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Laser : Theory and Modern Applications

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Homework No. 4 - Solution

4.1 ***Optimal output power of a laser*** For a laser cavity there is a trade-off between the reflectivity of the mirror and the amount of output power possible, high reflectivity will lead to low threshold but low output power, while low reflectivity will increase the pump threshold. Assume that the mirror has a scattering loss (s) and a transmission (t), that needs to be optimized. Here we want to derive the optimal transmission (t_{opt}) and the maximum possible output intensity I_ν^{out}.

We know from the class that $I_{\text{out}} = \frac{1}{2} I_{\text{sat}} t \left(\frac{2g_0(\nu)l}{t+s} - 1 \right)$ and by a setting the derivative to zero one can find the optimum value for transmission.

$$\frac{dI}{dt} = 0 \Rightarrow t_{\text{opt}} = \sqrt{2g_0(\nu)ls} - s \tag{1}$$

so the maximum output power is easily calculated :

$$I_{\text{out}} = I_{\text{sat}} \left[\sqrt{g_0(\nu)l} - \sqrt{s/2} \right]^2 \tag{2}$$

4.2 ***Laser output power with spontaneous emission*** The equation for the intracavity intensity I_ν assuming, for brevity, that the gain medium is filling the entire laser resonator

$$\frac{dI_\nu}{dt} = c\sigma(\nu)(N_2 - N_1)I_\nu - \frac{c}{2L}(1 - r_1 r_2)I_\nu, \tag{3}$$

where $\sigma(\nu)$ is the absorption cross-section, $N_{1,2}$ are the populations of the excited and the ground states of the medium, respectively, $r_{1,2}$ — transmittivities of the resonator mirrors and L — resonator (and the gain medium) length. Then relate the intracavity intensity to the intracavity photon number n_ν as

$$I_\nu = \frac{h\,\nu\,c}{2L\,S} n_\nu, \tag{4}$$

where h is the Plank constant, ν is the photon frequency and S is the mode cros-section inside the laser. In the left-hand side of the Eq. 3 the term $c\sigma(\nu)N_2 I_\nu$ is responsible for the light emission inside the cavity, so in terms of n_ν and accounting for the spontaneous process it will have a counterpart $c\sigma(\nu)N_2(n_\nu + 1)$. Thus, the equation for the intracavity photon number is

$$\frac{dn_\nu}{dt} = c\sigma(\nu)\Delta N\, n_\nu - \frac{c}{2L}(1 - r_1 r_2)n_\nu + c\sigma(\nu)N_2, \tag{5}$$

where $\Delta N = (N_2 - N_1)$.

To calculate the steady-state $\bar{n}_\nu$ far below threshold we treat $\sigma(\nu)$ as a power-independent constant and $\sigma_0(\nu)$ and set $dn_\nu/dt = 0$. We find

$$\bar{n}_\nu = \frac{c\sigma_0(\nu)N_2}{\frac{c}{2L}(1-r_1r_2) - c\sigma_0(\nu)\Delta N}. \tag{6}$$

Accounting for the gain saturation can be done by setting

$$\sigma(\nu) = \frac{\sigma_0(\nu)}{1 + \bar{n}_\nu/\bar{n}_{\text{sat}}}, \tag{7}$$

which yield the equation for the steady-state intracavity photon number, which is quadratic in $\bar{n}_\nu$

$$\left(\frac{1}{2L}(1-r_1r_2)\left(1+\frac{\bar{n}_\nu}{\bar{n}_{\text{sat}}}\right) - \sigma_0(\nu)\Delta N\right)\bar{n}_\nu = \sigma_0(\nu)N_2. \tag{8}$$

To plot the solution of the Eq. 8, we make the simplifying assumptions that the population of the ground level N_1 is always 0 ($\Delta N = N_2$) and that ΔN is proportional to pumping power P ($2L\sigma_0(\nu)\Delta N/(1-r_1r_2) = \alpha P$). Then

$$\left(\frac{1}{\alpha P}\left(1+\frac{\bar{n}_\nu}{\bar{n}_{\text{sat}}}\right) - 1\right)\bar{n}_\nu = 1. \tag{9}$$

Assuming the numerical parameters to be $\bar{n}_{\text{sat}} = 10^4$, $\alpha = 1$, we get the result displayed at the Fig. **??**.

4.3 ***Absorption of a single atom*** [1 2] Even a single atom can absorb all incident light of a laser beam. This counter-intuitive situation can be explained via interference of the atomic dipole emission and incoming light. This was experimentally demonstrated and analyzed. Explain the conditions under which a single atom can absorb all the incident light. In principle when shining light on an atom one can consider the interference between the residual input interacting with the emitted signal from the atom and as a result the transmitted signal is a sum of two vectors that can interfere destructive or instructively.

$$E_{\text{out}} = E_{\text{in}} + E_{\text{em}} \tag{10}$$

Where E_{em} is the emitted signal from the atom. Therefore, one can see that for the intensity at the detector we get :

$$I = |E_{\text{in}}|^2 + |E_{\text{em}}|^2 + 2Re[E_{\text{in}}E^*_{\text{em}}] \tag{11}$$

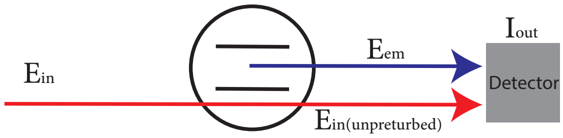

FIGURE 1 – Schematic of an atom in interaction with incident light field

The cross term can lead to destructive interference in the forward direction and this means that effectively all your light is reflected back from the atom. In more details one could write the equations of motion for atomic and field operators to model the the light field interaction with the atom [3] :

$$d\sigma^-/dt = -i\omega\sigma^- \frac{\Gamma}{2}\sigma^- + \sqrt{\Gamma}\sigma_z b_{\text{in}}(t) \tag{12}$$

$$d\sigma_z/dt = -\Gamma\cdot(1+\sigma_z)\sigma^- - 2\sqrt{\Gamma}\sigma^+ b_{\text{in}}(t) - 2\sqrt{\Gamma}\sigma^- b^\dagger_{\text{in}}(t) \tag{13}$$

where b is the field operator and σs are the atom operators. The input output relation for this system could be written as :

$$b_{\text{out}} = b_{\text{in}} - \sqrt{\Gamma}\sigma^- \tag{14}$$

which shows the origin of the interference picture that was introduced earlier. Here one can see that under some circumstances the output field can be totally canceled due to the interference of the input field and the emitted light (that is due to the atomic transitions σ^-).

It can be shown that this full cancellation of fields is possible only for a three level system (another atomic transition) [4] and the output field will be in the form of :

$$E_{\text{out}} = \beta - 2\beta\frac{\Gamma_{31}}{\Gamma_{31}+\Gamma_{32}} \tag{15}$$

where β is proportional to the input field (E_{in}). Here we can see that in the case of a two level system ($\Gamma_{32} = 0$) that there only is two transitions this equation cannot go to zero but in case of a third level transition being equal to the transition rate of the other, this can go to zero ($\Gamma_{32} = \Gamma_{31}$).

One important remark is that when shining a plane wave to an atom this does not mean that this picture holds and we have a very efficient interaction. In this case the dipole radiations of the atom is in all 4π possible directions in space so we need to mode match the incident light to this radiation and this is done by manipulating the

1. PRL 98.3 (2007) Strong extinction of a laser beam by a single molecule
2. PRA 69.4 (2004) Atoms, dipole waves, and strongly focused light beams.
3. PRA 69.4 (2004) Atoms, dipole waves, and strongly focused light beams.
4. PRL 100.9 (2008) Single photon absorption by a single quantum emitter.

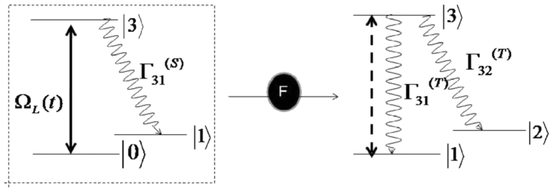

FIGURE 2 – Schematic of the atomic transitions

dipole radiations and enhancing the field using plasmonic antennas or in the case of this experiment aluminum fiber tip. This means that the radiations of the atom will be guided to a preferred direction that is enhanced via phase matching of the interacting light and the emitting atom.

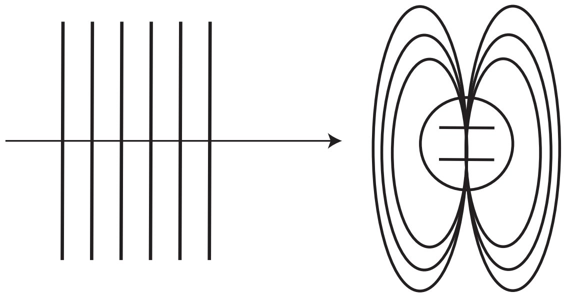

FIGURE 3 – Dipole radiations of an atom vs the planar excitation