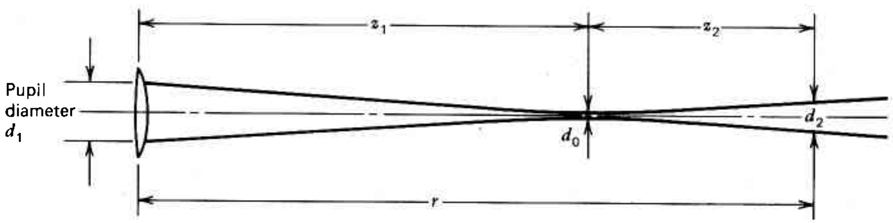


Figure 7.10. Geometry for the derivation of conjugate distances.

Thus, the minimum spot occurs when the product of the two distances (pupil to waist (z_1) and waist to target (z_2)) is equal to z_R^2. When related in this manner, these two distances are called *conjugate distances*. The other distinguishing feature of these conjugate points is that the radius of curvature of the wavefront R is the same value for both points except for a change in sign and it is equal to their sums. That is,

$$R = |z_1| + |z_2|. \tag{7.43}$$

The benefit of analyzing a Gaussian beam system by computing and using the conjugate points is that one can calculate optimum values on the basis of input parameters. One can show that beam diameters at the conjugate points are related by

$$d_1 d_2 = \frac{4\lambda R}{\pi} \tag{7.44}$$

and that [Problem 7.14]

$$\frac{d_1^2}{z_1} = \frac{d_2^2}{z_2} = \theta^2 R. \tag{7.45}$$

Although the optimum focal length for a lens may not be equal to the lens–target distance, in the case of tightly focused beams for cutting or welding ($z_R \ll z_1$), z_2 is so small that one might as well consider the conjugate point at the beam waist [Problem 7.15]. But in cases of scanning systems and collimating problems, the conjugate point can be removed a considerable distance from the waist and this must be taken into account in a design problem. The most extreme case is the Gaussian relay case, where $z_1 = z_2 = z_R$.

7.3.2. Calculation of Conjugate Distance Parameters

There are two general cases that should be considered. The first of these cases assumes that the size of the new beam waist *or* the focal length of the lens and its position are known. With either condition the value of α can be calculated. Since z_R is already known and $z'_R = \alpha^2 z_R$, the product of the conjugate distances is found from

$$z'_1 z'_2 = z'^2_R = \alpha^4 z_R^2. \tag{7.46}$$

Also, since

$$z'_1 = f + \alpha^2(z_2 - f), \tag{7.20}$$

z'_2, d'_2, and R' can be easily determined [Problem 7.16]. Beyond a certain point it does little good to carry out the calculations if α is small and $z'_1 \gg z'_R$. In such a case $z'_2 = z'^2_R/z'_1 \ll z'_1$ and we may as well set $z'_2 = 0$ and $z'_1 = R$. But, even if z'_2 is substantial, the calculations are simple.

The more difficult case to compute is one in which only the target distance, which is equal to R', and the beam diameter at the exit pupil, d'_1 $(=d_2)$, are given. Using Eq. 7.44 one can immediately check to see if the value of the spot at the conjugate point d'_2 is small enough for the application at hand:

$$d'_2 = \frac{4\lambda R'}{\pi d'_1}.$$

This can be used to find the ratio of the conjugate distances z'_2/z'_1. Using Eq. 7.45 and setting the ratio equal to γ, we obtain

$$\gamma \equiv \frac{z'_2}{z'_1} = \left(\frac{d'_2}{d'_1}\right)^2 = \left(\frac{4\lambda R}{\pi d'^2_1}\right)^2. \tag{7.47}$$

The quantity γ is useful in determining how important the effect of conjugate distances is. We can express z'_1 in terms of γ:

$$z'_1 = R' - z'_2 = R' - \gamma z'_1,$$

$$z'_1 = \frac{R'}{(1 + \gamma)}, \tag{7.48}$$

If $\gamma \ll 1$, we can ignore the effect of conjugate distance and set $z'_1 = R'$ and $z'_2 = 0$. Note that $\gamma = (4\lambda R/\pi d'^2_1)^2$ is a product of the initial quantities and thus can be determined first. If, however, γ is determined to be close to or greater than unity, then z'_R can be determined from $z'_R = z'_1 z'_2 = z'_1 \cdot \gamma z'_1 = \gamma R'^2/(1 + \gamma)^2$ and from this one can calculate that $\alpha = \sqrt{z'_R/z_R}$ [Problem 7.17].

7.4. LASER BEAM ILLUMINATION

Although most of this chapter has been concerned with finding the beam parameter relative to the waist region, not all problems require that the beam be focused on the surface in question. In some applications such as annealing and surface hardening, it is not necessary or useful to place the target at the beam waist. Such considerations as uniformity of illumination and tolerance of placement will play a part in the final design. It may be necessary to determine irradiance distribution in three dimensions to determine the best set up for a particular application.

7.4.1. Best Case Illumination

In contrast to those problems in which conjugate distances cannot be ignored, there are those in which the γ is small, and the new beam distance is approximated by $z'_1 = R' = f$ and $d'_2 = d_0$. These approximations in Eq. 7.44 gives

$$d'_0 \doteq d'_2 = \frac{d\lambda R'}{\pi d'_1} \doteq \frac{4\lambda f}{\pi d'_1}. \tag{7.49}$$

We see that, up to a point, as the lens is filled (d'_1 is made larger), the spot becomes smaller and thus the peak irradiance increases. Beyond this point the lens is overfilled and the focus spot begins to take on an Airy diffraction pattern (Section 1.7.3). In this region the spot size is the diameter of the central spot of the Airy diffraction pattern:

$$d'_{0\text{Airy}} = 2.44\frac{\lambda f}{D}, \tag{7.50}$$

where D is the diameter of the focusing lens. The problem is this: If the lens is underfilled ($d'_1 < D$) then the spot size will not be a minimum, yet the lens is overfilled ($d'_1 > D$) then there will be a loss of power through the lens. Obviously there is some maximum point where the beam irradiance at the focus is a maximum.

Following Rhyins (Ref. 7.5) we define a truncation ratio as $k = d'_1/D$. That is, when $k = 1$, the $1/e^2$ points on the Gaussian beam are at the edge of the lens. For $k < 1$ the lens is underfilled; for $k > 1$ the lens is overfilled. Comparing the Gaussian and Airy cases:

1. *Gaussian case.* $d'_1 = kD$, where $k < 1$, $d'_0 = 4/\pi k \cdot \lambda f/D = (1.27/k)\lambda F$, where $F = f/D$, the f-number of the lens. Because the lens is underfilled, the power through the lens would be, except for reflection losses, the total power in the beam P_0. Thus the irradiance of the spot is (using Eq. 7.49):

$$I = \frac{4P_0}{\pi d'^2_0} = \frac{4P_0}{\pi \times \frac{16}{\pi^2 k^2}\lambda^2 F^2} = \frac{\pi k^2 P_0}{4\lambda^2 F^2} \approx k^2\frac{P_0}{\lambda^2 F^2} \tag{7.51}$$

as plotted in Fig. 7.11 in the region $k < 1$.

2. *Airy case.* $d'_0 = 2.44\lambda f/D = 2.44\lambda F$. The power through the lens can be calculated by integrating the Gaussian distribution over the lens area. As an approximation, we assume

$$P_t = P_0 \times \frac{D^2}{d'^2_1} = \frac{P_0 D^2}{k^2 D^2} = \frac{P_0}{k^2}. \tag{7.52}$$

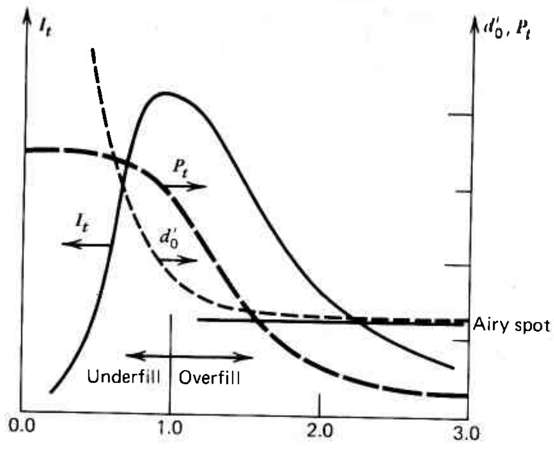


Figure 7.11. Plot of power (P_0), spot size (d'_0) and irradiance (I_t) as a function of lens filling factor, $k = d_2/D$ (d_2 = Gaussian beam diameter at the lens; D = lens diameter). $k < 1$ corresponds to an underfilled lens; $k > 1$ corresponds to an overfilled lens.

Since the spot size is constant, the irradiance is approximately

$$I = \frac{4P_t}{\pi d_0'^2} = \frac{4}{\pi d_0'^2}\frac{P_0}{k^2} = \frac{4}{\pi k^2}\cdot\frac{P_0}{[2.44\lambda F]^2} \approx \frac{1}{k^2}\frac{P_0}{\lambda^2 F^2} \tag{7.53}$$

as plotted in Fig. 7.11 for the region $k > 1$.

Thus in both regions the irradiance falls off as the truncation ratio departs from unity. More precise calculations show that maximum irradiance occurs at $k = 0.93$, a slight underfill. However, with this illumination, the spot has Airy rings and these may not be acceptable in certain applications. So the decision as to the size of a lens and the filling of its exit pupil is one that rests with the details of the application (including cost). In those cases where the Gaussian profile must be preserved, the lens must be underfilled to prevent any substantial diffraction due to the edge of the lens. As a rule of thumb, the beam diameter at the lens (d'_1) should be less than $\frac{5}{8}$ of the lens diameter (Ref. 7.6) corresponding to a truncation ratio of $k = 0.625$ [Problems 7.18 and 7.19].

7.4.2. The Irradiance Distribution of a Gaussian Beam

Equation 7.4 describes a boundary of the Gaussian beam, the hyperboloid of revolution for the focus of irradiance point of $1/e^2$ of the axis irradiance. This does not give a complete picture of the irradiance variation in the vicinity of the

beam waist. The irradiance distribution there can be described by an expression in terms of d_0 and z_R:

$$I(d, z) = \frac{I_0 \exp(-(2d^2/d_0^2)/(1 + (z^2/z_R^2))}{1 + (z^2/z_R^2)} \tag{7.54}$$

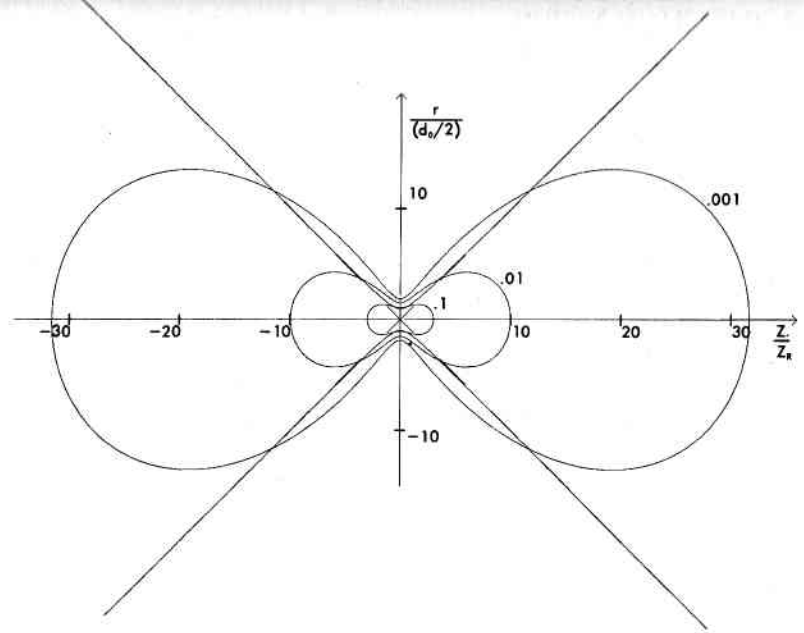
where $I_0 = I(0, 0)$, the irradiance at the center of the beam waist. A plot of the lines of constant irradiance (isophotes) is shown in Fig. 7.12. Note that there is a change in the character of contours from an elongated ellipsoid to a "dogbone." This inflection point occurs when $I/I_0 = 1/e = 0.36788$. In terms of optical design, any effect that has an irradiance dependent threshold, such as in a photographic medium, will be more tolerant of a shift of the target out of the focus area if the beam power is adjusted to maintain the threshold value at $\sim$37% of the central irradiance. The distribution shown here and the equations given here are for loosely (as opposite to tightly) focused beams; beams for which $\theta \leq \frac{1}{4}$ rad (or 14°).

7.4.3. Beam Shaping

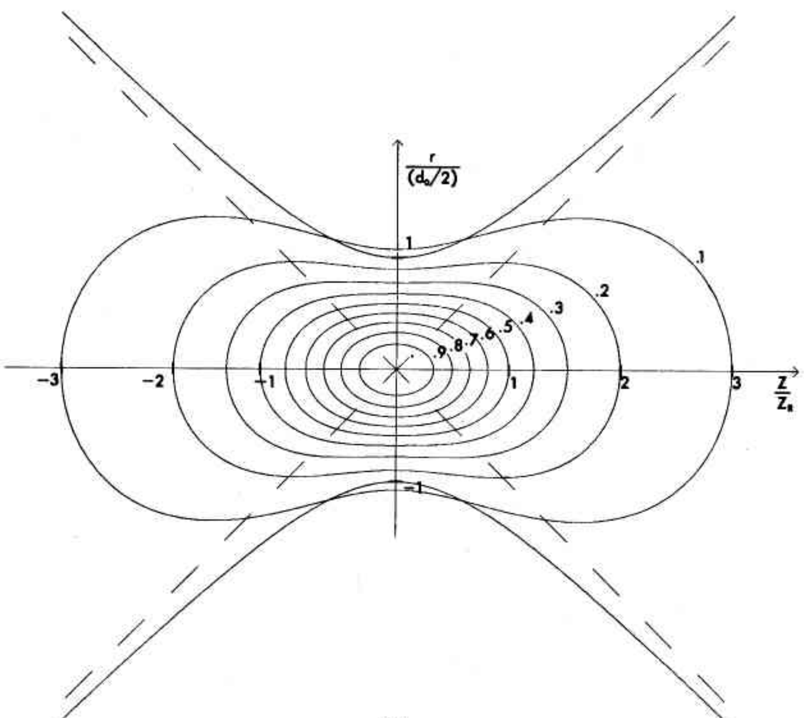
For some applications a circular Gaussian beam shape is not the most effective irradiance distribution. There are a number of methods of converting beam shapes from one form to another. The first of these, the cylindrical telescope, is used in some of the beam deflection schemes described in Chapter 8. In acousto-optic deflectors, the beam must be expanded and collimated, but only in one direction. If negative and positive cylindrical lenses are arranged to form a one-dimensional Galilean telescope, as shown in Fig. 7.13(a), the beam can be expanded and recollimated in one direction, while the original beam divergence and small beam cross sections are preserved in the other direction. It might seem easier to just use standard spherical lenses and expand the beam in two dimensions, but the economics of the beam deflectors require that the beam remain small so that it can pass through a thin slab of deflector material. Obviously, any telescope, when reversed serves as a beam compressor. Remember, however, that even though a telescope can compress and recollimate a beam, the divergence increases with decreased beam waist diameter. The restriction on the $d_0\theta$ product holds.

Another device for changing the beam size in one direction is the *prism beam compressor* (Ref. 7.7). Two prisms, arranged as shown in Fig. 7.13(b) reduce the beam width by refraction at the second slanted interface. If the refractive indices of the prisms are n_1 and n_2, the compression due to the first prism oriented at the Brewster angle (Section 1.6.1) $c_1 = d_2/d_1 = 1/n_1$ [Problem 7.20] and that due to the second prism, $c_2 = 1/n_2$. The overall compression is the product of the two compressions and thus the product of the reciprocals of the two refractive indices,

$$c = c_1 \cdot c_2 = \frac{1}{n_1 n_2}. \tag{7.55}$$



(a)



(b)

Figure 7.12. Irradiance distributions of Gaussian Beams near a beam waist. (a) Contours of equal irradiance for three orders of magnitude; (b) between 0.1 I_0 and I_0.

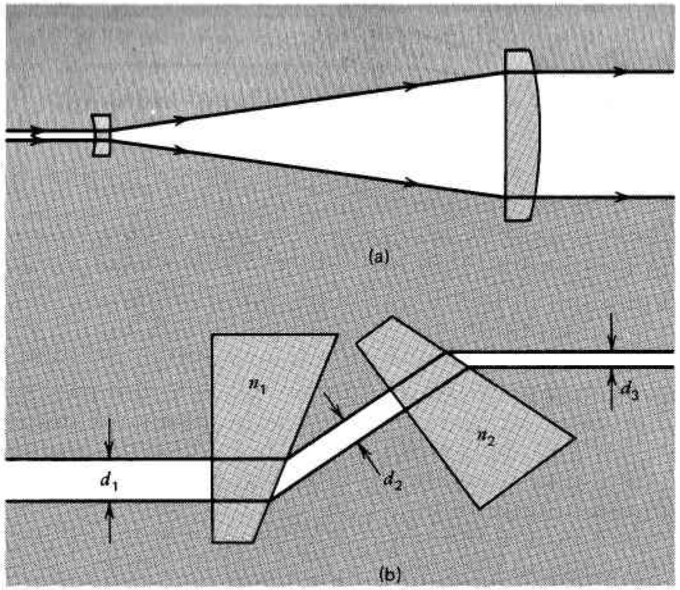


Figure 7.13. Beam expansion and compression. (a) One-dimensional telescope of cylindrical lenses used as a beam expander. (b) A two-prism beam compressor. Reversing the direction of light propagation changes the device from an expander into a compressor.

It is possible to arrange the prisms so that the output beam is parallel to the input beam. Since the prisms are cut to work at the Brewster angle, reflection losses are reduced. Some advantages to this assembly are that it is less costly to construct and more compact than a telescope, especially for modest compression ratios. Although the system can be used to vary the compression ratio by changing the angles of the prisms relative to the beams, this is done at the expense of reflection losses, especially if the surfaces where the beam is normally incident are anti-reflection coated. As was noted with the cylindrical telescope, the function of the system can be reversed by reversing the direction of the beam. The beam compressor is turned into a beam expander.

The two examples given above produce a fan-shaped beam, a beam with different divergences in two orthogonal directions. The beam spreads more rapidly in one dimension than in the other dimension, that is, θ_x is β times larger than θ_y ($\theta_x = \beta\theta_y$). This same situation is present in most semi-conductor diode (injection) lasers. In many designs it is necessary to convert the fan-shaped beam into a circular beam, with the same divergence for both dimensions. One method would be to use a cylindrical telescope or prism beam compressor to increase or decrease the divergence in one beam direction to match that of the other direction. Another method, shown in Fig. 7.14, uses a hybrid system, composed of both cylindrical and spherical lenses. Here a cylindrical lens increases the divergence of the beam in the more collimated direction to that of

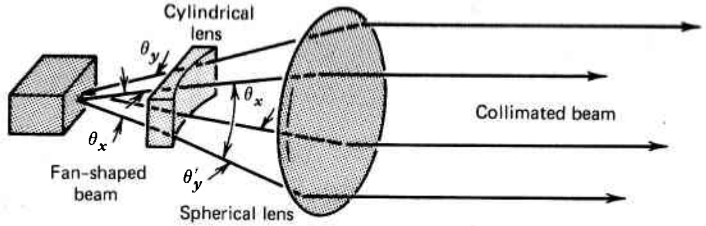


Figure 7.14. Hybrid system. A cylindrical diverging lens is used to diverge the more collimated of two divergences in a fan-shaped beam. A spherical lens then collimates the divergent beam.

beam in the less collimated direction. That is: $\theta'_y = \theta_x = \beta\theta_y$. The lens must increase the divergence of θ_y by β. Since $\alpha = \theta/\theta'$ and $\theta'_y = \beta\theta_y$, $\alpha_y = 1/\beta$. In addition, since a circular beam cross section is desired, the cylindrical lens should be located at a point where $d_{2x} = d_{2y}$. It can be shown that this location is $z_{2x} = \beta z_{Rx}$. Since the divergences in the two directions are different $\theta_x = \beta\theta_y$, so are the Rayleigh ranges. One can easily show that

$$z_{Ry} = \beta^2 z_{Rx}. \tag{7.56}$$

The Rayleigh range of the more collimated beam direction is large compared to the other direction, so that the cylindrical lens used to increase the divergence by β times would be inside of z_{Ry}. Thus the simple approximation of a short focal length lens inside the Rayleigh range can be used. For that case, $\alpha = f_y/z_{Ry}$ and $\alpha_y = 1/\beta$ determine the focal length of the lens: $f_y = z_{Ry}/\beta = \beta^2 z_{Rx}/\beta = \beta z_{Rx}$, where we have used Eq. 7.56 for the second equality. Once the divergences of the beam directions and the emerging beam diameters are the same, a spherical lens can be positioned, as in the collimation case, at $z_2 = f + z_{Rx}$ to produce a collimated spherical beam.

In some cases, (e.g., the diode laser), not only are the divergences different, but the location of the beam waists d_{0x}, d_{0y} are different. In this instance additional cylindrical lenses must be used to produce a common beam waist location and divergence (Ref. 7.8).

7.4.4. Flat-Top Distributions

Another irradiance distribution that is useful in some applications is the "flat-top" distribution, a variation across the beam that, at one edge, rises rapidly, remains constant across the center of the beam and falls just as rapidly, as shown in Fig. 7.15(a). If this distribution is rotationally symmetric about the beam direction, it is sometimes referred to as a "top hat" distribution, since a three-dimensional plot of irradiance in the x, y plane resembles a top hat (Fig. 7.15(b)). There are a number of different methods of producing this distribution. The simplest method is to use a mask to absorb or reflect the center of the beam and the edges of the beam. This method is inefficient since it reduces the total

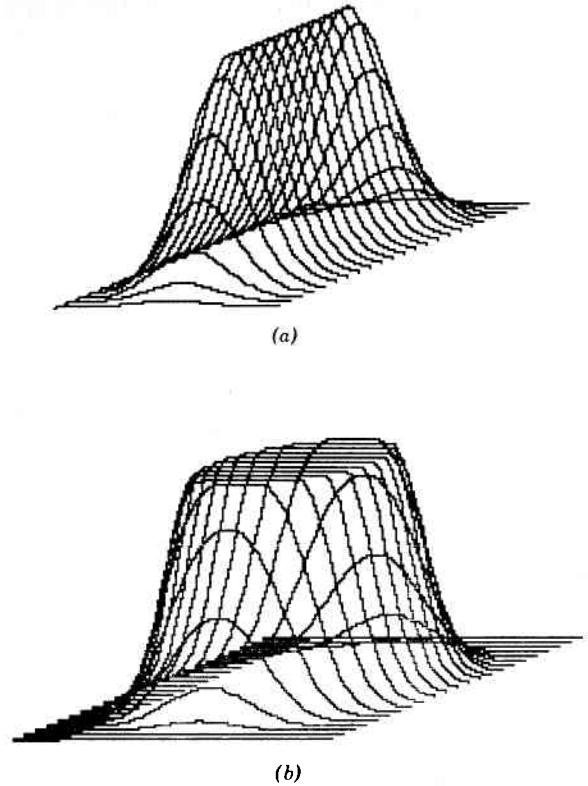


Figure 7.15. Non-Gaussian distributions. (a) Flat-top distribution; (b) Top-hat distribution.

energy in the beam. It also is difficult to use with high power lasers that could damage the mask. The work region must be located close to the mask, otherwise the distribution caused by diffraction will change in the far field.

One ingenious device that suffers only one of the drawbacks of the other methods (difficulty to produce) is the phase grating. The idea behind the phase grating is that a flat-top distribution in the far field is produced by an electric field distribution that has the form $\sin\phi/\phi$, as shown in Fig. 7.16(a). If a Gaussian beam illuminates a phase grating, a distribution of transparent material, which retards some parts of the beam by 180° with respect to the other parts, the transmitted electric-field amplitude will resemble the $\sin\phi/\phi$ function. In the far field, the distribution will close to a flat-top distribution, Fig. 7.16(b) (Ref. 7.9). By changing the periodicity of the grating and reducing the retardation to 144°, the flat-top distribution is enhanced. By combining the phase grating with a prism beam compressor a narrow flat-top distribution can be produced in the far field. Finally, it should be noted, a phase grating can be used to approximate other distribution. Since the relation between the phase grating output and the far-field pattern is that one is the Fourier transform

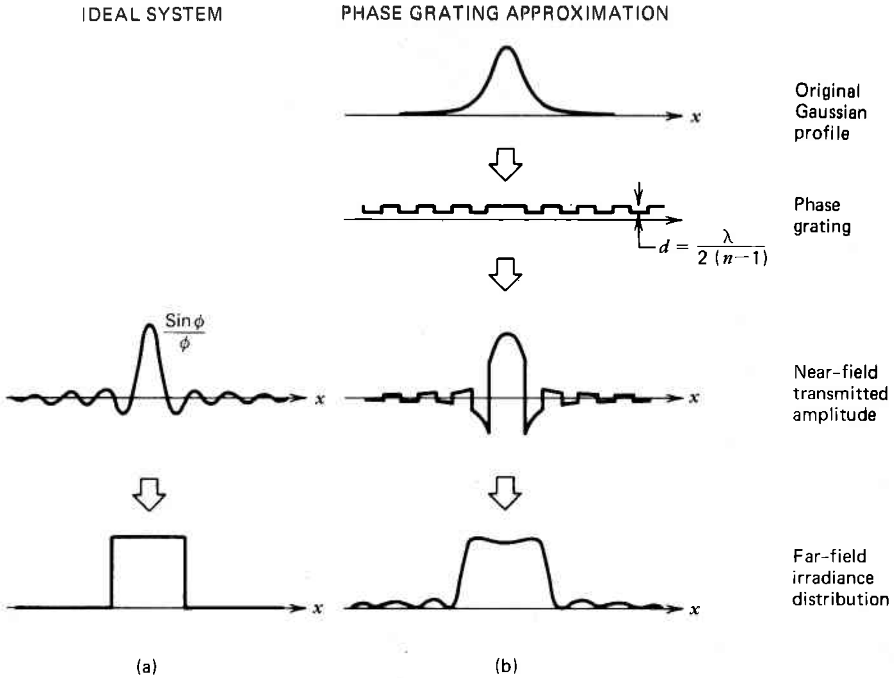


Figure 7.16. Binary phase grating. In an ideal system (a), a beam with an amplitude variation of sin ϕ/ϕ near a diffracting aperture will, in the far field, produce a flat-top irradiance distribution. An approximation (b) may be produced by illuminating a binary phase grating with a Gaussian beam.

of another, one must determine the closest phase grating approximation to the necessary output and construct a grating to produce it.

7.5. HIGHER ORDER MODES

The Gaussian beam is not the only irradiance distribution of radiation that can be produced by a laser, but it is the most fundamental. The resonant cavity referred to earlier may be able to support a number of other stable distributions, or modes, but none are as compact as the Gaussian beam and all contain nulls (regions of no radiation) within the irradiance distribution. The Gaussian beam is also called the TEM_{00} mode, a notation taken from the descriptions of electromagnetic field distributions in microwave cables and cavities. The other, higher order modes are labeled TEM_{mn}, where either m or n or both are greater than zero. In Fig. 7.17 there are shown a number of these distributions along with the TEM_{00} mode. Note the null lines in the distributions of the higher order modes. Calculations based on the electromagnetic field equations show that their intensity variations are described by polynomial functions multiplied by a Gaussian. A number of these have been plotted in Fig. 7.18. The

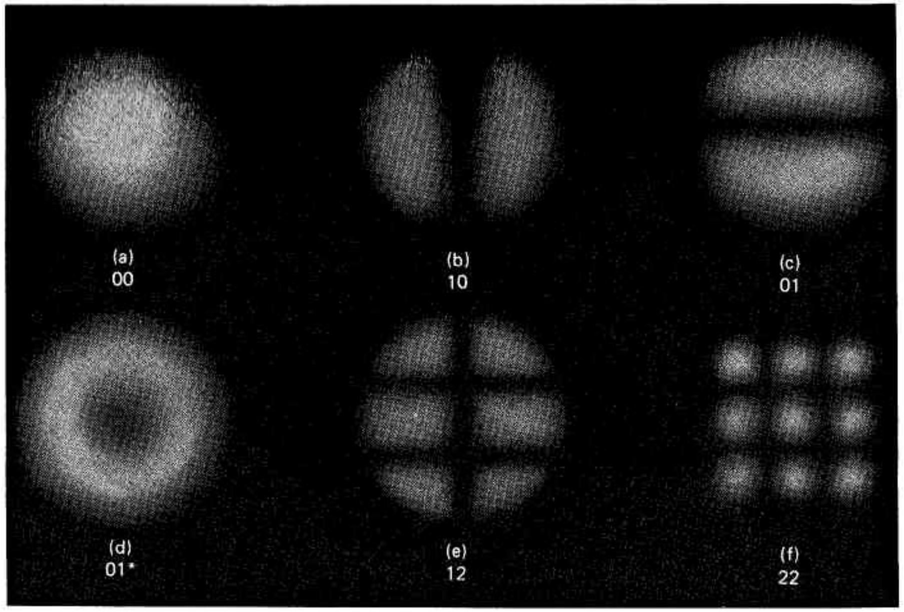


Figure 7.17. Transverse mode distributions.

descriptions of the propagation of higher order modes parallel those of the TEM_{00} modes, and many of the results can be derived by using the TEM_{00} mode values (Ref. 7.2).

For a higher order mode TEM_{mn} the beam spot size is a best fit to a rectangle of dimensions $d_x(0)$ by $d_y(0)$. There is a relation between the spot size and the divergence for higher order modes similar to that for the fundamental mode (Ref. 7.10).

$$d_{xm}(0)\theta_{xm} = \frac{4\lambda}{\pi}(2m + 1), \tag{7.57a}$$

$$d_{yn}(0)\theta_{yn} = \frac{4\lambda}{\pi}(2n + 1), \tag{7.57b}$$

where $d_{xm}(0)$ and $d_{yn}(0)$ are the beam diameters at $z = 0$ for the m and n orders respectively and

$$d_{xm}(0) = \sqrt{2m + 1}\,d_0, \tag{7.58a}$$

$$d_{yn}(0) = \sqrt{2n + 1}\,d_0, \tag{7.58b}$$

$$\theta_{xm} = \sqrt{2m + 1}\,\theta, \tag{7.58c}$$

$$\theta_{yn} = \sqrt{2n + 1}\,\theta, \tag{7.58d}$$

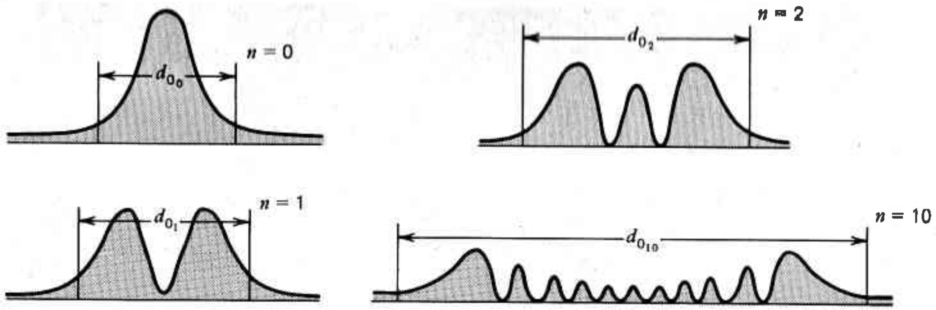


Figure 7.18. The profiles for higher-order modes. These are products of a Gaussian distribution and a polynomial (Hermite) function.

and d_0, θ are the TEM_{00} mode values [Problem 7.21]. Thus higher order modes have a larger spot size and divergence, but, by definition, the Rayleigh range

$$z_{Rm} = \frac{d_{xm}(0)}{\theta_{xm}} = \frac{d_0}{\theta} = z_R$$

remains the same. One difficulty with this measure of higher order mode spot size is that most of the energy is outside of the rectangle of dimensions $d_{xm}(0)$ by $d_{yn}(0)$. A practical alternative definition for higher order mode beam dimensions is the distance between the outermost radiance peaks (e.g., $n = 10$ in Fig. 7.18).

PROBLEMS

7.1. If the beam waist diameter of a TEM_{00} output of an argon laser operating at the 514.5 nm line is 1.5 mm, what are the divergence and Rayleigh range of the beam?

7.2. Derive the stability condition for laser resonators, $0 < g_1 g_2 < 1$, based on the definitions of g_1 and g_2 given in the text and the requirement that $z_R^2 > 0$.

7.3. A laser has mirrors whose radii of curvature are 3.33 m and −1.5 m. If they are 1 m apart, where is the beam waist located and what is the Rayleigh range?

7.4. A dye laser operating in the TEM_{00} mode at 620 nm has a full angle beam divergence of 1.2 mrad. The beam is to be focused to a diameter of 32 μm. Assume the initial beam waist is at the output mirror of the laser.

(a) If a 32-mm focal length lens is chosen, where should it be located?

(b) If a lens *must* be positioned 160 mm from the beam waist of the laser, what focal length is needed to produce the 32 μm spot size and where would the beam waist be located?

7.5. A 100-μm beam waist must be positioned 7.5 m from a CO_2 laser ($\lambda = 10.6$ μm) which has a divergence of 10 mrad. Find the focal length of the lens needed to accomplish this. (Assume beam waist is at the output mirror.)

7.6. A helium–cadmium laser ($\lambda = 442$ nm) has a full angle divergence of 0.562 mrad. A positive lens is to be used to produce a 10 μm spot at a distance of 50 mm from the beam waist which is located at the output mirror of the laser. What is the focal length of the lens and where should it be placed?

7.7. Determine the relay geometry for two cases other than those given at the end of Section 7.2.3. (a) $z_2 = 3/2f$ and (b) $z_2 = f/2$. Express the distance between beam waists in terms of z_R.

7.8. Show that (a) the Rayleigh range is a maximum when $z_2 = f$ and that (b) a refocused Gaussian beam has a maximum lens–waist distance, z_1', when $z_2 = f + z_R$. Note that z_R is always positive.

7.9 Plot the beam waist position as a function with lens position along the beam axis in the vicinity of $z_2' = f_o + z_R'$ (collimation point). Show that the collimation point is located at $z_1'' = f_o + z_R''$.

7.10. What is the maximum collimation distance for an argon ion laser operating in the deep blue at 457.9 nm with a 2 mm beam diameter given a 10× telescope made up of a +5 mm lens and a +50 mm lens?

7.11. A Gaussian beam whose wavelength is 442 nm and divergence is 3 mrad has its divergence increased by a thick plano-convex lens whose flat surface 15 cm from the beam waist faces the input beam, whose second radius of curvature is −30 mm, whose refractive index is 1.61 and whose thickness is 7.5 mm. What are the new beam characteristics and where is the new beam waist located? Use a paraxial calculation. How do these compare to a thin lens ray trace for an equivalent lens?

7.12. A 10× microscope objective ($f = 16$ mm) is used to focus a helium–neon laser beam into the center of a 100-μm thick specimen ($n = 1.359$) mounted on a microscope slide of crown glass ($n = 1.517$). If a 0.5-mm thick cover glass (crown, $n = 1.517$) is laid on the specimen, how will this affect the focus? Use the paraxial form to determine the effect.

7.13. Compare the results (d_0, θ, z_R) of focusing an argon laser ($\lambda = 488$ nm) beam ($\theta = 1.2$ mrad) with the lens in Example 5.D as a thick lens using the paraxial treatment with that of a thin lens. The lens is 50 mm from the laser beam waist.

7.14. Using the expressions for d_1, d_2, R, and the conjugate relation $z_1 z_2 = z_R$, derive Eqs. 7.44 and 7.45.

7.15. Show that under the conditions of a strong focus: $z_R \ll z_1$, that the other conjugate distance z_2 is effectively located at the beam waist.

7.16. Using the equations presented in this chapter and the conjugate condition, $z'_2 z'_1 = z'^2_R$, show that for a beam refocused by a thin lens of focal length f, that the conjugate variables in terms of the original beam variables d_0, z_2, z_R are

$$z'_2 = \frac{f^3 z_R^2}{[(z_2 - f)^2 + z_R^2][z_2^2 + z_R^2 - f z_2]},$$

$$d'_2 = \frac{d_0^2 f^2 (z_2^2 + z_R^2)}{[z_2^2 + z_R^2 - f z_2]^2},$$

$$R' = \frac{f(z_2^2 + z_R^2)}{z_2^2 + z_R^2 - f z_2}.$$

7.17. You are contracted to produce a laser show at an amusement park. Unfortunately, the best area for exhibit is a large sheet of cloth on the side of the park's star attraction, a roller coaster called "The Great National Screech Machine." The management would like to know how small of a spot you are going to project onto the sheet. To produce the size of display you want, you have to be 250 feet from the surface. The largest scanning mirrors you can use without slowing down the display noticeably are 10-mm-diameter mirrors. If you have a 5-watt argon ion laser whose output beam diameter for $\lambda = 488$ nm is 1.2 mm, what is the smallest spot you can project and what is the irradiance at the center of the spot? Is it useful to use conjugate distances here?

7.18. A helium–neon laser ($\lambda = 633$ nm, $\theta = 0.8$ mrad) is collimated by a 10 power telescope to illuminate a circular scanning mirror. How large must the mirror be so as not to change significantly the Gaussian profile?

7.19. A 10 power microscope objective ($f = 16$ mm) is used to converge a laser beam to a spot and then diverge. Variations and deviations in the irradiance of the beam, profile causes light to be directed away from the central spot and into an area some distance from it. If a pinhole of the appropriate diameter is inserted at the beam waist, the Gaussian beam will be transmitted while light due to the non-Gaussian irradiance variation will be blocked leaving a smoothed Gaussian beam. This technique is called spatial filtering (Ref. 7.11). Assuming that a helium–neon laser ($\lambda = 633$ nm) has a 1 mrad divergence, what is the diameter of the smallest pinhole that can be used to filter the beam without introducing Airy rings into the pattern? Show, in general, that for a laser with a 1 mrad beam divergence, the product of the power of the lens ($\phi = 160$ mm$/f$) and the diameter of the pinhole in microns is approximately 250 micrometers. (The power of a microscope objective, as defined here, is a dimensionless quantity.)

7.20. Show that the compression in dimension of a beam diameter by a prism oriented at the Brewster angle θ_B is equal to the reciprocal of the refractive index of the prism.

7.21. How much larger is the beam area at a distance of 10 m from a laser that is tuned to TEM_{59} than one tuned to TEM_{00}?

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