Laser: Theory and Modern Applications
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solution Set (for exercise Sheet 9): Phase-locking, Short Laser Pulses

9.1 Phase-locked Oscillators

This exercise investigates the general principle that a group of phase-locked oscillators produce a pulse train. This can be understood by considering the Fourier transform of the sum of the frequency components into the time domain.

The electric field $E(t)$ is given by:

$$E(t) = \sum_{j=-(N-1)/2}^{(N-1)/2} E_0 e^{i(\omega_0 + j\omega_R)t + i\Phi_j} = E_0 e^{i\omega_0 t} \sum_{m=0}^{N-1} e^{i\omega_R[m-(N-1)/2]t}$$

Simplifying the sum:

$$E(t) = E_0 e^{i\omega_0 t} e^{-i\omega_R[(N-1)/2]t} \sum_{m=0}^{N-1} e^{i\omega_R m t}$$

This leads to:

$$E(t) = E_0 e^{i\omega_0 t} e^{-i\omega_R[(N-1)/2]t} \frac{e^{iN\omega_R t/2}}{e^{i\omega_R t/2}} \frac{\sin(N\omega_R t/2)}{\sin(\omega_R t/2)}$$

After simplification,

$$E(t) = E_0 e^{i\omega_0 t} \frac{\sin(N\omega_R t/2)}{\sin(\omega_R t/2)}$$

Key Observations

The temporal distance between subsequent pulses is $\tau_{\text{rep}} = \frac{1}{\omega_R}$.

The pulse duration τ_{FWHM} can be approximated using the first zero crossing of the numerator:

$$\tau_{\text{FWHM}} = \frac{1}{N\omega_R}$$

A detailed explanation can be found in Chapter 11.3 of the script.

Including a Linearly Varying Phase

With a linearly varying phase $\Phi_j = j\Delta\Phi$, the field becomes:

$$E(t) = E_0 e^{i\omega_0 t} \frac{\sin[N(\omega_R t + \Delta\Phi)/2]}{\sin[(\omega_R t + \Delta\Phi)/2]}$$

This shows that a chirped spectrum forms a pulse train in the time domain.

The carrier-envelope offset frequency ω_{CEO} is defined as:

$$\omega_{\text{CEO}} = \omega_0 \mod \omega_R$$

For $\omega_{\text{CEO}} = 0$, the pulse train consists of identical envelopes, with the electric field oscillating consistently. For non-zero ω_{CEO}, a slow modulation term changes the carrier phase relative to the envelope.

Assuming the carrier frequency $\omega_C = z\omega_R$, $\omega_0 = \omega_C + \Delta\nu$ and $\Delta\Phi = 0$:

$$E(t) = E_0 e^{i\omega_C t + i\Delta\nu t} \frac{\sin(N\omega_R t/2)}{\sin(\omega_R t/2)} = \sum_{j=-(N-1)/2}^{(N-1)/2} E_0 e^{i(\omega_C + \Delta\nu + j\omega_R)t}$$

Calculating ω_{CEO}:

$$\omega_{\text{CEO}} = \omega_0 \mod \omega_R = z\omega_R + \Delta\nu \mod \omega_R = \Delta\nu$$

9.2 Shortest Laser Pulses from Ti:Sa and He:Ne Lasers

The pulse shape depends on the source's pulse-forming mechanism and can be modified by technical and experimental conditions. The time-bandwidth product $\tau_P \cdot f_{\text{BW}}$ measures pulse duration efficiency:

$$\tau_P \cdot f_{\text{BW}} = d$$

For Gaussian pulses, $d \approx 0.44$. Converting the Ti:Sa bandwidth to Hz:

$$c_0 \left(\frac{1}{775\text{ nm}} - \frac{1}{825\text{ nm}} \right) = 23.46\text{ THz}$$

Minimal pulse durations:

$$\tau_{P,\text{He:Ne}} = 258.823\text{ ps}, \quad \tau_{P,\text{Ti:Sa}} = 18.75\text{ fs}$$

9.3 Gaussian Laser Pulses

For a Gaussian pulse:

$$E(t) = \sum_{n=-\infty}^{\infty} E_n e^{i\omega_n t + i\Phi_n}$$

With frequencies $\omega_n = \omega_0 + n\omega_R$ and amplitudes:

$$E_n = E_0 \exp\left[-2\ln(2) \left(\frac{n\omega_R}{\Delta\omega_0} \right)^2 \right]$$

Intensity:

$$I(t) \propto |E(t)|^2 = |E_0 e^{i\omega_0 t}|^2 \cdot \left| \int_{-\infty}^{\infty} \exp\left[-2\ln(2) \left(\frac{\omega_R}{\Delta\omega_0} \right)^2 n^2 \right] e^{in\omega_R t}\, dn \right|^2$$

Using the Fourier transform property:

$$e^{-\alpha^2 n^2} \leftrightarrow \text{FT} \sqrt{\frac{\pi}{\alpha}} e^{-x^2/(4\alpha^2)}$$

Result:

$$I(t) \propto |E_0|^2 \cdot \frac{\pi\Delta\omega_0}{\sqrt{2\ln(2)}\omega_R} \exp\left[-\frac{\Delta\omega_0^2}{8\ln(2)} t^2 \right]$$

FWHM:

$$\tau_{\text{FWHM}} = \frac{4\ln(2)}{\Delta\omega_0}$$