Laser: Theory and Modern Application
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.10: Pulse characterization

10.1

$$I(t) = E^2(t) = e^{-(\frac{t}{t_0})^2}$$

1. τ is the FWHM of I(t) i.e.

$$\begin{aligned}
I(t) = \frac{I_{max}}{2} = \frac{I(t=0)}{2} &= \frac{1}{2} \\
\Rightarrow e^{-(\frac{t}{t_0})^2} &= \frac{1}{2} \\
\Rightarrow ln\left(e^{-(\frac{t}{t_0})^2}\right) = ln(\frac{1}{2}) &= -ln2 \\
\Rightarrow \frac{t^2}{t_0^2} &= ln2
\end{aligned}$$

The FWHM is then two times this value

$$\Rightarrow \tau = t_0 2\sqrt{ln2} = 1.66 \cdot t_0$$

2. The Fourier transform of E(t) is

$$\begin{aligned}
\mathcal{F}\left\{E_0(t)\right\} &= \int_{-\infty}^{-\infty} E_0(t) e^{-i\omega t} dt \\
&= \int_{-\infty}^{-\infty} e^{-\frac{t^2}{2t_0^2}} e^{-i\omega t} dt \\
&= \sqrt{2\pi} t_0 e^{-\frac{t_0^2}{2}(\omega-\omega_0)^2}
\end{aligned}$$

For a gaussian, fourier transform limited pulse

$$\begin{aligned}
\Delta\nu \cdot \tau &= 0.441 \\
\Delta\nu = \frac{0.441}{\tau} = \frac{0.441}{1.66} \cdot \frac{1}{t_0} &= \frac{0.27}{t_0}
\end{aligned}$$

3. The autocorrelator of the second-order, non colinear type gives an output from the photodiode which is the background-free intensity autocorrelation, i.e.

$$\begin{aligned}
G(\tau) &= \text{detector output} \\
&= 4 \cdot \int_{-\infty}^{\infty} I(t) \cdot I(t-\tau) dt \\
&= 4 \cdot \int_{-\infty}^{\infty} e^{-(\frac{t}{t_0})^2} \cdot e^{-(\frac{t-\tau}{t_0})^2} dt \\
&= 4 \left(\frac{1}{2}\sqrt{\frac{\pi}{2}} t_0 e^{-\frac{\tau^2}{2t_0^2}} erf\left(\frac{2t-\tau}{\sqrt{2}t_0}\right)\Big|_{-\infty}^{\infty} \right) \\
&= 2\sqrt{\frac{\pi}{2}} t_0 e^{-\frac{\tau^2}{2t_0^2}} (1+1) \quad \text{with } erf(\infty) = 1 \text{ and } erf(-\infty) = -1 \\
&= 4\sqrt{\frac{\pi}{2}} t_0 e^{-\frac{\tau^2}{2t_0^2}}
\end{aligned}$$

4. In order to compare the original function $I(t) = E^2(t) = e^{-(\frac{t}{t_0})^2}$ and the function $G(\tau)$, we normalize $G(\tau)$ i.e. we devide

$$G_{norm} = \frac{G(\tau)}{\int_{-\infty}^{\infty} I^2(t)dt}$$

Since

$$\int_{-\infty}^{\infty} I^2(t)dt = \sqrt{\pi} \cdot t_0$$

$$\Rightarrow G_{norm}(\tau) = \frac{4}{\sqrt{2}} e^{-\frac{\tau^2}{2t_0^2}}$$

$$\Rightarrow G_{norm,max}(\tau) = \frac{4}{\sqrt{2}}$$

The FWHM τ_{auto} is then given by

$$G_{norm}(\tau) = \frac{1}{2} G_{norm,max}$$

$$\frac{4}{\sqrt{2}} e^{-\frac{\tau^2}{2t_0^2}} = \frac{2}{\sqrt{2}}$$

$$\Rightarrow -ln2 = -\frac{\tau^2}{2t_0^2}$$

$$\Rightarrow \Delta\tau_{auto} = 2\Delta\tau = 2\underbrace{\sqrt{2ln(2)}t_0}_{\text{(see 1.1)}} = \sqrt{2} \cdot \tau$$

10.2

1.

$$
\begin{aligned}
I_2(\tau) &= \int \left| \left| E_0(t)e^{i(\omega t+\phi(t))} + E_0(t-\tau)e^{i(\omega(t-\tau)+\phi(t-\tau))} \right|^2 \right|^2 dt \\
&= \int \left| |E_0(t)|^2 + |E_0(t-\tau)|^2 + 2E_0(t)E_0(t-\tau)\cos\left(\omega t + \phi(t) - [\omega(t-\tau)+\phi(t-\tau)]\right) \right|^2 dt \\
&= \int \left| |E_0(t)|^2 + |E_0(t-\tau)|^2 + 2E_0(t)E_0(t-\tau)\cos\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \right|^2 dt \\
&= \int \left| |E_0(t)|^2 + |E_0(t-\tau)|^2 + 2E_0(t)E_0(t-\tau)\cos\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \right|^2 dt \\
&= \int \left(|E_0(t)|^2 + |E_0(t-\tau)|^2 \right)^2 + 4E_0^2(t)E_0^2(t-\tau)\cos^2\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \\
&\quad + 2\left((E_0^2(t) + E_0^2(t-\tau)) 2E_0(t)E_0(t-\tau)\cos\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \right) dt \\
&= \int E_0^4(t) + E_0^4(t-\tau) + 2E_0^2(t)E_0^2(t-\tau) + 4E_0^2(t)E_0^2(t-\tau)\cos^2\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \\
&\quad + 4\left((E_0^2(t) + E_0^2(t-\tau)) E_0(t)E_0(t-\tau)\cos\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \right) dt \\
&= \int 2E_0^4(t) + 2E_0^2(t)E_0^2(t-\tau) + 2E_0^2(t)E_0^2(t-\tau) \\
&\quad + 2E_0^2(t)E_0^2(t-\tau)\cos\left(2\omega\tau + 2\phi(t) - 2\phi(t-\tau)\right) \\
&\quad + 4\left((E_0^2(t) + E_0^2(t-\tau)) E_0(t)E_0(t-\tau)\cos\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \right) dt \\
&= \int 2E_0^4(t) + 4E_0^2(t)E_0^2(t-\tau) \\
&\quad + 2E_0^2(t)E_0^2(t-\tau)\cos\left(2\omega\tau + 2\phi(t) - 2\phi(t-\tau)\right) \\
&\quad + 4\left((E_0^2(t) + E_0^2(t-\tau)) E_0(t)E_0(t-\tau)\cos\left(\omega\tau + \phi(t) - \phi(t-\tau)\right) \right) dt
\end{aligned}
$$

2. If the average sweep time τ is much faster than the detector time response the cosine terms are averaged

$$cos(a) \to 0$$

From 2.1 follows then:

$$
\begin{aligned}
I_2(\tau) &= \int (2E_0^4(t) + 4E_0^2(t)E_0^2(t-\tau))dt \\
&= 2\int I^2(t)dt + 4\int I(t)I(t-\tau)dt
\end{aligned}
$$

3. Interferometric autocorrelation:

$$
\begin{aligned}
I_2(\tau \to \infty) &= 2\int E_0^4(t)dt \\
I_2(\tau = 0) &= 2^4 \int E_0^4(t)dt
\end{aligned}
$$

The ratio is then

$$\frac{I_2(\tau \to \infty)}{I_2(\tau = 0)} = \frac{2\int E_0^4(t)dt}{2^4 \int E_0^4(t)dt} = \frac{1}{8}$$

Intensity autocorrelation:

$$
\begin{aligned}
I_2(\tau \to \infty) &= 2\int E_0^4(t)dt \\
I_2(\tau = 0) &= 6\int E_0^4(t)dt
\end{aligned}
$$

The ratio is then

$$\frac{I_2(\tau \to \infty)}{I_2(\tau = 0)} = \frac{2\int E_0^4(t)dt}{6\int E_0^4(t)dt} = \frac{1}{3}$$

10.3

The field correlator is defined as the convolution of two functions:

$$S(\tau) = \int_{-\infty}^{\infty} I(t) f(t-\tau) dt. \tag{1}$$

The Fourier transform of the convolution is the product of the Fourier transforms

$$\mathcal{F}\{S(\tau)\} = (\mathcal{F}\{I(t)\})^* \cdot \mathcal{F}\{f(t)\}, \tag{2}$$

$$f[\omega] = \mathcal{F}\{f(t)\} = \frac{\mathcal{F}\{S(\tau)\}}{(\mathcal{F}\{I(t)\})^*}. \tag{3}$$

$f(t)$ then is the inverse Fourier transform of $f[\omega]$.

10.4

The tilt angle α can be given as

$$\tan(\alpha) = \frac{\overline{EA'}}{D'} = \frac{\overline{AA'} - \overline{AE}}{D'} = \frac{\overline{AA'} - \overline{AE}}{D} \frac{\cos(\gamma)}{\cos(\gamma')}. \tag{4}$$

Snell's law gives

$$\sin(\gamma) = n \cdot \sin(\gamma'), \tag{5}$$

$$\cos(\gamma') = \frac{1}{n}\sqrt{n^2 - \sin(\gamma)^2}. \tag{6}$$

$\overline{AA'} - \overline{AE}$ can be expressed in terms of phase and group velocity

$$\overline{AA'} - \overline{AE} = T_{phase} \cdot v_p - T_{phase} \cdot v_g = \left(\frac{c}{n} - v_g\right) \frac{D}{c} \tan(\gamma), \tag{7}$$

express the group velocity using material dispersion

$$k = n(\omega)\frac{\omega}{c}, \tag{8}$$

$$v_g = \left(\left. \frac{dk}{d\omega} \right|_{\omega_l} \right)^{-1} = \frac{c}{n(\omega) + \omega n'(\omega)}. \tag{9}$$

With this

$$\tan(\alpha) = \frac{\omega_l n'(\omega_l)}{n(\omega_l) + \omega_l n'(\omega_l)} \frac{\sin(\gamma) \cdot n}{\sqrt{n^2 - \sin(\gamma)^2}}. \tag{10}$$

10.5

The simple idea of any clock is based on a stable source of periodic signal and some means to count, accumulate and display the "ticks" of such source. Appearing in 1949, atomic clocks added the third component to these two parts of a standard clock – a narrow-linewidth resonance of a certain atomic transition which is used to control the source (oscillator) frequency. It is known,

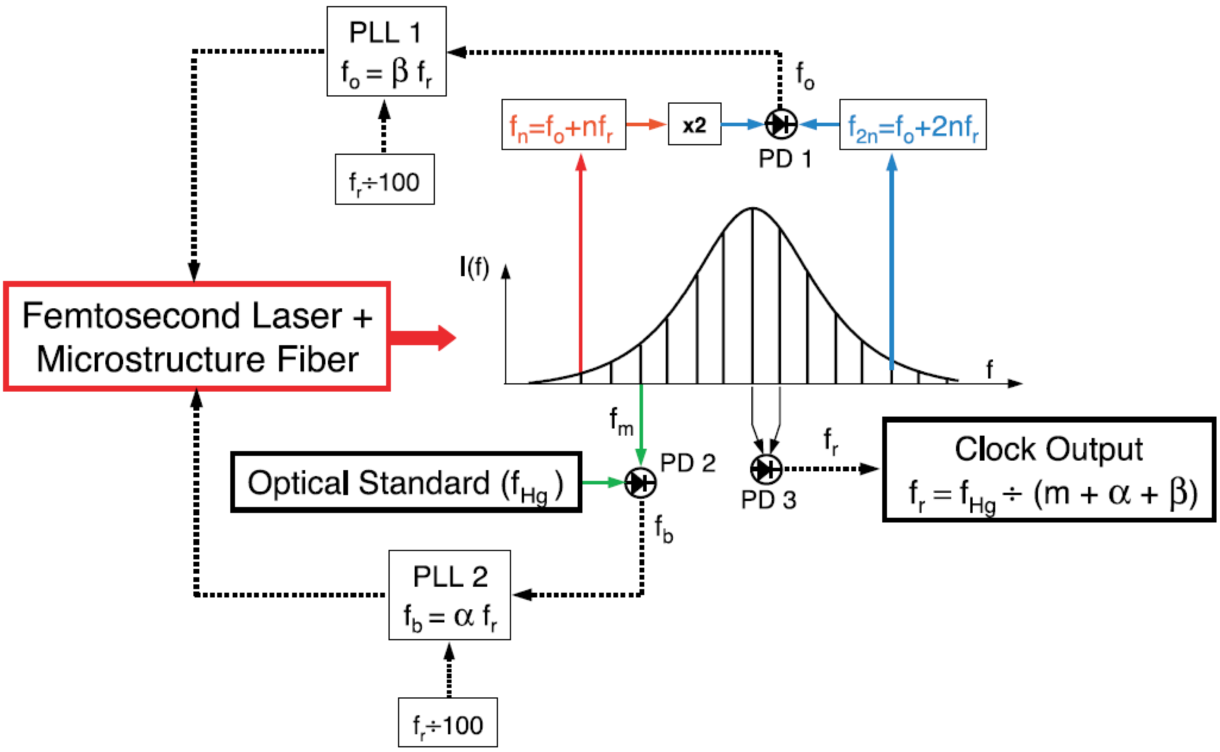

Figure 1: **Schematic of the self-referenced all-optical atomic clock. Solid lines represent optical beams, and dashed lines represent electrical paths. Photodiodes are designated by PD.**

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that for an oscillator locked to an atomic transition of frequency ν, the Allan deviation (a convenient measure of the clock instability) is inversely proportional to ν. For this reason having an atomic transition with frequency in the optical domain (optical standard) is much more preferable as it provides more stable clocks. The problem, however, with optical frequencies is that there is no electronic device which would be able to measure such fast oscillations. This problem was very difficult to solve until the invention of the optical frequency comb (OFC) – an optical source representing a set of equidistant frequencies f_m in the optical domain which satisfies the relation $f_m = f_r \cdot m + f_0$, where f_r is the repetition frequency, f_0 is the carrier offset frequency, and m is an integer number. OFC is able to provide a phase-coherent link between the microwave (f_r) and optical frequencies (f_m) and transfer the stability of the latter to the microwave domain. In order to employ an optical frequency comb for building an optical clocks, one can follow the route showed in the paper [1] (see Fig.1), where authors use two phase lock loops (PLL) to lock the carrier offset frequency f_0, obtained through the self-referencing of the comb and the beatnote frequency $f_b = f_{Hg} - f_m$ (where f_{Hg} is the frequency of 199Hg+ optical standard frequency and f_m - is the comb line closest to it), to the *integer ratios* of the repetition rate f_r: $f_0 = \alpha f_r$ and $f_b = \beta f_r$. Fixing α, β, m guarantees the phase-coherent link between the highly-stable optical standard f_{Hg} and the clock output f_r, which can be also be seen using the above mentioned equations for the PLLs and comb frequencies together: $f_{Hg} = (\alpha + \beta + m) \cdot f_r$. In summary, the main idea of the optical clock is to lock the optical frequency comb to the optical standard and eliminate one of its two "degrees of freedom" - f_0 - by referencing it to the comb's repetition rate, such that the repetition rate has a phase coherent link to the optical standard.

[1]. Diddams, Scott A., et al. "An optical clock based on a single trapped 199Hg+ ion." *Science* 293.5531 (2001): 825-828.

10.6

1. Solve the given equation in the Fourier domain by taking the Fourier transform of both parts:

$$i\frac{\partial \tilde{E}(z,\omega)}{\partial z} = \frac{\beta_2}{2}\omega^2 \tilde{E}(z,\omega)$$

Solving this partial differential equation one obtains:

$$\tilde{E}(\omega,z) = \tilde{E}(\omega,0)\exp\left(\frac{i}{2}\beta_2\omega^2 z\right)$$

2. Using the definition of the Fourier transform: $\tilde{E}(\omega,0) = \int_{-\infty}^{\infty} E(t,0)\exp(i\omega t)dt$, we can calculate $\tilde{E}(\omega,0)$ for the Gaussian pulse:

$$\tilde{E}(\omega,0) = t_0 \exp\left(-\frac{\omega^2 t_0^2}{2}\right)$$

Then using the inverse Fourier transform: $E(t,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\omega,0)\exp(-i\omega t)d\omega$, we can compute the pulse propagation in time the domain:

$$E(t,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t_0 \exp\left(-\frac{\omega^2 t_0^2}{2}\right)\exp\left(\frac{i}{2}\beta_2\omega^2 z\right)d\omega = \frac{t_0}{\sqrt{(t_0^2 - i\beta_2 z)}}\exp\left(\frac{t_0^2}{2(t_0^2 - i\beta_2 z)}\right)$$

3. From the last equation the duration of the pulse during the propagation changes as:

$$t_0'(z) = t_0\sqrt{\left(1 + \left(\frac{z}{L_D}\right)^2\right)}$$

where $L_D = t_0^2/|\beta_2|$. Recalling the link between $D(\lambda)$ and β_2: $D(\lambda) = -\frac{2\pi c}{\lambda^2}\beta_2$, we obtain pulse durations after 1m of SMF28: 920 fs for the 25-fs Gaussian pulse, and 250 fs for the 100-fs Gaussian pulse.