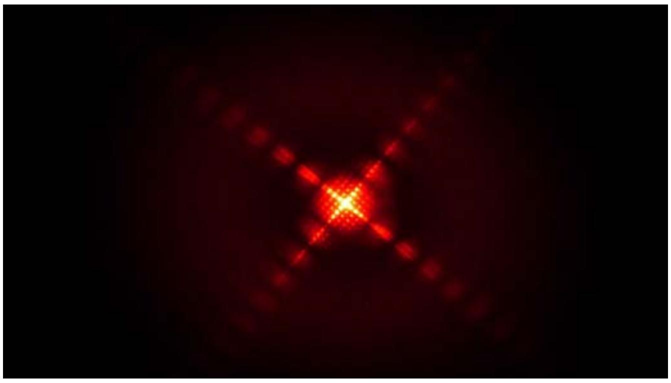
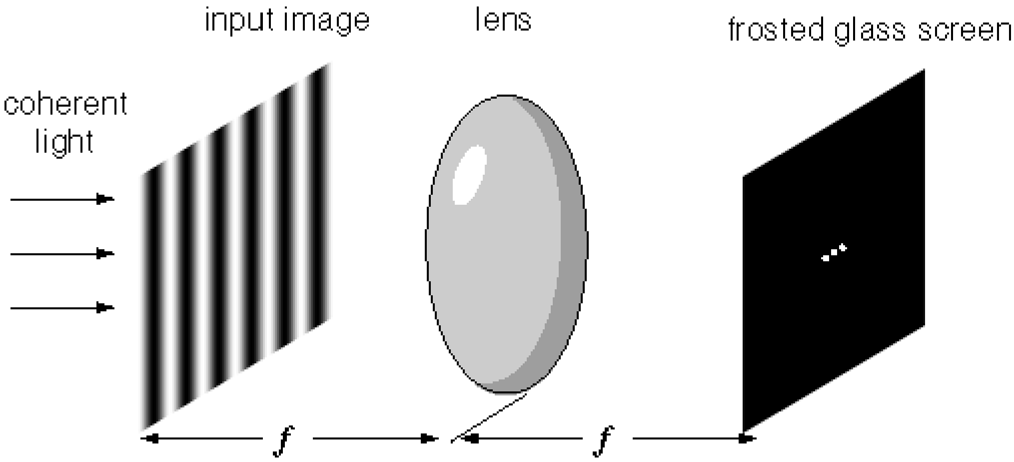
OPTICS LABORATORIES

MICRO 423-424

FOURIER OPTICS





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1. Objectives and safety

The aim of this laboratory is for you to acquire practical experience in Fourier optics by carrying out measurements using an optical set-up. To this end, a theoretical framework is first developed in order to achieve a basic understanding of the topic, so that the experimental work can then be carried out in a more straightforward and conscious way, allowing you to also compare the theoretical predictions with the experimental results. You will also have the chance to familiarise with simple optical instrumentation and basic alignment procedures.

You have to carefully perform your tasks, always bearing in mind that, even when working with low-power lasers, possible damage to the skin and the eye might occur. Therefore, it is important to always remember to:

- Never look directly into the laser beam
- Always keep your eyes higher than the plane of the laser while working on the running set-up
- Remove any shiny and reflecting object (e.g. your watch, rings, bracelets…) from your hands and arms

2. Introduction

Fourier optics exploits the Fourier transform (E1) to express any arbitrary signal (e.g. a wavefront, the intensity pattern of a digital image…), given in the spatial coordinates (x, y, z), into the spatial frequency domain (k_x, k_y, k_z). This allows us to represent any arbitrary waveform as a linear combination of plane waves propagating in different directions, with weighting factors determining the importance of each spatial frequency k. The arbitrariness of the signal studied makes Fourier optics a very versatile tool in modern physical optics, with applications covering the description of interference phenomena, spectral analysis, holograms, the determination of diffraction patterns and the description of imaging systems. For example, Fourier transform infrared spectroscopy (FTIR) is a case where the Fourier transform has been successfully applied to study the transmitted infrared light through a sample of interest. This light bears the fingerprint of the molecules dissolved in the sample, providing a quick and non-invasive way to study its composition.
As you will see in these lab notes, lenses are commonly employed as Fourier transformers in optical systems while, in commercial instruments, numerical methods such as fast Fourier transform (FFT) calculations are usually used to perform computational Fourier transforms.

3. Theoretical aspects of Fourier optics

Fourier transform

Given a complex-valued function g of two independent variables x and y which

1. Is absolutely integrable over the infinite (x, y) plane
2. Has only a finite number of discontinuities and a finite number of maxima and minima in any finite rectangle
3. has no infinite discontinuities

its Fourier transform $F[g] = G(k_X, k_Y)$ is defined as:

$$F[g] = G(k_X, k_Y) = \iint_{-\infty}^{\infty} g(x, y) e^{-j2\pi(k_X x + k_Y y)} dxdy \tag{E1}$$

G is itself a complex-valued function of two independent variables k_x and k_y, which are generally referred to as frequencies. If x and y are distance, k_x and k_y are referred to as spatial frequencies. Here, j is of course the imaginary unit. Similarly, the inverse Fourier transform of a function $G(k_x,k_y)$ is $F^{-1}[G]$, defined as:

$$F^{-1}[G] = g(x, y) = \iint_{-\infty}^{\infty} G(k_X, k_Y) e^{j2\pi(k_X x + k_Y y)} dk_X dk_Y \tag{E2}$$

In the case of a periodic function, the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients (see https://www.intmath.com/fourier-series/fourier-graph-applet.php). This way, the Fourier series is used to represent a periodic function by a discrete sum of complex exponentials, while the Fourier transform is used to represent a general, possibly even non-periodic function by a continuous superposition of complex exponentials. The Fourier transform can thus be viewed as the limit of the Fourier series of a function with the period approaching infinity. Some simple examples of Fourier transforms are shown in Fig. 1.

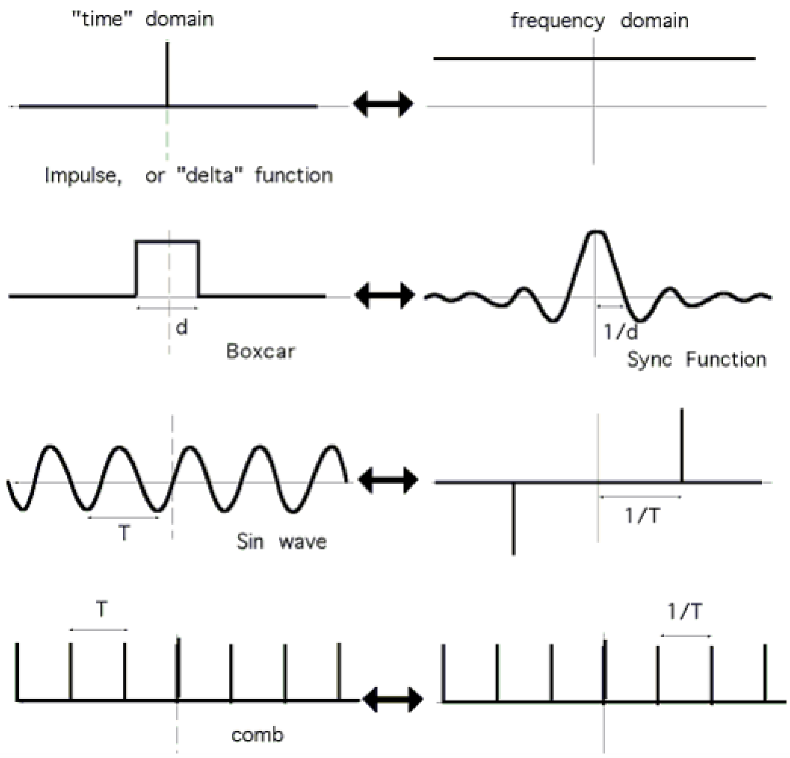

Fig. 1: Fourier transforms of different signals in the time domain.

The lens as a Fourier transform

As already mentioned, lenses are one of the most common tool to perform the Fourier transform of an optical signal. We can see how by considering the phase that a light beam accumulates as it propagates through a thin lens, which can be written as (see Figure 2, more details are presented in reference 4)

$$\varphi = k\, n\, \Delta + k(\Delta_0 - \Delta) = k\, \Delta_0 + k(n-1)\Delta \tag{E3}$$

Therefore, for an incident field U_l on the lens, the outgoing field will be

$$U_{l'}(x,y) = U_l(x,y)\, e^{-j\varphi} = U_l(x,y)\, e^{-jk\Delta_0} e^{-jk(n-1)\Delta} \tag{E4}$$

Δ can be rewritten, under the paraxial approximation, as

$$\Delta = \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{E5}$$

where $R_{1,2}$ are the radii of curvature of the two sides of the lens. By knowing that the focal distance f can be written as

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{E6}$$

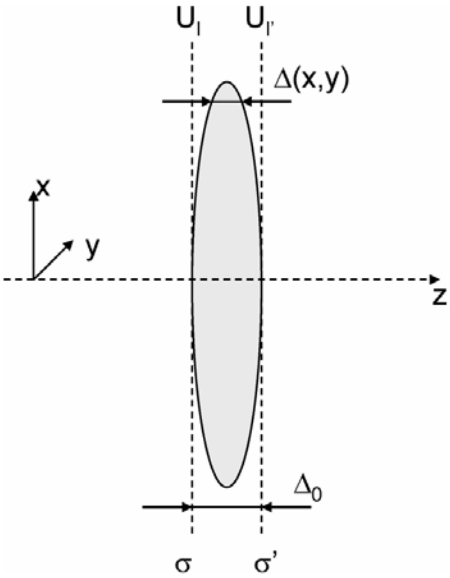

Fig. 2: Schematic of a thin lens.

we can insert (E5) and (E6) into (E4) to obtain

$$U_{l'}(x,y) = U_l(x,y)e^{-jkn\Delta_0}e^{j\frac{k}{2f}(x^2+y^2)} \tag{E7}$$

Here, the first exponential simply gives a constant phase retardation, while the second one describes the curvature of a wavefront converging towards a point on the z-axis at a distance f from the lens. Each point on this wavefront can be treated as a point source of secondary waves (Huygens principle) whose interference pattern in the near field, for $z = f$, can be calculated using Fresnel equation (with the appropriate quadratic phase dispersion), yielding

$$U_f(\xi,\eta) = \frac{ke^{-jkf}}{f}e^{jkf}e^{-j\frac{k}{2f}(1-\frac{d}{f})(\xi^2+\eta^2)} \iint U_{l'}(x,y)\, e^{-j\frac{k}{2f}(x^2+y^2)}e^{-j\frac{2\pi}{f\lambda}(x\xi+y\eta)}dxdy \tag{E8}$$

Here, (ξ,η) are the spatial coordinates in the focal plane of the lens and d is the distance of the object from the lens. Substituting $U_{l'}$ we get, after neglecting the constant phase factor $e^{-jkn\Delta_0}$

$$U_f(\xi,\eta) = \frac{k}{f}e^{-j\frac{k}{2f}(1-\frac{d}{f})(\xi^2+\eta^2)} \iint U_l(x,y)e^{-j\frac{2\pi}{f\lambda}(x\xi+y\eta)}dxdy \tag{E9}$$

This results clearly shows that the field intensity in the focal plane of a lens $I_f = \left|U_f\right|^2$ is proportional to the Fourier transform of the incident signal. The quadratic phase factor in front of the integral vanishes for $d = f$ in such a way that, for an object placed in the front focal plane of the lens, the signal at the back focal plane will be its exact Fourier transform.

Fourier filtering

Now that we know we have access to the Fourier transform of an optical signal at the focal plane of a lens, we can see what happens when we modify this plane, for example by applying a mask or filter. In general, it is obvious that applying (E2) to a full Fourier-transformed signal will result in the original input, as shown in Figure 3.

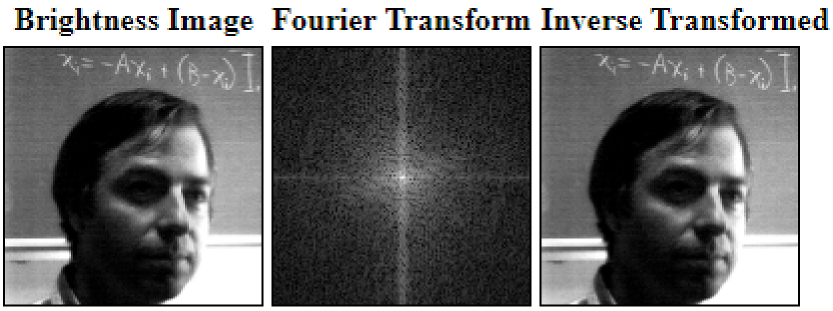

Fig. 3: Double Fourier transform of an image.

However, it is possible to calculate the inverse transform only on a restricted domain of the Fourier plane, so that the back-transformed image will contain only some of the frequencies of the original image. This procedure is called *Fourier filtering*. It is for example possible to retain only the low frequencies close to the DC point, while blocking all the higher frequencies[1]. On the other hand, one can also realise a high-pass filter that blocks only the low frequencies, as well as band pass filters, as shown in Figure 4.

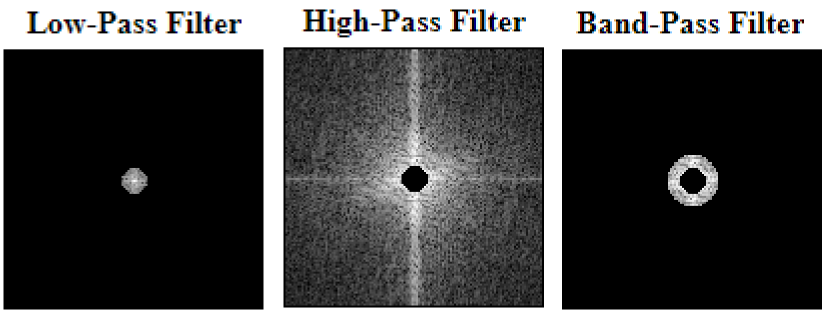

Fig. 4: Different types of Fourier filters.

Different filters can be used to study how different frequency components take part into the image formation process, as you will shortly see in one of the following exercises. These are meant to help you familiarise with the concept of the Fourier transform and help you verify and interpret the results you will obtain in the lab.

Theoretical part to prepare at home ⌂

[1] The DC point is the bright spot at the center of the Fourier plane where all the constant and slowly varying (low frequency) components of the image are mapped to by the Fourier transform.

Exercise 1:

For a circular aperture of radius *a*, show that the intensity obtained at the focal plane of a lens placed in front of the aperture is:

$$I(r,\theta) = \pi^2 a^4 A^2 \left| \frac{2J_1\left(\frac{kar}{f}\right)}{\left(\frac{kar}{f}\right)} \right|^2 \qquad \text{(E11)}$$

You might like to use the Bessel function defined below to simplify the integral:

$$J_n(x) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{-ixcos\gamma} e^{in\gamma} d\gamma \qquad \text{(E12)}$$

Exercise 2:

Try also extending this result to a one-dimensional array of *N* identical apertures of width *2a* and equally separated by a distance *2b* to show that the measured intensity will be:

$$I(x) = 4a^2C^2 \left[\frac{sin^2\left(\frac{kNbx}{f}\right)}{sin^2\left(\frac{kbx}{f}\right)} \right] sinc^2\left(\frac{kax}{f}\right) \qquad \text{(E13)}$$

Exercise 3:

Do the same for a rectangular aperture defined with a width of *2a* and a length of *2b*, showing that the intensity in the Fourier plane will be:

$$I(x,y) = 16a^2b^2 sinc^2\left(\frac{kax}{f}\right) sinc^2\left(\frac{kby}{f}\right) \qquad \text{(E14)}$$

Exercise 4:

What do you think happens to the image in the Fourier plane when we take an object and its inverse, for example a square aperture and a square object? Think in terms of the properties of a Fourier transform.

Exercise 5:

Suppose that we compare the Fourier spectra of an image taken before and after uniformly increasing its brightness. In what way do you think the two spectra will differ from each other? Think in terms of relative intensity.

Exercise 6:

Figure 5 shows sinusoidal brightness images obtained by adding higher harmonics (of spatial frequency 3, 5 and 7) to the fundamental one (of spatial frequency 1). The respective Fourier transforms are also shown. We can see that, as more harmonics are

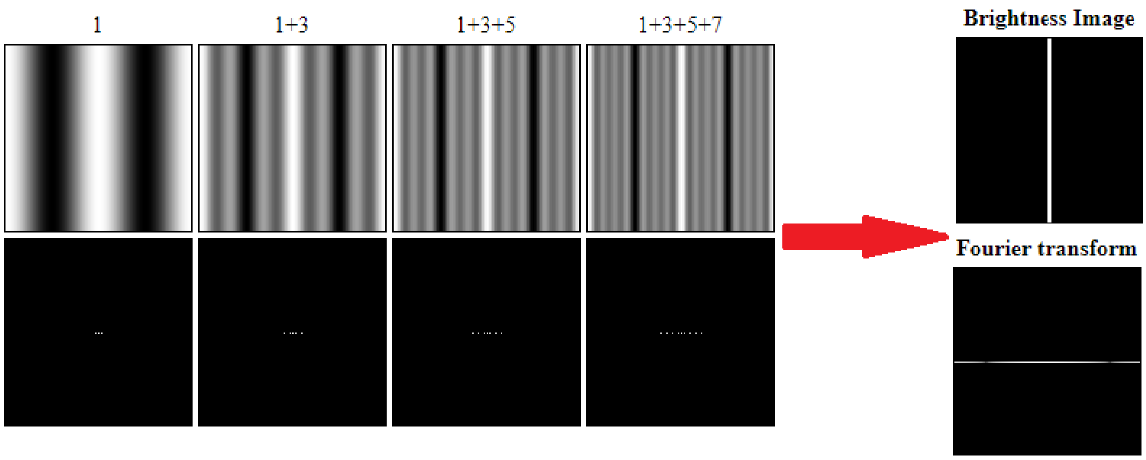

Fig. 5: Different types of Fourier filters.

added, the central vertical band gets sharper and stronger, while the background drops down towards a uniform dark field. In the Fourier plane, new peaks appear that spread outwards from the fundamental one.

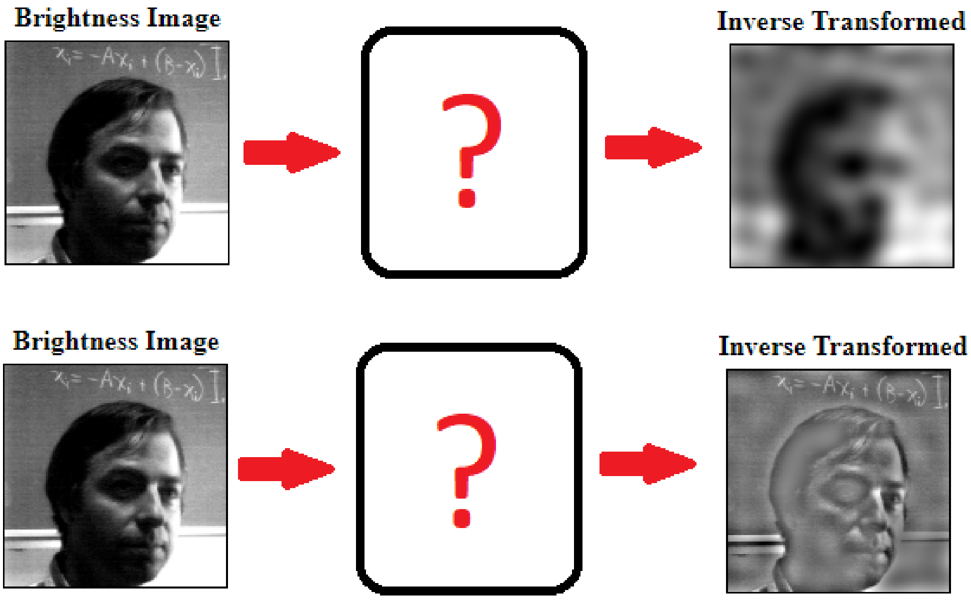

Fig 6: Different Fourier filtering of an image.

In the limiting case of a vertical bright band in the image, with very sharp contrast, the Fourier transform exhibits an "infinite" series of harmonics or higher order terms[2]. Based on this example, which filter, from the ones presented in Figure 4, would you

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[2] In this limiting case we reach the so called nyquist frequency, the highest spatial frequency that can be encoded in the digital image, which is limited by the resolution (size of the pixels).

apply to the Fourier plane to obtain a blurred back-transformed image? And to obtain a sharper one? Notice also, in the sharper image, the loss of dark and bright regions.

4. Experimental part

Fourier 4-f setup

In this part, you will have to assemble a 4-f setup to carry out the experiments. The setup you will find, depicted in Figure 3, consists of a light source emitting laser light which, after going through a diaphgram (diameter of the pinhole: 50 μm), needs to be guided towards the two cameras at the other end of the setup. One of these cameras will be used to record the image plane, while the other will probe the Fourier plane.

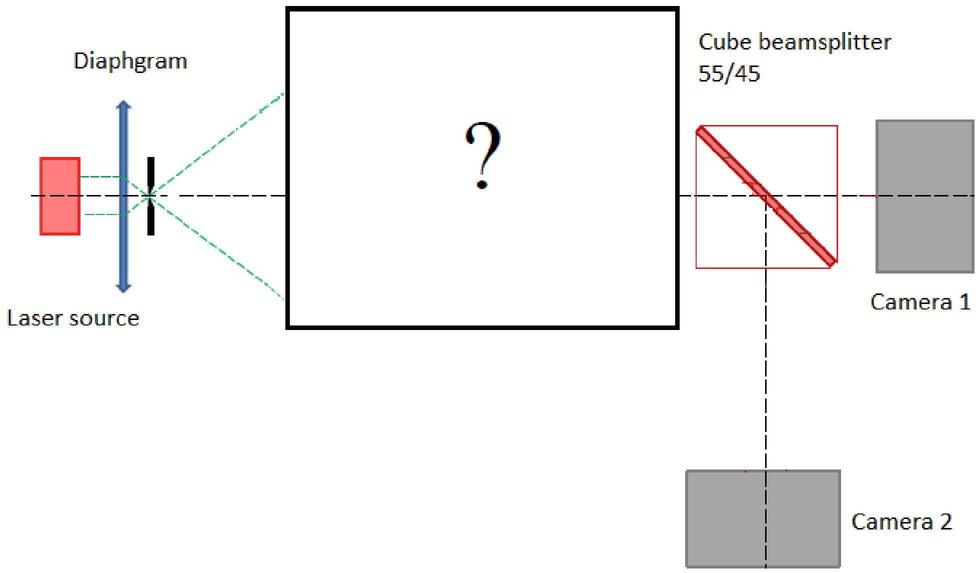

Fig. 7: Scheme of the experimental set-up.

Task 0 to prepare at home ⌂:
Let's have a look at the light source provided. It is a 635 nm, 1 mW continuous laser source. What laser class does it belong to? What colour will the light be? Recall the definition of maximum permissible exposure (MPE) and provide a value for this setup.

Task 1:
You will find, in the box provided, 5 plano-convex lenses, a sample holder and a few post holders. Of these lenses, one has a focal length of 25.4 mm, while the other four have a focal length of about 150 mm. With the help of a ruler, find the focal length of each lens (hint: you can consider the neon lamps on the ceiling as a far-field source of collimated light).

Task 2:

Use one lens to collimate the light beam and the others to build the 4-f setup and imaging system (hint: the plane side of the plano-convex lenses has to be oriented towards the focal point). How many lenses do you need in total? Provide a sketch of the final setup, clearly marking the object plane and every image and Fourier plane. Build the setup in in order to minimise the phase difference between the image and Fourier plane. Align the beam so that it is centered on the cameras.

Task 3:

Work out the magnification of the image and Fourier planes on the cameras with respect to the object and the Fourier plane in the 4f system. Knowing that each camera has got 1280×1024 pixels on its CMOS chip taking up an area of 6.66×5.32 mm^2, calculate the size of a pixel in the spatial domain (r_{pixel}). How much does this correspond to in the Fourier domain? For this task, use Fig. 8 to figure out how to convert a spatial length in the object plane into frequency units in the Fourier plane and find out the size of a pixel in the Fourier domain (k_{pixel} , hint: this size depends, among other things, on the focal length of the lens and on the wavelength used).

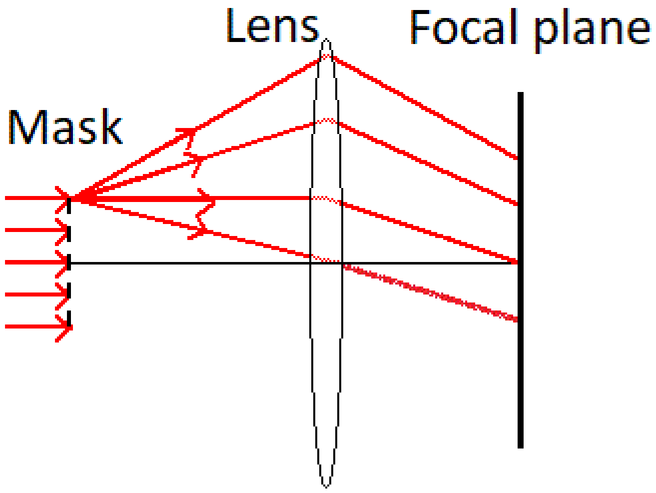

Fig. 8: Ray optics diagram showing how a lens maps each diffracted ray coming from a mask (object) into different positions in the Fourier plane. This results in a 1:1 correspondence between a ray's incident angle on the lens and its position in the focal plane.

Measurements

The goal is now to place different objects in the object plane and study their diffraction patterns in the image and Fourier planes. Using the two cameras, make sure to always take a picture of both planes simultaneously and name the files accordingly. For some samples, knowing the analytical Fourier transform of the object, you will be able to deduce its spatial dimensions (hole diameter, slits period etc…) of the object by studying the Fourier spectrum and compare your values with the ones measured in the image plane and the ones given in Table 1.

To analyse the images, you can use ImageJ, MATLAB or Mathematica. The Thorlabs software also allows you to plot intensity profiles in real time during the experiment. Table 1 lists the available samples, which you will find in the round plastic box.

#	Type	Dimensions	Remarks
1	Single hole	∅ = 225 μm	Broken
2	Double hole	∅ = 175 μm d = 750 μm	
3	Slit array	w = 385 μm p = 850 μm 1 x 6	Vertical
4	Single slit	w = 500 μm	
7	Grid of slits	w = 150 μm p = 1000 μm 8 x 8	Inverse structure to #15
9	Single disk	∅ = 225 μm	Inverse structure to #1 Filter
10	Line array	w = 360 μm p = 850 μm 6 x 1	Structure #11 horizontal
11	Line array	w = 360 μm p = 850 μm 1 x 6	Structure #10 vertical
12	Line array	w = 280 μm p = 500 μm 1 x 6	Vertical
13	Line array	w = 280 μm p = 500 μm 1 x 11	Vertical
15	Grid of lines	w = 150 μm p = 1000 μm 8 x 8	Inverse structure to #7 Broken
16	Single line		Horizontal
	Three holes		
	Fourier House		
	Two disks		Do not use
	Single slit		Do not use Broken

Table 1: List of the objects and filters provided. w and p mean the width and period of the slits and lines, respectively. All slits and lines have a length l of 1 cm.

Task 4:

Place a pinhole (sample 1) in the sample holder and record its image and Fourier plane. Use the theoretical Fourier spectrum (obtained in Exercise 1) of a pinhole to fit the Fourier image obtained in the experiment and calculate the diameter of the pinhole (hint: rather than intensity, think about the position of the minima). Does it match with the theoretical value in Table 1? Measure the diameter of the hole in the image plane and compare it with the value obtained from the Fourier spectrum and the one in Table 1. Explain and discuss any discrepancy between these values.

Is the dynamic of the camera sufficient to see more than one secondary maxima without saturating the central one? Adjust the acquisition settings so as to take one image of the Fourier plane without saturating the central maxima and another one where it is saturated, so that you can observe as many secondary maxima as possible.

Task 5:

Place a filter (sample 9) in the Fourier plane and record the image and Fourier plane. What changes do you see in the images? Comment.

Task 6:

Work out the diameter of the three larger holes (sample without number, with 3 holes on the same substrate) in the same way as you did for sample 1.

Task 7:

Place an object with 2 pinholes (sample 2) in the object plane. What Fourier pattern do you expect? Explain the differences when compared to the single hole case. Derive the distance between the holes and their diameter from the experimental Fourier image and from the image plane, then compare these values with the actual ones mentioned in Table 2.

Task 8:

Place sample 4 in the object plane. Record the experimental Fourier image of the single slit and compare it with the calculated Fourier image plotted from (E14). Evaluate the width of the slit from the experimental Fourier image and compare it with the actual value mentioned in Table 2, as well as with the value extrapolated from the image plane.

Exercise 9:

Place sample 3 in the object plane, measure and compare the experimental curve with the determined function. Characterize the size and pitch of the objects.

Task 10:

Place sample 7 and its inversion (sample 15) in the object plane. Compare the different Fourier spectra and explain your observations with the results of Exercise 4.

Task 11:
Place the "Fourier house" sample in the object plane. Observe the Fourier plane and explain the origin of the different diffraction patterns. As in Task 5, apply a filter in the Fourier plane to suppress the DC component and record the new Fourier spectrum. After removing the filter, use a piece of paper to cover each of the four parts of the object and record the resulting image and Fourier planes. Comment on your findings.

<u>Content of the report</u>

For all measured samples, you must include:

a) An image of both the image and Fourier plane, with their intensity profile (line-cut) through the center of the pattern
b) For the Fourier plane, measure the distance between the central maximum and the adjacent zeros (or maxima). Using this value and the analytical formulas derived in the exercises at home, calculate the dimensions of the object and compare them to the measured values from the image plane. Comment on the matching between the two, especially concerning the position of the zeros and the relative amplitude of the maxima.
Can you find an easier (but less elegant!) way of calculating the object's size from its Fourier plane data?
c) When asked, the computed spatial dimensions *with an error analysis* of the object from both the image and Fourier plane images.
d) For samples 1 and the Fourier house, images of the filtered Fourier plane and the resulting image in the image plane.

5. References

1. *The New Physical Optics Notebook: Tutorials in Fourier Optics,* G. O. Reynolds, J. B. DeVelis, G. B. Parrent and B. J. Thompson
2. *Fundamentals of Photonics,* B. E. A. Saleh and M. C. Teich (Chapter 4)
3. *Introduction to Fourier optics,* J. W. Goodman
4. *Optics*, E. Hecht and A. Zajac
5. http://www.photonics.intec.ugent.be/download/ocs130.pdf