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Laser: Theory and Modern Application

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.12: Nonlinear frequency conversion - I

12.1 Coherence length for phase matching

Coherence length for phase matching

$$L_{coh} = \frac{1}{|\Delta k|} - \left(-\frac{1}{|\Delta k|}\right) = \frac{2}{|\Delta k|}$$

For a SHG system, this yields into the following

$$\Delta k = 2k_1 - k_2 = \frac{2\omega}{c}(n_e^{2\omega} - n_o^{\omega})$$

So we get the following coherence length

$$\begin{aligned} L_{coh} &= \frac{2}{|\Delta k|} \\ &= \frac{c}{\omega(n_e^{2\omega} - n_o^{\omega})} \\ &= \frac{\lambda}{2\pi(n_e^{2\omega} - n_o^{\omega})} \end{aligned}$$

With $n(\omega) = 1.496044$ and $n(2\omega) = 1.514928$ we get the following coherence length

$$L_{coh} = \frac{\lambda}{2\pi(n_e^{2\omega} - n_o^{\omega})} = 8.428\ \mu m$$

12.2 Angular sensitivity of phase matching

Taylor expansion of a function at a point (a) is:

$$f(x) \approx f(a) + f'(a)(x-a)$$

For extraordinary refractive index we have

$$\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}$$

$$\Rightarrow n_e(\theta) = \frac{n_e n_o}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$$

Taylor expansion of the extraordinary refractive index near the phase matching angle θ_m is

$$n_e(\theta) \approx \frac{n_e n_o}{\sqrt{n_o^2 \sin^2\theta_m + n_e^2 \cos^2\theta_m}} + n_e'(\theta_m)(\theta - \theta_m)$$

And

$$n_e'(\theta_m) = -\frac{n_e n_o}{2}(n_o^2 \sin^2\theta_m + n_e^2 \cos^2\theta_m)^{-3/2}(n_o^2 - n_e^2)\sin 2\theta_m$$

$$\Rightarrow n_e(\theta) = n_e(\theta_m) - n_e(\theta_m)\frac{(n_o^2 - n_e^2)\sin 2\theta_m \Delta\theta}{2(n_o^2\sin^2\theta_m + n_e^2\cos^2\theta_m)}$$

On the other hand for hase mismatch we have:

$$\Delta k = \frac{2\omega}{c}(n_o^{\omega} - n_e^{2\omega}(\theta))$$

Therefore, for phase matching condition we have:

$$n_e^{2\omega}(\theta_m) = n_o^{\omega}$$

Simplifying the relationship for $n_e(\theta)$ in phase mismatch relationship, we come up with:

$$\frac{\Delta kL}{2} = \frac{-\omega n_{o,\omega}^3 L}{2c}\left(\frac{1}{n_{e,2\omega}^2} - \frac{1}{n_{o,2\omega}^2}\right)\sin 2\theta_m\Delta\theta$$

12.3 Angle phase matching in non-linear crystals

Angle phase matching in nonlinear crystals The phase matching condition in BBO crystal is

$$2k_1 = k_2$$

From this follows then

$$2\frac{\omega n_o(\omega)}{c} = 2\frac{\omega n(\theta, 2\omega)}{c}$$
$$n_o(\omega) = n(\theta, 2\omega)$$

Calculating the extraordinary refractive index

$$\frac{1}{n^2(\theta, 2\omega)} = \frac{\cos^2(\theta)}{n_o^2(2\omega)} + \frac{\sin^2(\theta)}{n_e^2(2\omega)}$$
$$\Rightarrow \sin^2(\theta) = \frac{\frac{1}{n_o^2(\omega)} - \frac{1}{n_o^2(2\omega)}}{\frac{1}{n_e^2(2\omega)} - \frac{1}{n_o^2(2\omega)}}$$

With $n_o(\omega) = 1.667$, $n_o(2\omega) = 1.729$ and $n_e(2\omega) = 1.592$

$$\Rightarrow \sin^2(\theta) = 0.4221$$
$$\theta = 40.52^\circ$$

12.4 Coupled-mode equations for four-wave mixing

1. We need to solve the wave equation

$$\nabla^2 E - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}E = \frac{\chi^{(3)}}{c^2}\frac{\partial^2}{\partial t^2}|E|^2 E,$$

where the full field is the sum of pump, Stokes and anit-Stokes fields. Let's condsider the left hand side of this equation. Using the slowly varying approximation ($\partial_z F_{(s,a,p)} \gg k_{(s,a,p)}\partial^2_{zz}F_{(s,a,p)}$), and the identity $k^2_{(s,a,p)} = n^2/c^2\omega_{(s,a,p)}$, one obtains the left hand side of the form

$$2i\Big(A_a' k_a \Psi_a e^{i(k_a z - \omega_a t)} + A_s' k_s \Psi_s e^{i(k_s z - \omega_s t)} + A_p' k_p \Psi_p e^{i(k_p z - \omega_p t)}\Big),$$

where $A_{(s,a,p)} = F_{(s,a,p)}/2N_{(s,a,p)}$, and $A' = dA/dz$.
Neglecting the quadratic and cubic terms of A_s and A_a, one gets the relation

$$|E|^2E = |A_p\Psi_p|^2\left(A_p\Psi_p e^{i(k_pz-\omega_pt)} + 2A_s\Psi_s e^{i(k_sz-\omega_st)} + 2A_a\Psi_a e^{i(k_az-\omega_at)}\right)$$
$$(A_p\Psi_p)^2(A_a\Psi_a)^* e^{i((2k_p-k_a)z-(2\omega_p-\omega_a)t)} + (A_p\Psi_p)^2(A_s\Psi_s)^* e^{i((2k_p-k_s)z-(2\omega_p-\omega_s)t)},$$

where * stands for complex conjugation.

2. In roder to get the set of $1-$st order ODE's we use the orthogonality relation $\int \Psi_i\Psi_k^* dS = \delta_{ik}$. Multyplying the reduced equation by $\Psi^*_{(s,a,p)}$ and integrating it, we obtain the following system

$$2ik_pA_p' = -\frac{\chi^{(3)}}{c^2}\Big[|A_p|^2A_p\omega_p^2\int|\Psi|^4dS + 2\omega_a^2|A_p|^2A_a\int|\Psi_p|^2\Psi_a\Psi_p^*dSe^{i((k_a-k_p)z-(\omega_a-\omega_p)t)}$$
$$+2\omega_s^2|A_p|^2A_s\int|\Psi_p|^2\Psi_s\Psi_p^*dSe^{i((k_s-k_p)z-(\omega_s-\omega_p)t)} + (2\omega_p-\omega_a)^2A_p^2A_a^*\int|\Psi_p|^2\Psi_p\Psi_a^*dSe^{i((k_p-k_a)z-(\omega_p-\omega_a)t}$$
$$+(2\omega_p-\omega_s)^2A_p^2A_s^*\int|\Psi_p|^2\Psi_p\Psi_s^*dSe^{i((k_p-k_s)z-(\omega_p-\omega_s)t)}\Big], \tag{1}$$
$$2ik_aA_a' = -\frac{\chi^{(3)}}{c^2}\Big[|A_p|^2A_p\omega_p^2\int|\Psi|^2\Psi_p\Psi_a^*dSe^{i((k_p-k_a)z-(\omega_p-\omega_a)t)} + 2\omega_a^2|A_p|^2A_a\int|\Psi_p|^2|\Psi_a|^2dS$$
$$+2\omega_s^2|A_p|^2A_s\int|\Psi_p|^2\Psi_s\Psi_a^*dSe^{i((k_s-k_a)z-(\omega_s-\omega_a)t)} + (2\omega_p-\omega_a)^2A_p^2A_a^*\int\Psi_p^2(\Psi_a^2)^*dSe^{i((k_p-k_a)z-(\omega_p-\omega_a)t)}$$
$$+(2\omega_p-\omega_s)^2A_p^2A_s^*\int\Psi_p^2\Psi_a^*\Psi_s^*dSe^{i((2k_p-k_a-k_s)z-(2\omega_p-\omega_a-\omega_s)t)}\Big],$$
$$+2\omega_a^2|A_p|^2A_a\int|\Psi_p|^2\Psi_a\Psi_s^*dSe^{i((k_a-k_s)z-(\omega_a-\omega_s)t)} + (2\omega_p-\omega_s)^2A_p^2A_s^*\int\Psi_p^2(\Psi_s^2)^*dSe^{i((k_p-k_s)z-(\omega_p-\omega_s)t)}$$
$$+(2\omega_p-\omega_s)^2A_p^2A_a^*\int\Psi_p^2\Psi_s^*\Psi_a^*dSe^{i((2k_p-k_a-k_s)z-(2\omega_p-\omega_a-\omega_s)t)}\Big]. \tag{2}$$

3. Keeping only resonant terms (i.e. neglecting all the terms with the fast oscillating exponentials), one obtains the following set of equations on varying A_a and A_s (for constant A_p)

$$\begin{cases} A_s' = ic_s\chi^{(3)}PA_s + ic_4\chi^{(3)}PA_a^*e^{ikz}, \\ (A_a')^* = ic_a\chi^{(3)}PA_a^* + ic_4\chi^{(3)}PA_se^{-ikz}, \end{cases}$$

where $P = |A_p|^2$ is the pump intensity, $k = \Delta k - c_s\chi^{(3)}P - c_a\chi^{(3)}P$, $\Delta k = 2k_p - k_a - k_s$, and coefficients $c_{(s,a,4)}$ are obtaied from the previous equations.

4. Evolution of the Stokes field

$$A_s(z) = \left(Ae^{gz} + Be^{-gz}\right)e^{i(\kappa z/2 + c_s\chi^{(3)}Pz)}, \tag{3}$$
$$g = \left((c_4\chi^{(3)}P)^2 - (k/2)^2\right)^{1/2}. \tag{4}$$

5. Maximum gain occures for $k = 0$, and the bandwidth of the interaction ($g > 0$) depends on the pump power.

12.5 Nonlinearity enhancement in ENZ materials

Consider total polarization: $P^{TOT}(t) = \epsilon_0\chi_{eff}E(t)$:

$$P^{TOT}(t) = P_L(t) + P_{NL}(t) = \epsilon_0\chi^{(1)}E_0cos(\omega t) + \epsilon_0\chi^{(3)}(E_0cos(\omega t))^3 =$$
$$= \epsilon_0\chi^{(1)}E_0cos(\omega t) + \frac{\epsilon_0}{4}\chi^{(3)}E_0^3cos(3\omega t) + \frac{3\epsilon_0}{4}\chi^{(3)}E_0^3cos(\omega t)$$

In the expression above only the first and the third terms represent the contributions to the total polarization at the frequency ω. Thus:

$$P^{TOT}(t,\omega) = \epsilon_0(\chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2)E_0 cos(\omega t)$$

and comparing to the first equation, we have: $\chi_{eff} = (\chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2)$. Now, using $n^2 = \chi_{eff} + 1$, and $n = n_0 + n_2 I$ we can obtain:

$$(n_0 + n_2 I)^2 = 1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2$$
$$n_0^2 + 2n_0 n_2 I + n_2^2 I^2 = 1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}\frac{2I}{n_0}\epsilon_0 c$$

Neglecting the small term $\sim n_2^2$ on the left-hand-side, we get for the term proportional to the intensity I of the light:

$$n_2 = \frac{3}{4n_0^2\epsilon_0 c}\chi^{(3)}$$

For ENZ materials $n_0^2 = \epsilon_0 \to 0$ suggesting a significant enhancement of n_2 at certain wavelengths ω.