Optical Scanning

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Gaussian Laser Beam Diameters and Divergence

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1.1 INTRODUCTION

Lasers are notably intense sources of electromagnetic radiation. A laser radiates at one or more wavelengths in the spectrum according to its design, but is essentially considered monochromatic. It emits a very narrow cone of radiation that is referred to as a laser beam (Figure 1.1). A typical helium-neon (HeNe) laser emits red light at a wavelength of 632.8 nm [25 μin], and has a nominal divergence of 1 mrad [3.44 min of arc] and a nominal diameter of 1 mm [39.4 thousandths of an inch, or mils]. Measuring the diameter and divergence of a laser beam may be likened to determining the width and spread of a smoke plume or the dimensions of a fog patch. Where do the boundaries begin and end? In this chapter the meaning and definitions of the diameter and divergence of a laser beam are addressed.

1.1.1 Laser Modes

Lasers operate, and emit beams, in different modes. The mode that shall be addressed is the simplest, and the one in which the beam has a gaussian wavefront [1, pp. 156–158], namely, the lowest-order transverse electromagnetic (TEM_{oo}) mode. Typically in this mode the characteristics of the beam are radially symmetric about its axis, which is the principal direction of propagation.

1.1.2 Gaussian Characteristics

At any perpendicular cross section of the beam the *irradiance* (power density) E distribution is gaussian. It is at a maximum E_0 on the axis and decreases with distance from the axis (Figure 1.1). The term *power density* is customarily used by laser scientists and engineers, instead of irradiance (Section 1.6.1). The word *density* connotes three dimensions, but in this context it is *areal* density, which denotes two dimensions. To assist the reader who is more familiar with the term *power density* rather than *irradiance*, the expression *power density* is included in parentheses.

1.1.3 Ambiguous Terms

There are several terms in vogue used to express and define the diameter of a laser beam. Some of these terms are misnomers, and some are ambiguous; to the uninitiated both are as misleading as the directions given by some road signs, unless one is thoroughly familiar with the territory. Interrelated with this confusion

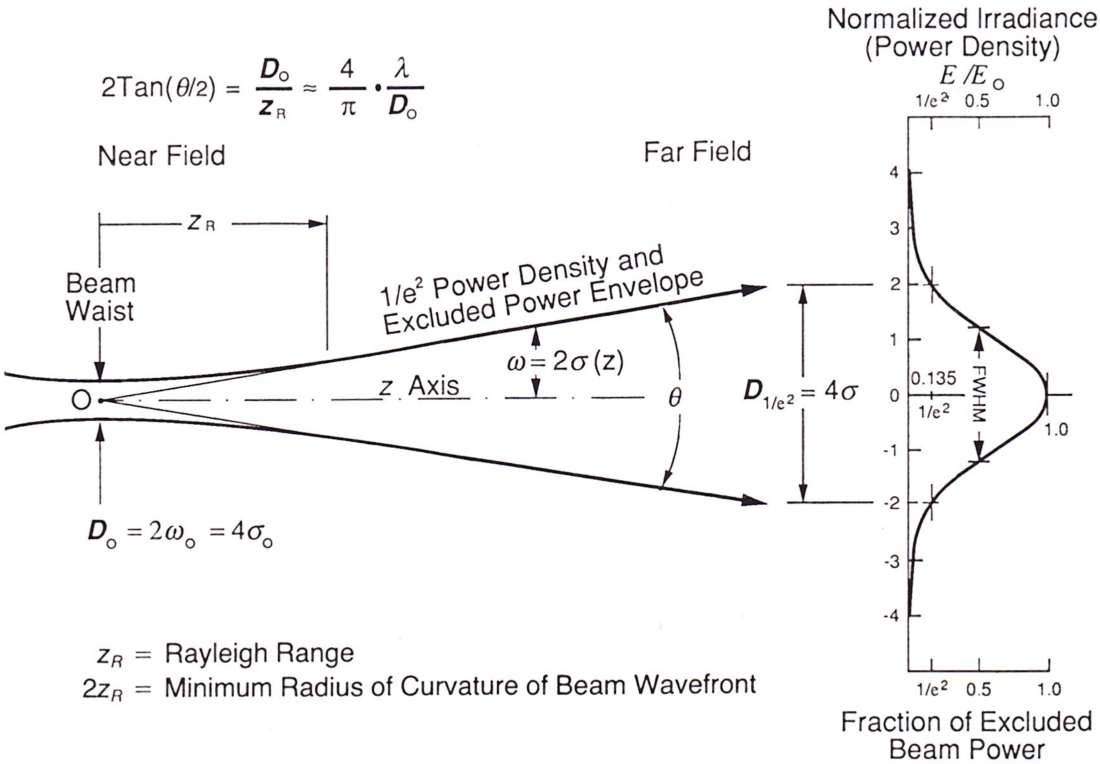


Figure 1.1 Gaussian laser beam divergence angle θ is the far-field full-cone angle of the $1/e^2$ (0.135) irradiance (power density) envelope. The apex of the conical envelope is at the beam waist. At any beam cross section the irradiance (power density) has a radially symmetric gaussian distribution profile, as depicted on the right.

in defining the diameter of a laser beam is the seeming misuse of the more accepted radiometric terminology [2, 3]. The terminology itself is seemingly always in a "state of flux" because it is frequently under revision [4, pp. 289–301]. The origin of these misnomers may be attributed to the cross fertilization of terminology from scientists and engineers with backgrounds of experience in other disciplines, wherein the terms are valid. To avoid misunderstandings, awareness and tolerance of these apparent inconsistencies are important.

1.1.4 Section Outlines

The definitions of beam diameters and divergence are deliberately presented in the first few sections so that readers will not have to wade through mathematics before reaching what they want to know at a glance.

1. The three most useful laser beam diameters and their definitions and an outline of the procedures for measuring beam diameters are presented in Section 1.2. See Table 1.1.
2. The beam divergence definition and an outline of the procedure for measuring beam divergence are given in Section 1.3.
3. The three types of angles used in beam divergence and radiometry are defined in Section 1.4.
4. The relevant CIE (Commission International de l'Eclairage) radiometric terms that are most widely accepted internationally, and which are used to develop the beam definitions, are reviewed and clarified in Section 1.5 and summarized in Table 1.3.
5. The mathematics and calculations that are used in the earlier sections to define the beam diameters and divergence are covered in Section 1.6.

1.2 BEAM DIAMETER DEFINITIONS

From Table 1.1 and Figure 1.2 several different radii may be chosen to define the diameter of a gaussian laser beam. The three most widely used definitions are given.

1.2.1 Full-Power Beam Diameter $D_{99.97\%}$

The *full-power beam diameter* may be effectively approximated to that diameter of the laser beam core that contains all but $1/e^8$ ($<0.034\%$) of the total beam power (Figure 1.2). See Section 1.2.3. It also corresponds identically to the distance between diametrically opposite points of an irradiance (power density) distribution profile in a section normal to the beam axis, at which the irradiance (power density) is $1/e^8$ ($<0.034\%$) of the axial irradiance (power density); see Figure 1.1.

Table 1.1 Irradiance, Excluded Power, Encircled Power, and Beam Diameter

Beam diameter in standard deviations σ_z $2k$	Exponential function $\exp[-k^2/2]$ =	Normalized irradiance (power density) (%) E/E_0 =	Excluded beam power fraction (%) $g(k) = 1 - h(k)$	Encircled beam power fraction (%) $h(k)$	Symbology for the beam diameter $D_{h(k)\%}$	$D_{1/e^{(k^2/2)}}$
0	$1/e^0$	100	100	0	Axis	
1.3490	$1/e^{0.228}$	79.65	79.65	20.35	$D_{20\%}$	
2	$1/e^{0.5}$	60.65	60.65	39.35	$D_{39\%}$	
$2\sqrt{2 \ln 2}$	$1/e^{0.693}$	50	50	50	$D_{50\%}$	D_{FWHM}
2.8102	$1/e^{0.987}$	37.26	37.26	62.74	$D_{63\%}$	
$2\sqrt{2}$	$1/e^1$	36.79	36.79	63.21	$D_{63\%}$	$D_{1/e}$
4	$\mathbf{1/e^2}$	**13.53**	**13.53**	**86.47**	$\mathbf{D_{86.5\%}}$	$\boldsymbol{D_{1/e^2}}$
6	$1/e^{4.5}$	1.11	1.11	98.89	$D_{99\%}$	
8	$1/e^8$	0.034	0.034	99.966	$D_{99.97\%}$	D_{1/e^8}
∞	$1/e^{\infty}$	0	0	100	$D_{100\%}$	

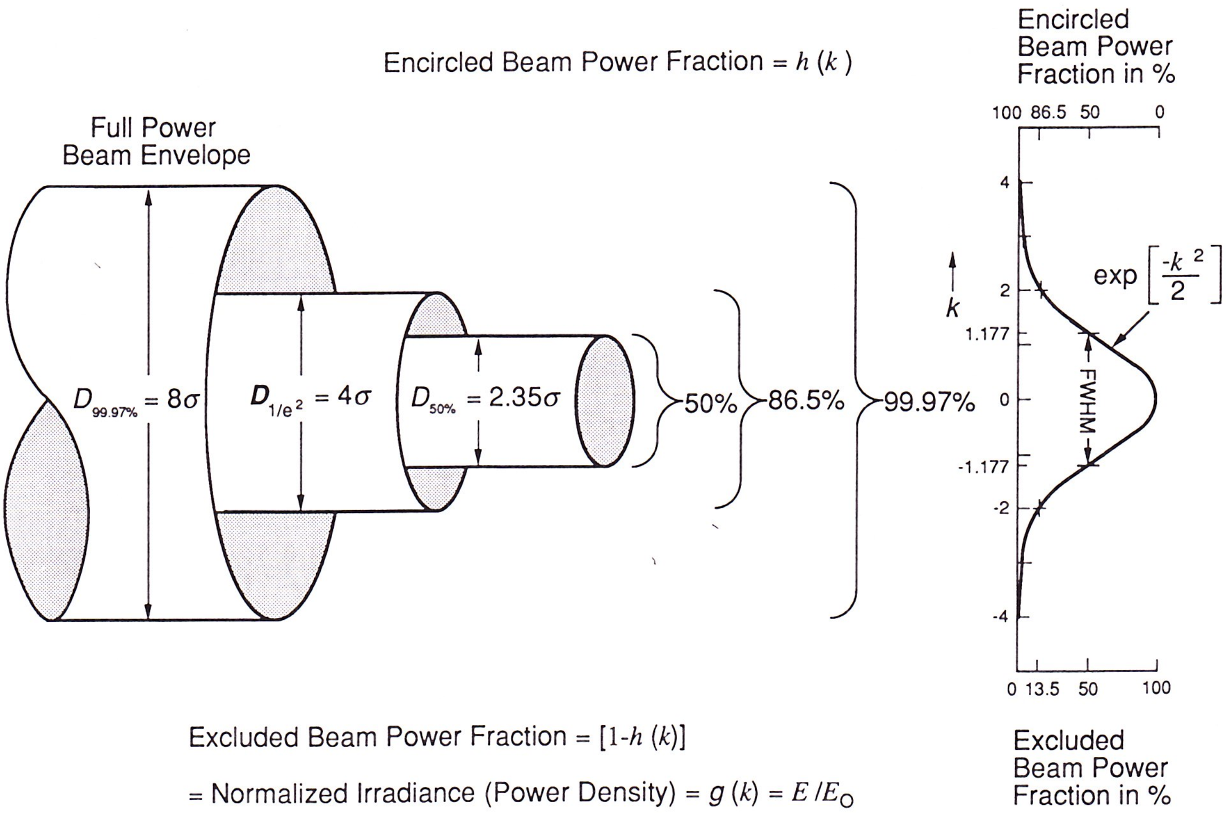


Figure 1.2 The normalized irradiance (power density) $g(k)$ value at a gaussian beam envelope identically equals the fraction of excluded beam power. For example, the $1/e^2$ (0.135) normalized beam irradiance (power density) envelope excludes the $1/e^2$ (0.135) fraction of the total beam power ϕ_∞. Therefore, the envelope must encircle 86.5% of the total beam power. Likewise, the 0.5 (50%) normalized beam irradiance (power density) envelope excludes and encircles 50% of the total beam power.

Let σ represent the standard deviation. If k is a multiplying factor that expresses the beam radius in units of the standard deviation σ, then for $k = 4$ (see Table 1.1) the beam diameter is expressed by

$$2k\sigma = 8\sigma = D_{1/e^8} = D_{\phi=99.97\%} \approx D_{\phi=100\%} \tag{1}$$

1.2.2 Half-Power Beam Diameter $D_{50\%}$

The *half-power beam diameter* is that diameter of the laser beam core that contains, and therefore also excludes, 50% of the total radiant beam power (Figure 1.2). It also corresponds identically to the distance between diametrically opposite points of an irradiance (power density) distribution profile in a section normal to the beam axis, and at which the irradiance (power density) is half (50%) of the axial irradiance (power density). Thus, this beam diameter is frequently called the *full-width at half maximum* (FWHM) diameter (Figure 1.1).

Let σ represent the standard deviation. If k is a multiplying factor that expresses the beam radius in units of the standard deviation σ, then for $k = 1.1774$ (see Table 1.1) the beam diameter is expressed by

$$2k\sigma = 2.355\sigma = D_{1/e^{0.693}} = D_{\phi = 50\%} \tag{2}$$

1.2.3 $1/e^2$ Excluded Power Beam Diameter *D*

The *$1/e^2$ excluded power beam diameter* is that diameter of the laser beam core that contains all but $1/e^2$ (13.5%) of the total radiant beam power (Figure 1.2). It also corresponds identically to the distance between diametrically opposite points of an irradiance (power density) distribution profile in a section normal to the beam axis, at which the irradiance (power density) is $1/e^2$ (13.5%) of the axial irradiance (power density) (Figure 1.1).

Let σ represent the standard deviation. If k is a multiplying factor that expresses the beam radius in units of the standard deviation σ, then for $k = 2$ (see Table 1.1) the beam diameter is expressed by

$$2k\sigma = 4\sigma = D_{1/e^2} = D_{\phi = 86.5\%} = \mathbf{D} \tag{3}$$

This $1/e^2$ excluded power beam diameter is one-half the effective full-power beam diameter D_{1/e^8}. Hence,

$$\mathbf{D} = \frac{D_{1/e^8}}{2} = \frac{D_{\phi = 99.97\%}}{2} \tag{4}$$

Unless otherwise stated, the $1/e^2$ excluded power beam diameter $\mathbf{D}$ is *the* diameter of a laser beam most generally understood when referring to the diameter of a laser beam. However, the expression is customarily contracted to and referred to as the *$1/e^2$ diameter*.

$\mathbf{D}_0$ represents the $1/e^2$ excluded power beam diameter at the beam waist, $z = 0$.

1.2.4 Measurement of Beam Diameters

The beam diameter may be determined in several different ways using suitably calibrated radiometric detectors (Figure 1.3). Alignment is important for all methods to obtain accurate results. There are a number of commercially available instruments that are designed to measure beam diameters and display beam irradiance (power density) profiles [5].

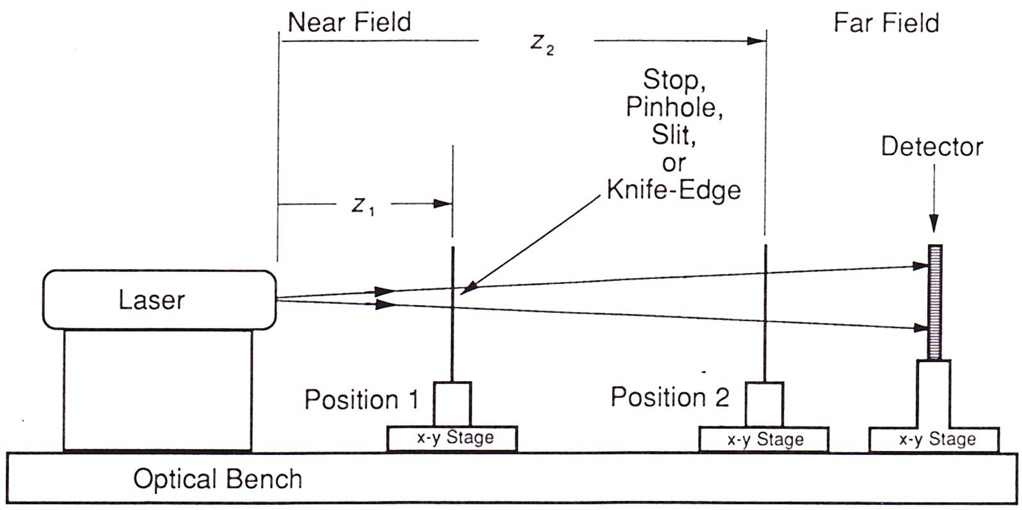


Figure 1.3 Test arrangement for measuring laser beam diameters and divergence by stop, pinhole, knife-edge, and slit methods.

Stops

Measure and plot the truncated beam power ϕ as a percentage of the total measured beam power against the diameter D_ϕ of a set of apertures as the beam power is stopped down to less than 50% (Figure 1.4). A calibrated iris may be used instead of aperture stops. Then

$$\boldsymbol{D} = D_{\phi=86.5\%} \approx 1.70 D_{\phi=50\%} \tag{5}$$

and

$$D_{50\%} = D_{\phi=50\%} = D_{\mathrm{FWHM}} \tag{6}$$

Pinhole

Measure and plot the transmitted beam power ϕ distribution as a percentage of the peak power against micrometer position displacement x of a pinhole aperture as it moves along a beam diameter (Figure 1.4). Then

$$\boldsymbol{D} = [(x_2)_{\phi=13.5\%} - (x_1)_{\phi=13.5\%}] \tag{7}$$

and

$$D_{50\%} = [(x_2)_{\phi=50\%} - (x_1)_{\phi=50\%}] = D_{\mathrm{FWHM}} \tag{8}$$

The pinhole diameter should be no more than about 10% of the expected beam diameter $\boldsymbol{D}$ being measured; see Figure 1.4.

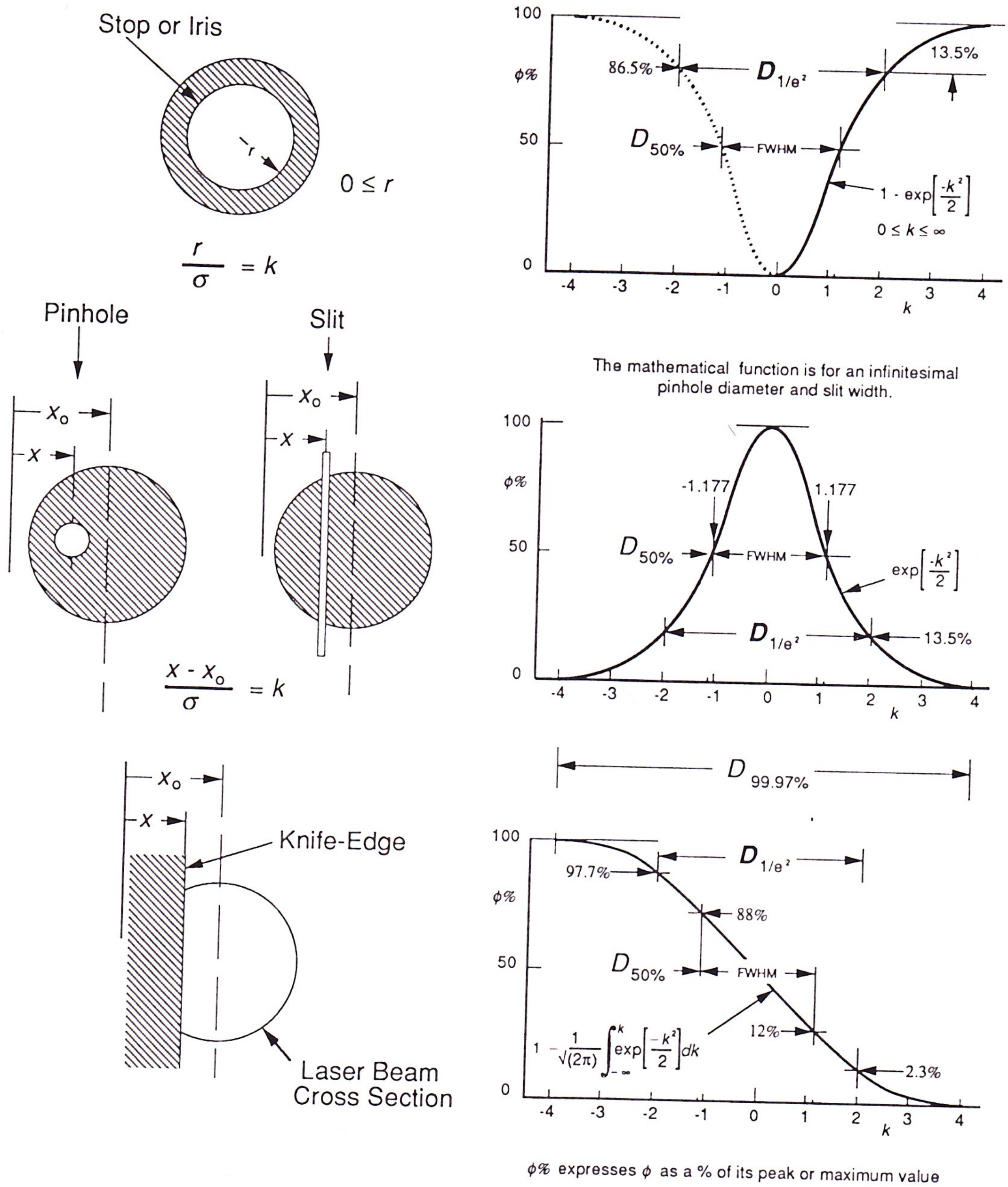


Figure 1.4 Stop, pinhole, knife-edge, and slit methods for measuring the beam diameters produce different radiant flux distribution characteristics.

Knife-Edge

Measure and plot the residual clipped beam power ϕ as a percentage of the total measured beam power against micrometer position displacement x of a knife-edge as it eclipses the beam. See Figure 1.4 and Table 1.2. Then

$$D = [(x_2)_{\phi=2.3\%} - (x_1)_{\phi=97.7\%}] = D_{1/e^2} \tag{9}$$

and

$$D = 2[(x_2)_{\phi=15.9\%} - (x_1)_{\phi=84.1\%}] = 2D_{1/e^{0.5}} \tag{10}$$

and

$$D_{50\%} = [(x_2)_{\phi=12\%} - (x_1)_{\phi=88\%}] = D_{\mathrm{FWHM}} \tag{11}$$

Table 1.2 Encircled Power Beam Diameter Determined from Beam Power Eclipsed by a Knife-edge

Radius	Diameter	Beam power fraction (%)			Symbology for the beam diameter	
in standard deviations σ		Uneclipsed position of knife-edge		Encircled diameter		
k	$2k$	x_1	x_2	$(x_2 - x_1)$	$D_{h(k)\%}$	$D_{1/e^{(k^2/2)}}$
0	0	50	50	0	Axis	
0.6745	1.3490	75	25	20.35	$D_{20\%}$	
1	2	84.134	15.866	39.35	$D_{39\%}$	
$\sqrt{2 \ln 2}$	2.3548	88.048	11.952	50	$D_{50\%}$	D_{FWHM}
1.4051	2.8102	92	8	62.74	$D_{63\%}$	
$\sqrt{2}$	$2\sqrt{2}$	92.135	7.865	63.21	$D_{63\%}$	$D_{1/e}$
2	**4**	**97.725**	**2.275**	**86.47**	$\boldsymbol{D_{86.5\%}}$	$\boldsymbol{D_{1/e^2}}$
3	6	99.865	0.135	98.89	$D_{99\%}$	
4	8	99.997	0.003	99.966	$D_{99.97\%}$	D_{1/e^8}
∞	∞	100	0	100	$D_{100\%}$	

Slit

The slit is equivalent to two rigidly coupled knife-edges. Measure and plot the transmitted beam power ϕ distribution against micrometer displacement x of a narrow parallel slit as it scans through the beam. Then

$$D = [(x_2)_{\phi=13.5\%} - (x_1)_{\phi=13.5\%}] = D_{1/e^2} \tag{12}$$

and

$$D_{50\%} = [(x_2)_{\phi=50\%} - (x_1)_{\phi=50\%}] = D_{\mathrm{FWHM}} \tag{13}$$

The width of the slit should be less than 20% of the expected beam diameter $\boldsymbol{D}$ being measured in order to keep the convolution error below 2% [6], and a length of at least twice $\boldsymbol{D}$.

Linear Array

Measure and display the beam power density distribution along a beam diameter incident on a suitably calibrated detector array. The pixel elements of the array need to be small compared with the diameter being measured, and the array should have a length no less than twice the expected beam diameter D being measured.

Two-Dimensional Array

Measure and display the beam power density distributions of a beam cross section incident on a suitably calibrated two-dimensional detector array, say 32×32. The pixel elements of the array need to be small compared with the cross-sectional area of the beam being measured, and the array should have an area containing no less than twice the expected beam diameter D being measured.

1.2.5 Measurement Sensitivity

From each of these methods data are obtained that have to be evaluated and analyzed to determine the desired beam diameter. Although the $1/e^2$ beam diameter D is desirable, in practice it is not easy to accurately determine the associated 13.5% of excluded beam power or irradiance (power density). This is because of "noise" and because of the slow rate of change in irradiance (power density) with distance along the tails of the gaussian distribution characteristic (Figures 1.2 and 1.4).

Half-Power Beam Diameter

By far the simplest is the determination of the half-power beam diameter $D_{50\%}$, FWHM beam diameter. This is for three good reasons, namely:

1. The 50% encircled power diameter or the FWHM beam diameter can be read almost directly by the stops, pinhole, and slit methods given in preceding subsections as can, in principle, the $1/e^2$ beam diameter $\boldsymbol{D}$.
2. At the half-power positions in the gaussian characteristic profile the rate of change in irradiance (power density) along a diameter is close to its maximum rate of change, and thereby provides a high sensitivity for measurement (Figures 1.2 and 1.4).
3. For diameters $D_{50\%}$ and D there is, respectively, 50% and 13.5% of the excluded beam power, or irradiance, respectively, in the two tails of the generated gaussian characteristic that is being clipped (Figure 1.2). The residual 13.5% of the power in the tails, which has to be detected and clipped, is closer to being comparable to the percentage level of experimental error of measurement.

The method in the subsection on knife-edges requires careful analysis to determine the half-power beam diameter $D_{50\%}$ (D_{FWHM}).

$1/e^2$ Excluded Power Beam Diameter

All methods require careful alignment, measurement, calibration, and rigorous analysis to determine the $1/e^2$ beam diameter $\boldsymbol{D}$ because of the relative measurement insensitivity mentioned in the preceding subsection.

Knife-edge Method. In the knife-edge method, which progressively truncates segments from the entire beam, note that the percentage power clip levels, 2.3% and 97.5% for $\boldsymbol{D}$, are markedly different to the 13.5% irradiance (power density) level for $\boldsymbol{D}$ when using the stop and pinhole methods. See Figure 1.4, Tables 1.1 and 1.2. Likewise, note that the percentage power clip levels for a beam diameter $D_{50\%}$ are 12% and 88%.

Similarly, the percentage clip levels for a beam diameter $D_{1/e^{0.5}}$, which corresponds to a beamwidth of 2σ, are approximately 16% and 84%. See equation (10), Figure 1.4, and Table 1.2.

$$\boldsymbol{D} = 2D_{1/e^{0.5}} \tag{14}$$

Slit Method. In the slit method, which progressively apertures long narrow samples of flux across the entire beam, the resultant measured distribution is also gaussian. This is because of the unique mathematical properties of the gaussian function. Thus the percentage clip level is also at 13.5% of the peak power level, as in the case of the pinhole method.

1.3 BEAM DIVERGENCE DEFINITION

The divergence θ of a gaussian laser beam is the plane angle (Section 1.4.2.), expressed in milliradians (mrad), that the $1/e^2$ beam diameter $\boldsymbol{D}$ subtends at the laser. This definition presumes that the beam is continuously diverging and that the beam waist is located at or near the laser output port. Furthermore, it assumes that the beam is uninterrupted by optical elements and that the diameter is measured at a distance far away from the laser, in the far-field of the beam $z > 3z_R$, where z_R represents the Raleigh range.

1.3.1 Rayleigh Range z_R

The *Rayleigh range* z_R is the distance to that position on the beam axis at which the beam wavefront has a minimum radius of curvature $R_{\min}$ on either side of the beam waist. $R_{\min}$ equals twice the Rayleigh range z_R; also $D_R = \sqrt{2}\boldsymbol{D}_0$. Also,

$$z_R \approx \frac{\pi \boldsymbol{D}_0^{\,2}}{4\lambda} \tag{15}$$

1.3.2 Calculation of Divergence θ

The divergence θ (in mrad) is given by

$$2\,\mathrm{Tan}\left(\frac{\theta}{2}\right) = \frac{D_0}{z_R} \approx \frac{4\lambda}{\pi D_0} \tag{16}$$

For small angles, say $\theta < 10°$, the divergence (in mrad) is $4/\pi$ times the laser beam output wavelength λ (in nm) divided by the nominal beam diameter D_0 (in mm), at the laser output port where the beam waist is presumed to be. Thus

$$\theta \approx \frac{D_0}{z_R} \approx \frac{4\lambda}{\pi D_0} \tag{17}$$

1.3.3 Measurement of Beam Divergence θ

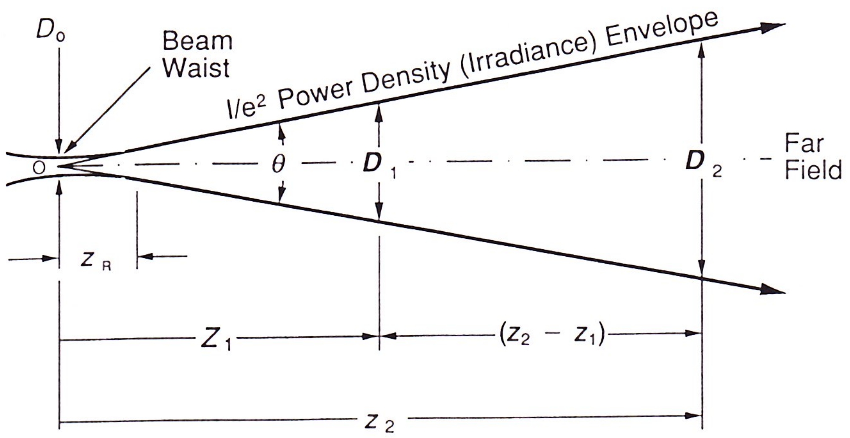
The beam divergence may be determined by measuring the beam diameter D at positions z_1 and z_2 in the uninterrupted beam, and at positions at which the distance z is at least three times greater than the Rayleigh distance z_R. The distance $z_1 - z_2$ between the positions in the far field of the beam should be chosen such that one diameter is at least twice the other diameter. The divergence is then the arcsine of the difference in diameters divided by the distance between the positions (Figure 1.5):

$$\theta = \mathrm{Arcsin}\left[\frac{D_2 - D_1}{z_2 - z_1}\right] \times 1000\ \text{(mrad)} \tag{18}$$

1.3.4 Beam Quality

As stated in Section 1.1, this chapter deals only with a laser beam that has a symmetrical spherical gaussian wavefront of the lowest-order TEM_{oo} mode. In practice, beams are not likely to be pure gaussian. Real beams are most likely to contain modes of higher orders that reduce the quality of the beam with respect to characteristics of a pure gaussian beam. One such dimensionless measure of beam quality is the quantity M^2 [7].

The M^2 value of a beam compares the real beam characteristics to that of a theoretically pure gaussian beam of the type that is being discussed in this chapter. The pure gaussian beam is chosen to have the same waist diameter at the same position. An M^2 value of 1.0 indicates perfect correlation with a pure gaussian beam. Larger values of M^2 suggest the presence of higher-order modes and provide a measure of the increase in the angle of divergence over that of a pure gaussian beam with the same waist diameter [8].



$z_1 >> z_R$ $\qquad$ $D_2 > 2D_1$

z_R = Rayleigh Range

$$\theta = \frac{D_2 - D_1}{z_2 - z_1} \times 1{,}000 \text{ mr}$$

Figure 1.5 The beam divergence may be determined by measuring the beam diameter D in the far field at distances z_2 and z_1 from the beam waist in an uninterrupted beam. The ratio of z_2 and z_1, and therefore the beam diameters D_2 and D_1, should be at least 2.

1.4 ANGLES

Several angles are referred to in the course of defining divergence and radiometric terms. These are (1) the angle of view, (2) plane angle, and (3) solid angle.

1.4.1 Angle of View α

The *angle of view* is the angle between a specified direction and the normal to the emitting surface.

1.4.2 Plane Angle β

Consider the plane angle that a circular arc subtends at its center of curvature. The plane angle in circular measure is the ratio of the length of the arc to its radius of curvature (Figure 1.6).

$$\beta_{\text{rad}} = \frac{\text{Length of the arc}}{\text{Radius of curvature of the arc}} \tag{19}$$

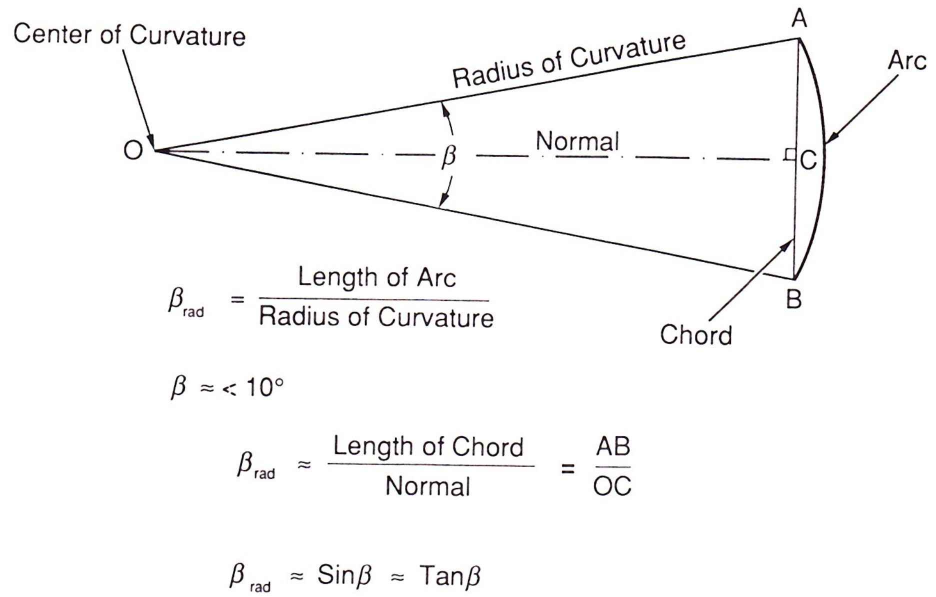


Figure 1.6 Plane angle β defined as a ratio expressed in radians.

The unit in circular measure of a plane angle is the radian, and is dimensionless. A plane angle β_{deg} expressed in degrees is converted to a plane angle β_{rad} expressed in radians by the following relationship:

$$\beta_{rad} = \frac{\pi}{180}\beta_{deg} \tag{20}$$

For small angles, say to $\beta \approx 10°$ (175 mrad),

$$\beta_{rad} \approx \text{Sin}\ \beta \approx \text{Tan}\ \beta \tag{21}$$

When $\beta = 10°$ (175 mrad), the approximation $\beta_{rad} \approx \text{Sin}\ \beta$ in equation (21) leads to a lower value of β_{rad} that is in error by about 0.5%.

Correspondingly, when $\beta = 10°$ (175 mrad), the approximation $\beta_{rad} \approx \text{Tan}\ \beta$ in equation (21) leads to a higher value of β_{rad} that is in error by about 1.0%. Likewise, when β is small, the length of the arc approaches the length of the chord of the arc, that is, the projected length of the arc. Thus, for small plane angles,

$$\beta_{rad} = \frac{\text{Length of the chord of an arc}}{\text{Distance to the chord}} \tag{22}$$

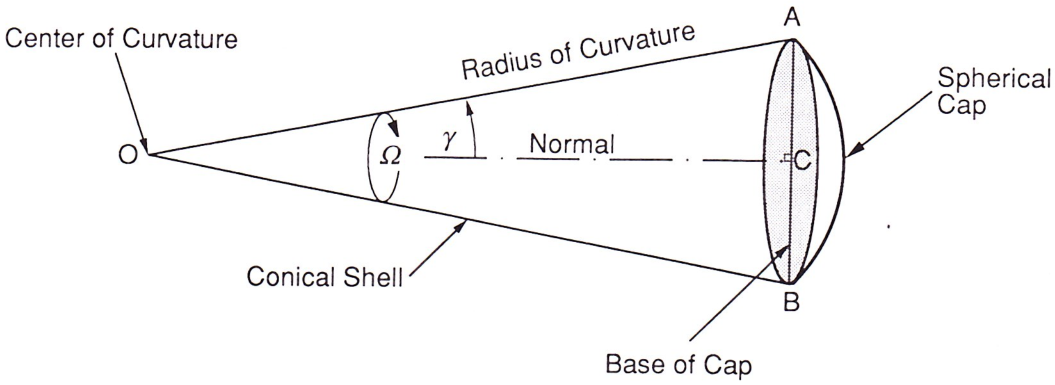
As a reference, the diameters of the Moon and the Sun both subtend at the Earth's surface an angle of approximately 1/2°, which corresponds to 8.7 mrad. Thus a laser beam with a divergence of 1 mrad (3′ 26″) and the beam directed at such a distant object as the Moon would irradiate (illuminate) an area on the Moon's surface that has a diameter one-ninth that of the Moon.

1.4.3 Solid Angle, Ω

Inherent in the definitions of radiometric terms is the concept of solid angle Ω. A solid angle is the three-dimensional counterpart of the two-dimensional plane angle. Instead of the ratio of lengths in a plane, it is the ratio of areas in a space that is determined (Figure 1.7).

Consider the solid angle that a spherical cap subtends at its center of curvature. The solid angle in circular measure is the ratio of the surface area of the cap to its radius of curvature squared (Figure 1.7):

$$\Omega = \frac{\text{Surface area of the cap}}{\text{Radius of curvature squared}} \tag{23}$$



$$\Omega = \frac{\text{Surface Area of Cap}}{(\text{Radius of Curvature})^2} = 2\pi(1 - \text{Cos}\ \gamma) = 4\pi\ \text{Sin}^2(\gamma/2)$$

$$\gamma \approx < 10°, \quad \Omega \approx \frac{\text{Area of Base}}{(\text{Normal})^2} = \frac{\pi}{4}\left[\frac{AB}{OC}\right]^2$$

Figure 1.7 Solid angle Ω defined as a ratio expressed in steradians.

The unit in circular measure of a solid angle is the steradian, and is dimensionless, as is the radian for the circular measure of a plane angle.

A spherical cap also subtends a regular cone at its center of curvature. If the full cone angle is 2γ, then the solid angle subtended at the apex, the center of curvature of the cap, is given by

$$\Omega = 2\pi(1 - \text{Cos}\,\gamma) = 4\pi\,\text{Sin}^2\left(\frac{\gamma}{2}\right) \tag{24}$$

For small angles, say to $\gamma = 10°$, $\Omega = 95.5$ mstr.

$$\Omega \approx \pi\,\text{Sin}^2\,\gamma \approx \pi\,\text{Tan}^2\,\gamma \tag{25}$$

When $\gamma = 10°$ (175 mrad), the approximation $\Omega \approx \pi\,\text{Sin}^2\,\gamma$ in equation (25) leads to a lower value of Ω that is in error by about 0.77%. Correspondingly, when $\gamma = 10°$ (175 mrad), the approximation $\Omega \approx \pi\,\text{Tan}^2\,\gamma$ in equation (25) leads to a higher value of Ω that is in error by about 2.25%. Likewise, when γ is small, the surface area of the cap approximates the area of the base of the cap, which is the projected area of the cap or the apparent area as seen by the eye at the viewing distance at the center of curvature. Thus, for small solid angles,

$$\Omega = \frac{\text{Projected area of an object}}{\text{Distance to the object squared}} \tag{26}$$

Equation (26) corresponds to $\Omega \approx \pi\,\text{Tan}^2\,\gamma$, given in equation (25).

As a guideline reference, the Moon's disk has a diameter that subtends approximately 1/2° (8.7 mrad) at the Earth's surface. Substituting $\gamma = 0.25°$ (4.36 mrad) into equation (23) leads to the Moon's disk subtending a solid angle of 59.8 μstr at the Earth's surface. The whole visible sky, a celestial hemisphere, subtends 2π radians at the Earth's surface.

1.5 RADIOMETRIC TERMINOLOGY

Radiometry is the technology surrounding the measurement of electromagnetic radiant power (watts) and energy (joules) across the entire spectrum of wavelengths. Photometry, which involves the sensitivity of the eye and has its own units, is that part of radiometry that deals only with the visible portion of the electromagnetic spectrum ranging approximately from 400 to 720 nm [15.8 to 28.4 μin] [9, p. 189]. In this wavelength band, because of the spectral response characteristic of the eye, the conversion factor between photometric and radiometric units has a nonlinear relationship with wavelength [1, pp. 45–46]. Since radiometry covers the whole spectrum, the subject shall be discussed in the radiometric terms, and for reference the photometric terms are included in parentheses, where appropriate. A summary of the radiometric terms is presented in Table 1.3.

Table 1.3 Summary of Relevant Radiometric Terms

Term	Symbol	Unit
Radiant flux Radiant power	ϕ	Watts
Radiance Radiant power emitted per unit solid angle in a specified direction θ, per unit projected area in that specified direction, of a source, a receiver, or an intermediate intersecting reference surface.	L_α	Watts/meter^2/steradian
Radiant intensity Radiant power emitted per unit solid angle in a specified direction α from a source.	I_α	Watts/steradian
Irradiance Radiant power incident per unit area at a surface or at an intermediate intersecting reference surface.	E	Watts/meter^2

1.5.1 Radiometric Terms

The important radiometric properties to consider are those associated with

1. Radiation emanating from a source
2. Radiation traversing a medium
3. Radiation incident at a surface

The CIE terms to review are given in the following.

1.5.2 Radiant (Luminous) Flux ϕ

The term *radiant flux*, symbolized by ϕ, represents the radiant power in a beam expressed in watts, and is often simply referred to as flux (Figure 1.8). The term

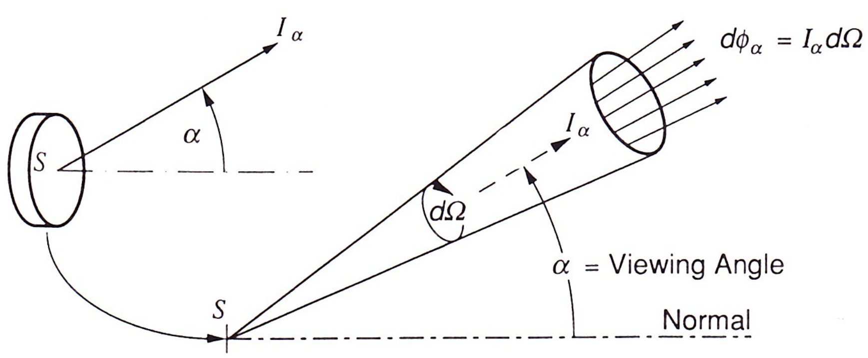


Figure 1.8 Viewing angle α and radiant flux ϕ.

luminous flux represents the luminious power in a beam expressed in lumens, the photometric unit. Provided that there is no ambiguity with radiant flux, these terms are simply abbreviated to flux.

There is no simple relationship between photometric and radiometric units, but luminance flux ϕ_v and spectral radiant flux $\phi_{e\lambda}$ may be related as follows:

$$\phi_v = \int_{380\text{ nm}}^{780\text{ nm}} \phi_{e\lambda} V(\lambda)\, d\lambda \tag{27}$$

where $V(\lambda)$ is the spectral luminous efficiency function expressed in lumens per watts [10, 11]. Beyond each side of the visible waveband, into the invisible ultraviolet and infrared regions, $V(\lambda) = 0$ and, therefore, no lumens are contributed to ϕ_v in equation (27).

1.5.3 Radiance (Luminance) L_α

Radiance, symbolized by $\boldsymbol{L_\alpha}$, corresponds nonlinearly to the photometric term *luminance* and the subjective term *brightness* that is perceived with the eye in the visible region of the spectrum. Radiance is a function of the angle of view α and is therefore a vector quantity (Figure 1.9). Radiance (luminance) $\boldsymbol{L_\alpha}$ is also an invariant parameter for the conservation of flux ϕ within a bundle of rays that emanate from a small source object of finite area as it traverses an optical system of nonabsorbing media [9, pp. 104, 105].

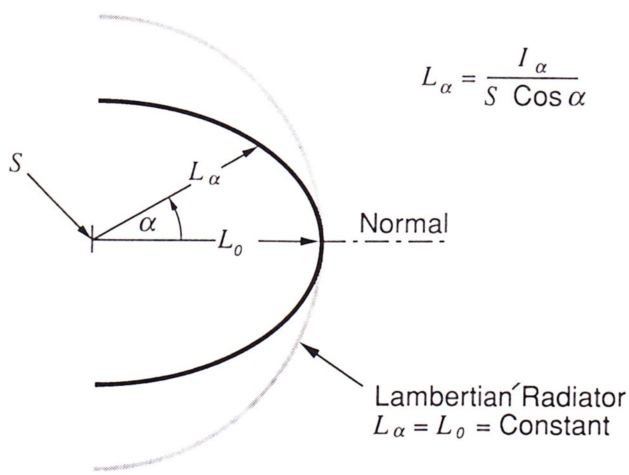


Figure 1.9 Radiance L_α. A lambertian radiator has the same radiance L_o in all directions and therefore is independent of the viewing angle α.

1.5.4 Radiant (Luminous) Intensity I_α

The term *radiant intensity*, symbolized by $\boldsymbol{I_\alpha}$, and the term *radiance* ($\boldsymbol{L_\alpha}$) both relate to the radiant properties of a source and, therefore, to one another by

$$(\text{Radiant intensity})_\alpha = \iint_S (\text{Radiance}_\alpha \text{ Cos } \alpha)\, dS \tag{28}$$

$$I_\alpha = \iint_S (L_\alpha \text{ Cos } \alpha)\, dS \tag{29}$$

where dS is an elementary area of a finite source of area S and α represents the angular direction from the normal (Figure 1.10).

1.5.5 Lambertian Radiator

The surface of an isotropically diffuse radiator in the form of a uniformly radiating luminous disk will look equally bright from any angle of view α from the normal to the disk. That means L_α is a constant and, therefore, equals L_0. Such a uniform radiator is referred to as a *Lambertian radiator* or *emitter* (Figures 1.8 and 19). Then because L_α = constant = L_0 for a Lambertian radiator, from equation (29),

$$I_\alpha = L_0 S \text{ Cos } \alpha \tag{30}$$

$$I_\alpha = L_0(\text{projected area of the disk}) \tag{31}$$

The projected area (S Cos α) of the disk is also the apparent area of the disk that is seen by the eye from a viewing direction α.

When $\alpha = 0$,

$$I_0 = L_0 S \tag{32}$$

Substituting into equation (31) leads to

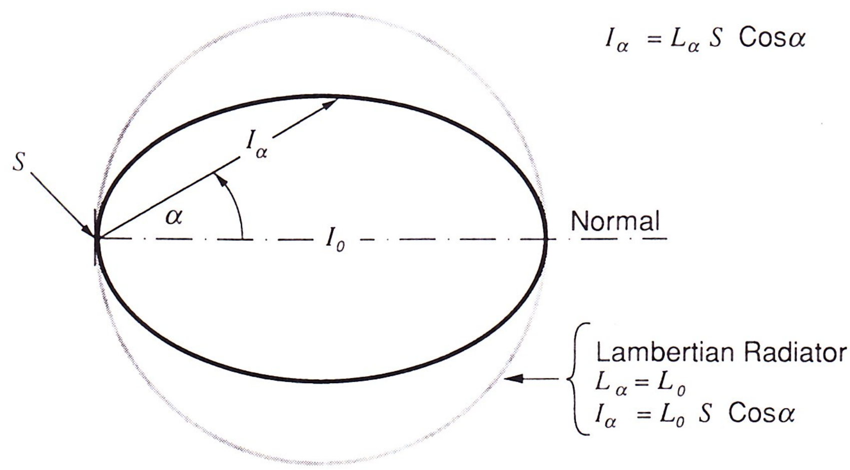


Figure 1.10 Radiant intensity I_α. a lambertian radiator has a radiant intensity that is a cosine function (I_o Cos α) of the viewing angle α.

$$I_\alpha = I_0 \operatorname{Cos} \alpha \tag{33}$$

Thus, from equation (33), the radiant intensity I_α of a Lambertian radiator decreases according to the cosine of the angle of view α. Hence, radiators and reflective surfaces are said to be Lambertian when their radiant intensity distributions with respect to the angle of view closely approximate a cosine function (Figure 1.10).

1.5.6 Irradiance *E*

Irradiance, symbolized by E, is the radiant *power density*—that is, the radiant power per unit area associated with a bundle of light rays as it traverses a plane or is incident on a plane (Figure 1.11).

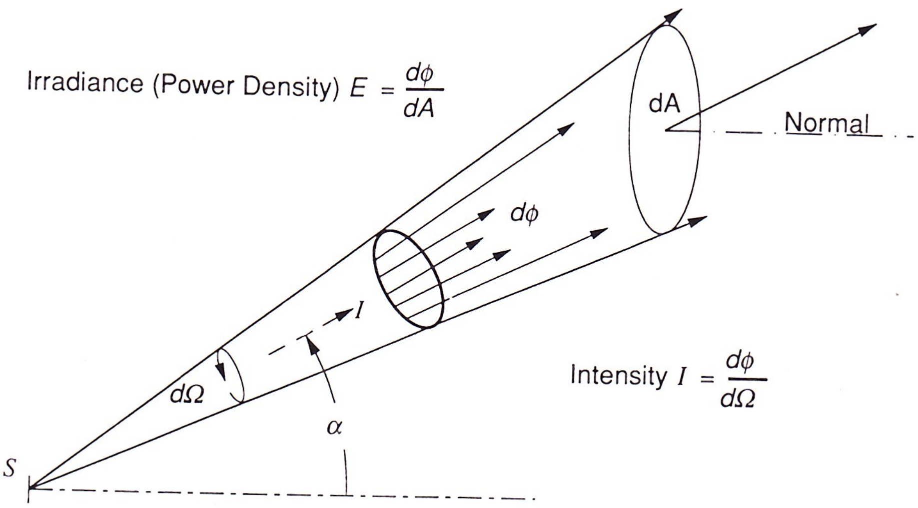


Figure 1.11 Irradiance E is the radiant power per unit area associated with a bundle of light rays as it traverses a plane or is incident on a plane.

1.6 GAUSSIAN BEAM

With respect to cylindrical coordinates, let the direction of propagation of a laser beam be collinear with the z-axis with the origin O at the beam waist (Figure 1.12). for a radially symmetric beam, all points P with radial coordinate r are expressed by $P(r,z)$.

Consider any perpendicular cross section of the beam through the axial point $Q(0,z)$. The radial distribution of radiant flux density traversing this plane is gaussian with a standard deviation σ. Because the beam is either diverging or converging, σ is a function of z; therefore

$$\sigma = \sigma(z) = \sigma_z \tag{34}$$

This is expressed by

$$\sigma(z) = \sigma_0 \left[1 + \left(\frac{\lambda z}{4\,\sigma_0^2} \right)^2 \right]^{1/2} \tag{35}$$

where λ = wavelength of the radiation

σ_0 = beam waist standard deviation for $z = 0$.

σ_z = beam standard deviation at a distance z.

Figure 1.12 shows a laser beam envelope for which $r(k,z) = k\sigma(z)$ and in which k is a chosen constant. Since r is always a function of z, it shall henceforth be dispensed with as a subscript. Thus $r_k = k\sigma_z$.

Hence, the beam envelope diameter $D = 2r_k = 2k\sigma_z$ at any beam section may be defined by a multiple ($k > 1$) or a fraction ($k < 1$) of the standard deviation σ_z of the gaussian beam power density distribution. In other words, the laser beam diameter may be normalized and expressed in terms of the standard deviation σ_z at any cross section by choosing a value for k. It is now the selected value of k, $k = 1, 2, \ldots$, that defines the threshold criterion for the beam boundaries. Hence, the diameter of a laser beam at any cross section is in units of standard deviation σ_z.

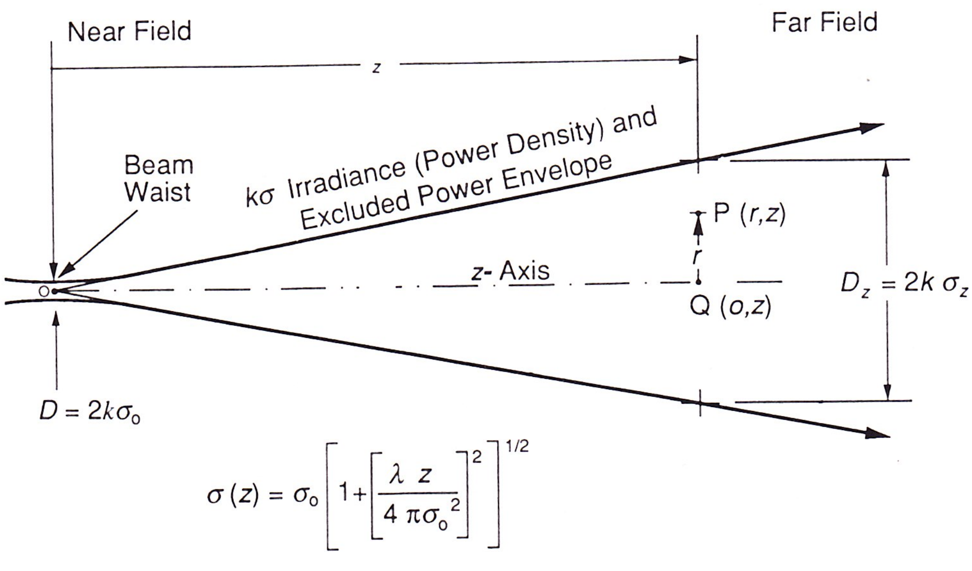


Figure 1.12 Beam diameter D is expressed in multiples $2k$ of the standard deviation $\sigma(z)$ of the irradiance (power density) E distribution profile.

1.6.1 Beam Irradiance

The radiant flux $\Delta\phi$ in a beam can be intercepted and explored with a radiometric detector that has a minute active area ΔA compared with the cross-sectional area of the laser beam, for example, at the point $P(r,z)$ (Figure 1.12). By the definition already given in Section 1.5.6 and Table 1.3, the quotient

$$\frac{\Delta\phi}{\Delta A} = \frac{d\phi}{dA} \tag{36}$$

represents the irradiance $E(r,z)$ incident on the detector surface and, therefore, the irradiance of the beam at all points $P(r,z)$. The three-dimensional radially symmetric gaussian distribution of $E(r,z)$ in the perpendicular plane through the point $Q(0,z)$ resembles a Mexican sombrero with a flattened brim [12, p. 295, Figure 2]. Mathematically it is expressed by

$$E(r,z) = E(0,z)\exp\left[\frac{-(r/\sigma_z)^2}{2}\right] \tag{37}$$

in which $E(0,z)$ represents the axial and peak beam irradiance. Dividing equation (37) by $E(0,z)$ leads to equation (38), which expresses the irradiance $g(k)$ that is normalized to the peak irradiance and shown in Figure 1.1.

$$g\left(\frac{r}{\sigma_z}\right) = \frac{E(r,z)}{E(0,z)} = \exp\left[\frac{-(r/\sigma_z)^2}{2}\right] \tag{38}$$

Let $r = k\sigma_z = r_k$ and substitute into equation (38), to give

$$g(k) = \frac{E(r_k,z)}{E(0,z)} = \exp\left(\frac{-k^2}{2}\right) \tag{39}$$

Understandably, because of the interdisciplinary influences, other terms and symbology than the acknowledged and preferred radiometric term for irradiance (power density) and its appropriate symbol E are sometimes heard and written, such as intensity with its symbol I [13]. See Section 1.1.3.

1.6.2 Beam Diameter Criteria

For a diameter $D = 2r_k$ of a laser beam, any value of k may be chosen. As the beam diverges from the beam waist or converges toward a beam waist, the beam diameter increases or decreases, respectively, but the criteria for defining the beam diameter remain the same. The important criteria for selecting the value of k for the beam envelope diameter are likely to be governed by

1. Encircled beam power, that is, the radiant beam flux ϕ_k contained within the beam envelope of relative diameter k.
2. Total beam power, that is, the total radiant beam flux ϕ_∞, $k = \infty$.
3. Fraction $h(k)$ of the encircled radiant flux ϕ_k.
4. Fraction $1 - h(k)$ of the excluded radiant flux $\phi_\infty - \phi_k$.
5. Irradiance (power density) variation $\Delta E(r,z)$ across a chosen beam diameter.

1.6.3 Beam Power ϕ_k

The total beam power is determined by substituting $k = \infty$. Then the total beam power (total radiant flux) is expressed by ϕ_∞. For any chosen beam envelope that is determined by a k value, the sum of the encircled beam power ϕ_k and the excluded beam power $\phi_\infty - \phi_k$ must equal the total beam power ϕ_∞. Both the encircled and the excluded beam powers are fractions of the total beam power.

1.6.4 Encircled Beam Power Fraction *h*(*k*)

The *encircled beam power fraction* $h(k)$ of the encircled radiant flux ϕ_k with respect to the total radiant flux ϕ_∞ is given by

$$h(k) = \frac{\phi_k}{\phi_\infty} \equiv \frac{\text{Encircled beam power}}{\text{Total beam power}} \tag{40}$$

See Figure 1.2.

1.6.5 Excluded Beam Power Fraction 1 − *h*(*k*)

The complementary *excluded beam power fraction* $1 - h(k)$ of the excluded radiant flux $\phi_\infty - \phi_k$ with respect to the total radiant flux ϕ_∞ is given by

$$1 - h(k) = \frac{\phi_\infty - \phi_k}{\phi_\infty} \equiv \frac{\text{Excluded beam power}}{\text{Total beam power}} \tag{41}$$

It will now be shown that, for a gaussian beam, the normalized irradiance $g(k)$ given in equation (39) also exactly corresponds to the excluded fraction $1 - h(k)$ of the beam power, such that the righthand sides of equations (39) and (53) are identical; thus

$$g(k) \equiv 1 - h(k) \tag{42}$$

1.6.6 Encircled Beam Power Derivation

Consider the radiant flux $d[\phi(r)]$ encircled between two envelopes of radii r and $r + dr$ at any plane section such as that through the axial point $Q(0,z)$:

$$d[\phi(r)] = 2\pi r\, dr\, E(r,z) \tag{43}$$

Since

$$2\pi r\, dr \equiv \pi d(r^2) \tag{44}$$

equation (43) then becomes

$$d[\phi(r)] = \pi E(r,z)\, d(r^2) \tag{45}$$

Substituting for $E(r,z)$ from equation (37) gives

$$d[\phi(r)] = \pi E(0,z)\left\{\exp\left[\frac{-(r/\sigma_z)^2}{2}\right]\right\} d(r^2) \tag{46}$$

Integrating from $r = 0$ to $r = r_k = k\sigma_z$ gives

$$\int_0^{\phi_k} d[\phi(r)] = \pi E(0,z) \int_0^{r_k} \left\{\exp\left[\frac{-(r/\sigma_z)^2}{2}\right]\right\} d(r^2) \tag{47}$$

$$[\phi(k\sigma_z)]_0^{\phi_k} = \pi E(0,z)\left[\exp\left[\frac{-(r/\sigma_z)^2}{2}\right]\right]_0^{r_k} \tag{48}$$

Substituting $r_k = k\sigma_z$ gives

$$\phi_k = \pi(\sqrt{2}\sigma_z)^2\, E(0,z)\left[1 - \exp\left(\frac{-k^2}{2}\right)\right] \tag{49}$$

in which ϕ_k represents the encircled radiant flux for a beam envelope of radius r_k.

The total radiant flux ϕ_∞, that is, the total beam power, is obtained by substituting $k = \infty$ into equation (49). This leads to

$$\phi_\infty = \pi(\sqrt{2}\sigma_z)^2\, E(0,z) \tag{50}$$

Dividing equation (49) by equation (50) gives the fraction of encircled beam power $h(k)$. This leads to

$$h(k) = \frac{\phi_k}{\phi_\infty} = 1 - \exp\left(\frac{-k^2}{2}\right) \tag{51}$$

Therefore, for a fixed value of k, $h(k)$ is constant, independent of z, and satisfies all sections of the beam envelope.

1.6.7 Determination of Axial Irradiance from Total Beam Power

The coefficient $\pi(\sqrt{2}\sigma_z)^2$ of the axial irradiance (power density) $E(0,z)$ in equation (50) represents a beam cross-sectional area of radius $\sqrt{2}\sigma_z$, which is also equal to $D/(2\sqrt{2})$. Thus the axial irradiance (power density) may be derived from a determination of the standard deviation σ_z and the total power ϕ_∞ of a gaussian beam. Rearranging equation (50) leads to

$$\text{Axial irradiance} = \frac{\text{Total beam power}}{\text{Beam cross-sectional area}} = \frac{\phi_\infty}{\pi(\sqrt{2}\sigma_z)^2} \tag{52}$$

1.6.8 Excluded Beam Power Derivation

As stated in Section 1.6.5, it follows that if $h(k)$ is the encircled beam power then the excluded beam power fraction must be $1 - h(k)$. By substituting for $h(k)$ from equations (51) into equation (41), the fraction $1 - h(k)$ of excluded radiant flux from the beam envelope is given by

$$1 - h(k) = \frac{\phi_\infty - \phi_k}{\phi_\infty} = \exp\left(\frac{-k^2}{2}\right) \tag{53}$$

1.6.9 Equating 1 − *h*(*k*) to *g*(*k*)

Now equation (39) is

$$g(k) = \frac{E(r_k,z)}{E(0,z)} = \exp\left(\frac{-k^2}{2}\right) \tag{54}$$

Inspection and comparison of equations (53) and (54) show that

$$1 - h(k) \equiv g(k) \equiv \exp\left(\frac{-k^2}{2}\right) \tag{55}$$

Thus the fraction of excluded beam power exactly corresponds to the relative power density, normalized to the axial peak power density (Figure 1.2).

The function $\exp(-k^2/2)$ in equation (55) is a gaussian distribution independent of z. Hence, the fraction of the excluded beam power and the normalized irradiance (power density) decrease in an identical manner as increasing values for the relative beam radius k are chosen, where k is a measure of the number of standard deviations σ. This convenient identity is entirely due to the unique mathematical properties of the gaussian function.

Table 1.1 gives $1 - h(k)$ and $g(k)$ for selected values of k. Inspection of Table 1.1 shows that for all practical purposes the effective beam envelope of the total beam power may be limited to a diameter of eight standard deviations, that is, 8σ, $k = 4$, at which the beam envelope encircles all but 0.034% of the total beam power. Further study of Table 1.1 shows that the beam envelope which contains 86.5% of the total beam power corresponds to a diameter of four standard deviations, that is, 4σ, $k = 2$, at which the beam envelope encircles all but 13.5% ($1/e^2$) of the total beam power. Similarly, the full width at the 13.5% ($1/e^2$) level of the normalized irradiance distribution profile corresponds to a width of four standard deviations, that is, 4σ, $k = 2$.

1.6.10 Conclusion

The accepted diameter D of a gaussian laser beam may be defined in two different ways that give identical results provided that the laser operates in the TEM_{oo} mode.

1. **The beam diameter D** is the diameter of the beam envelope that encircles 86.5% of, and excludes $1/e^2$ (13.5%) of, the total beam power.
2. **The beam diameter D** is the full width of the normalized irradiance (power density) distribution profile at the $1/e^2$ (13.5%) level of the peak value.

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