Laser: Theory and Modern Applications

Homework 5: Relaxation oscillations, Wiener Khinchine Theorem, Allan Deviation

Fall 2024

Questions

1. **Relaxation Oscillations:**

   The rate equations which describe a laser are fundamentally nonlinear, due to the stimulated emission process which depends on the density of excited state atoms N_0 and the light intensity in the cavity I_ν:

$$\frac{dI_\nu}{\mathrm{d}t} = c\sigma(\nu)N_2 I_\nu - c g_t I_\nu \tag{1}$$

$$\frac{dN_2}{dt} = -\sigma(\nu)N_2\frac{I_\nu}{h\nu} - \Gamma_{21}N_2 + K_2 \tag{2}$$

   We have defined a small perturbation to the steady state values $\varepsilon(t)$, $\eta(t)$:

$$I_\nu(t) = \overline{I_\nu} + \varepsilon(t)$$

   where

$$|\varepsilon(t)| << \overline{I_\nu}$$

   and

$$N_2(t) = \overline{N_2} + \eta(t)$$

   where

$$|\eta(t) << \overline{N_2}$$

   We have derived the rate of change of the perturbation of the light intensity $\varepsilon(t)$ starting from equation (1):

$$\frac{d\varepsilon(t)}{\mathrm{d}t} = c\sigma\overline{I_\nu}\eta(t) \tag{3}$$

   (a) Derive a relation for the rate of change of the perturbation of the density of atom in the excited state $\eta(t)$, starting from equation (2).

(b) Combine equation (3) with the relation derived in (a) to find a second order differential equation for $\varepsilon(t)$

$$\frac{d^2\varepsilon(t)}{\mathrm{d}t^2} + \frac{d\varepsilon(t)}{\mathrm{d}t}\frac{\sigma(\nu)K_2}{g_t} + \frac{g_t c\sigma\overline{I_\nu}\varepsilon(t)}{h\nu} = 0$$

Verify that the solution to this equation is $\varepsilon(t) = Ae^{-\gamma t/2}cos(\omega t + \phi)$ with $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

(c) Now, we would like to modulate the pump power: $K_2(t) = \overline{K_2} + \kappa_0 e^{i\Omega t}$ Starting from equation (2) derive the following relation:

$$\frac{d\eta(t)}{\mathrm{d}t} = -\frac{g_t\varepsilon(t)}{h\nu} - \frac{\sigma(\nu)\overline{K_2}\eta(t)}{g_t} + \kappa_0 e^{i\Omega t}$$

(d) From equation (c), derive the following expression:

$$\frac{d^2\varepsilon(t)}{\mathrm{d}t^2} + \frac{d\varepsilon(t)}{\mathrm{d}t}\frac{\sigma(\nu)K_2}{g_t} + \frac{g_t c\sigma(\nu)\overline{I_\nu \varepsilon t}}{h\nu} = c\sigma(\nu)\kappa_0\overline{I_\nu}e^{i\Omega t}$$

(e) Assume that the perturbation of the light intensity also varies at the same modulation frequency i.e. $\varepsilon(t) = i_0(\Omega)e^{i\Omega t}$. Derive the amplitude frequency response $i_0(\Omega) = \frac{\kappa_0'}{(\omega_0^2 - \Omega^2 + i\gamma\Omega)}$ by using equation (d), plot the modulus square as a function of modulation frequency.

Bonus (f) Assume that the perturbation of the density of atom in excited state also caries at the same modulation frequency i.e $\eta(t) = \eta_0(\Omega)e^{i\Omega t}$. Derive the amplitude frequency response $\eta_0(\Omega) = \frac{1}{c\sigma(\nu)\overline{I_\nu}}\frac{i\kappa_0'\Omega}{(\omega_0^2 - \Omega^2) + i\gamma\Omega}$. Plot the modulus square as a function of modulation frequency.

What does this result mean practically, when modulating the output laser power of a laser via modulation of the inversion?

2. **Modulation bandwidth of a GaAs/GaAlAs Laser diode:**

The plot of the laser amplitude response to the period modulation of the inversion in the last question shows that only for frequencies below the relaxation oscillation frequency can the laser cavity respond. A unique aspect of a diode laser is that the inversion can be directly modulated by varying the pump current entering the semiconductor p-n junction gain region.

• Calculate the expected modulation bandwidth (i.e. the relaxation oscillation frequency) of a diode laser. For this assume that the laser is emitting a power of $P = 5 \cdot 10^{-3} W$ at $850nm$ with a cross section of the beam inside the $L = 120\mu m$ long cavity od $3\mu m \times 0.1\mu m$ and has material losses given by the attenuation coefficient of $30cm^{-1}$ The laser cavity has one partially reflecting facet (mirror) which has a reflectivity of

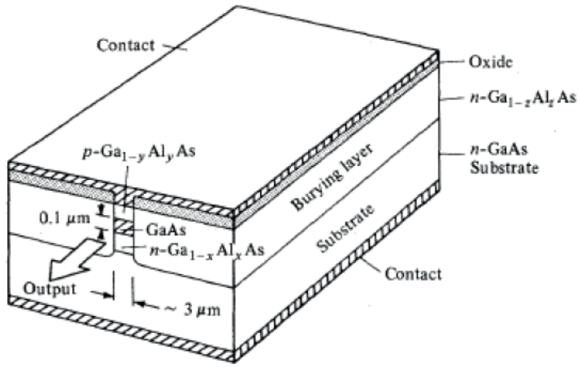

Figure 1: A buried heterostructure laser.

$R = 0.3$ (i.e 70% transmission) and and the laser gain medium a refractive index of $n = 3$. The upper state lifetime of the inverted population in the gain region is short, only $\tau = (\tau_{21}^{-1}) = 4 \cdot 10^{-9} sec.$

3. **1st order autocorrelation measurements and the Wiener Khinchine Theorem:** Introduction: If one has an laser or light source, an important characterization is the optical spectrum $S(\omega)$. The spectrum describes the distribution of the power as a function of frequency, i.e. the quantity $P = S(\omega)d\omega$ describes the average power contained in the frequency interval $[\omega, \omega + d\omega]$. A laser has a very narrow spectrum with a single line, or it can oscillate multimode exhibiting several peaks. It is therefore central in many applications to measure the spectrum of the laser experimentally. Mathematically the spectrum is related to the Fourier transform of the electric field, via the proportionality relation $S(\omega) \propto |E(\omega)|^2$. [1] powerful and simple method to measure the spectrum is a Fourier transform analyzer (see figure 2). In this device, an unknown beam (whose spectrum is to be characterized) is send into an interferometer. The two arms of the interferometer are allowed to interfere, and the interference is detected with a broadband photodetector. If the two arms differ by a length δl then the signal at the detector is given by $I(\tau) \propto \langle |E(t) + E(t-\tau)|^2 \rangle$ where $\tau = \frac{2\delta l}{c}$, c is the speed of light and $\langle ... \rangle$ denotes the temporal average, i.e. $\langle ... \rangle = \frac{1}{T}\int_0^T ...\mathrm{d}t$. This signal includes a correlation term $C(\tau) = \int E(t) \cdot E^*(t-\tau))\mathrm{d}t$.

   (a) Proof, that if you record the function $I(\tau)$ and therefore $C(\tau)$ by varying the delay length δl (and thereby the delay time τ), you can determine the optical spectrum of the light. Concretely show that the following relation holds true:

$$S(\omega) = \int C(\tau)\, e^{-i\omega\tau}\mathrm{d}\tau \tag{4}$$

[01] Where $E(\omega)$ is the Fourier transform of the light field, i.e.$E(\omega) = \int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\mathrm{d}t$

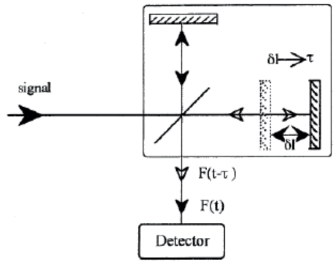

Figure 2: Fourier transform analyzer.

For $C(\tau)$ being a pure e autocorrelation, this relation is known as the Wiener-Khintchine theorem and among the most important theorems in signal processing. It is of practical importance since one can construct a spectrograph using this principle with a simple interferometer (with a variable arm) and a detector. [2]

(b) What is the signal $C(t)$ produced at the detector as a function δl, when the signal inserted into interferometer is given by a single laser of frequency ω_1?[Hint: use complex notation for the fields to simplify the calculation]. What happens if you Fourier transform $C(t)$?

(c) Assume a multimode laser (oscillating on two modes with frequency ω_1 and ω_2 with equal amplitudes). What is the autocorrelation function $C(t)$? What is the periodicity of the two signals?

(d) Assume that you have a laser whose correlation function is exponentially decaying $C(t) = E_0^2 \cos(\omega_0 \tau)\, e^{-\tau/\tau_c}$. What is the spectrum of the laser in this case? Show that the spectrum is Lorenzian and has a linewidth $\delta\omega = \frac{1}{tc}$.

4. **Allan Deviation:**

The Allan deviation is a measure for the frequency stability of an oscillator and is used to characterize the type of frequency noise as a function of time. It is defined via the relation:

$$\sigma_\nu^2 = \frac{1}{2}\langle(\overline{\nu}_2 - \overline{\nu}_1)^2\rangle$$

Here $\overline{\nu}_{1,2}$ denotes the average (over a time τ) frequency in two consecutive time intervals ($t = 0...\tau$ and $t = \tau...2\tau$). The brackets $\langle...\rangle$ denote the measurement of many of such consecutive time intervals. Thus the Allan deviation is similar to the standard deviation, with the important difference that the the deviations to adjacent intervals are considered and not those from the mean value of the entire measurement time.[3]

[02] Hint: use the fact that $lim_{T->\infty} = \frac{1}{T}\int e^{i\omega t}\mathrm{d}t = 2\pi\delta(\omega)$.
[03] The standard deviation in contrast, is defined as : $\sigma^2 = 1/N \cdot \sum_{i=1...N}(\overline{\nu} - \overline{\nu_i})^2$

- What is the Allan deviation for a frequency that experiences linear drift, i.e. $\nu(t) = \nu_0 + at$, where t denotes time? Show that for such a signal the Alla Deviation is given by:

$$\sigma_\nu(\tau) = \frac{a\tau}{\sqrt{2}}$$