Laser: Theory and Modern Applications

Homework 5: Relaxation oscillations, Wiener Khinchine Theorem, Allan Deviation

Fall 2023

Solutions

1. **Relaxation Oscillations:**

a) Relaxation Oscillations The nonlinear differential equation system is derived in chapter 7.2.2 of the script. It is a simplification for a lasing system with strong population inversion ($N_2 \gg N_1$) where N_2 and N_1 are populations per unit volume (unit m^{-3}. Thus also the lower pumping rate is negligible and only the upper level pumping rate K (unit $\mathrm{m}^{-3}\ \mathrm{s}^{-1}$) remains. The cavity loss coefficient $\alpha = cg_t$ (unit s^{-1}) accounts mainly for the mirror losses and $\sigma(v)$ (unit m^2) is the absorption cross section.

As there is no analytical solution to this nonlinear equation system, it is analyzed by linearization around a working point that is a solution of the steady state equations $\left(\frac{\mathrm{d}}{\mathrm{dt}} \ldots = 0\right)$:

$$
\begin{aligned}
0 &= -\sigma(v)\bar{N}_2\frac{\bar{I}_0}{hv} - \Gamma_{21}\bar{N}_2 + K_2 \\
0 &= +c\sigma(v)\bar{N}_2\bar{I}_0 - \alpha\bar{I}_0
\end{aligned}
$$

From (2) the steady state upper level population $\bar{N}_2$ can be derived, which plugged into (1) yields the steady state intensity I_0 :

$$
\begin{aligned}
\bar{N}_2 &= \frac{\alpha}{c\sigma} \\
\bar{I}_0 &= \frac{-\Gamma_{21}hv}{\sigma} + \frac{cK_2hv}{\alpha}
\end{aligned}
$$

Next the differential equation for the upper level population density is rewritten for small deviations η about the stable, steady state working point just calculated:

$$
\begin{aligned}
\frac{d}{dt}N_2 &= \frac{d}{dt}\left(\bar{N}_2+\eta\right) = \frac{d}{dt}\left(\eta\right) \\
&= \frac{-\sigma(v)}{hv}\left(\bar{N}_2+\eta\right)\left(\bar{I}_0+\varepsilon\right) - \Gamma_{21}\left(\bar{N}_2+\eta\right) + K_2 \\
&= \frac{-\sigma(v)}{hv}\left(\bar{N}_2\bar{I}_0+\eta\bar{I}_0+\varepsilon\bar{N}_2+\ldots\right) - \Gamma_{21}\left(\bar{N}_2+\eta\right) + K_2 \\
&= -\frac{\sigma(v)}{hv}\left(\left(\bar{I}_0+\Gamma_{21}\right)\eta - \bar{N}_2\varepsilon\right) \\
\implies \frac{d}{dt}\left(\eta\right) &= -\frac{\sigma c K_2}{\alpha}\eta - \frac{\alpha}{chv}\varepsilon
\end{aligned}
$$

In the upper calculation we have used (1) to cancel $\frac{-\sigma(v)}{hv}\bar{N}_2\bar{I}_0$ and $-\Gamma_{21}\bar{N}_2 + K_2$ and substituted $\bar{N}_2$ and $\bar{I}_0$ with (3) and (4). For the intra cavity intensity the differential equation changes to:

$$
\begin{aligned}
\frac{d}{dt}I &= \frac{d}{dt}\left(\bar{I}_0+\varepsilon\right) = \frac{d}{dt}(\varepsilon) \\
&= \sigma(v)N_2 I - \alpha I \\
&= c\sigma(v)\left(\bar{N}_2\bar{I}_0 + \delta N\bar{I}_0 + \varepsilon\bar{N}_0 + \ldots\right) - \alpha\left(\bar{I}_0+\varepsilon\right) \\
&= \left(c\sigma(v)\bar{N}_2 - \alpha\right)\varepsilon + \left(c\sigma(v)\bar{I}_0\right)\eta \\
\implies \frac{d}{dt}(\varepsilon) &= \left(c\sigma\bar{I}_0\right)\eta \Leftrightarrow \eta = \frac{1}{c\sigma\bar{I}_0}\frac{d}{dt}(\varepsilon)
\end{aligned}
$$

As can be seen neglecting of higher order term effectively turns the nonlinear differential equation system into a linear differential equation system.

b) In order to show that the small signal behavior of the laser is essentially the same as a damped harmonic oscillator the last expression of (6) is plugged into (5):

$$
\begin{gathered}
\frac{d}{dt}\left(\frac{1}{c\sigma\bar{I}_0}\frac{d}{dt}(\varepsilon)\right) = -\left(\frac{\sigma c K_2}{\alpha}\right)\left(\frac{1}{c\sigma\bar{I}_0}\frac{d}{dt}(\varepsilon)\right) - \left(\frac{\alpha}{chv}\right)\varepsilon \\
\implies \frac{d^2}{dt^2}(\varepsilon) + \gamma\frac{d}{dt}(\varepsilon) + \omega_0^2\varepsilon = 0 \\
\text{with: } \gamma = \frac{\sigma c K_2}{\alpha} \text{ and } \omega_0^2 = \frac{\alpha\sigma I_0}{hv}
\end{gathered}
$$

The same result can be obtained for the small deviations of the population density η with the same values for γ and ω_0^2.

c,d) In order to introduce the small signal - a harmonic pumping term with frequency Ω-, one realizes that the pumping term K_2 in the formulae above was just a constant

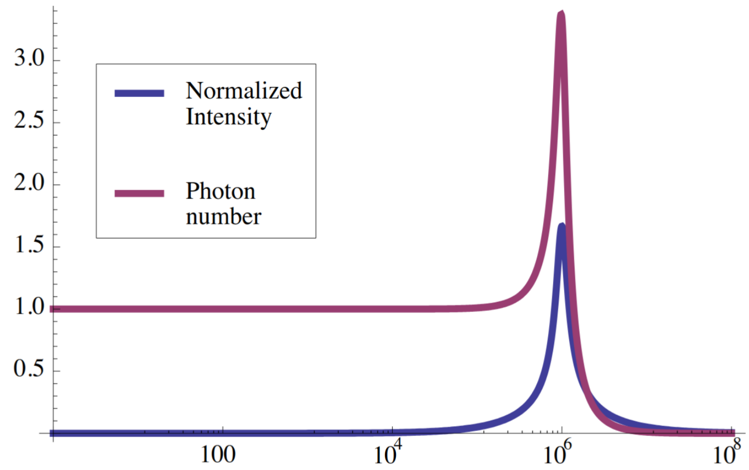

Figure 1: Figure 1: Normalized intensity and photon number versus frequency for an intrinsic frequency of $\omega_0 = 1$MHz and bandwidth of $\gamma = 300$kHz

pump. Adding a small pump modulation term is the same as turning the damped oscillator equations we found before into harmonically driven, damped oscillator equations. We can therefore write (using the same γ and ω_0) :

$$\frac{d^2\varepsilon}{dt^2} + \gamma\frac{d\varepsilon}{dt} + \omega_0^2\varepsilon = \kappa_0 e^{i\Omega t} \Longrightarrow e^{i\Omega t}\left(-\Omega^2\iota_0 + i\Omega\gamma\iota_0 + \omega_0^2\iota_0\right) = \kappa_0 e^{i\Omega t}$$

e) Substituting $\eta(t) = \eta_0 e^{i\Omega t}$ and $\varepsilon(t) = \iota_0 e^{i\Omega t}$ into their according now driven, damped harmonic oscillator equations, and after executing the simple temporal derivatives, the equations have been transformed into the Fourier domain and read:

$$\eta_0(\Omega) = \frac{1}{c\sigma\bar{I}_0}\frac{i\Omega\kappa_0}{(\omega_0^2 - \Omega^2) + i\gamma\Omega}$$
$$\iota_0(\Omega) = \frac{\kappa_0}{(\omega_0^2 - \Omega^2) + i\gamma\Omega}$$

Looking at Fig. 1 the response of the laser system to fast pump modulation can be understood. The laser system can only respond to low frequencies (frequencies lower than ω_0) and the response drops down dramatically for modulation frequencies (Ω) higher than ω_0. This is due to the intrinsic "relaxation oscillation frequency" of the laser (ω_0), which depends on the intrinsic time constants of the laser system. If the frequency of the pump modulation is higher than ω_0, the laser can not follow it and the laser output power remains constant (I_0) even with modulated pump power.

Note: the relationship between the relaxation frequency of a laser and the physical dimensions can be easily seen from Eq. 9.: $\omega_0^2 = \frac{\alpha\sigma I_0}{hv}$, where α is the loss rate (i.e. the loss in one round trip of the light bouncing in the cavity). This can be approximated by $\alpha = c\frac{1-r_1r_2}{2L}$ where r_1, r_2 are the reflectivity of the mirrors and L is the length of the cavity. It is clear that the smaller the cavity, the faster the response is. For example, small diode lasers can be used for GHz range communications, while the relaxation oscillation for a He-Ne laser in the order of kHz

f) ...

2. **Modulation bandwidth of a GaAs/GaAlAs Laser diode:**

According to the result of the previous question first we will calculate the intensity of the laser. For simplification we assume that the power is uniformly distributed in the cavity aperture, then we would have:

$$I_0 = \frac{P}{A} = \frac{5 \times 10^{-3}\ \mathrm{W}}{3 \times 10^{-6} \cdot 1 \times 10^{-7}\ \mathrm{m}^2} = 1.66 \times 10^{10} \frac{\mathrm{W}}{\mathrm{m}^2}$$

Next the total intensity loss rate α is calculated which is equal to the loss in the material and at the mirrors. Therefore the total loss rate would be:

$$\begin{aligned}\alpha = \alpha_{\text{material}} + \alpha_{\text{mirror}} &= c/n \cdot \text{loss}_{\text{material}} + \frac{c\,(1 - r_1 r_2)}{2Ln} \\ &= \frac{3 \times 10^8\,\mathrm{m\,s^{-1}}}{3} \cdot 3000 + \frac{3 \times 10^8\,\mathrm{m\,s^{-1}}(1 - 0.3)}{2 \cdot 120\,\mu\mathrm{m} \cdot 3} = 5.91 \times 10^{11}\,\mathrm{s}^{-1}\end{aligned}$$

The third step is to calculate the absorption cross section in the material. According to the first lecture the cross section in the lasing frequency (v_0) would be:

$$\sigma\,(v_0) = \frac{(\lambda/n)^2}{4\pi} = \frac{\left(850 \times 10^{-9}/3\right)^2}{4\pi} = 6.38 \times 10^{-15}\ \mathrm{m}^2$$

Now we can calculate the relocation oscillation frequency of Ga/As diode laser:

$$\begin{aligned}\omega_0 &= \sqrt{\frac{\alpha\sigma I_0}{h\nu}} = \sqrt{\frac{\alpha\sigma I_0 \lambda}{hcn}} = 9.44 \times 10^{12} \mathrm{rad} \cdot \mathrm{s}^{-1} \\ &\implies v_0 = \frac{\omega_0}{2\pi} = 1.5\mathrm{THz}\end{aligned}$$

3. **1st order autocorrelation measurements and the Wiener Khinchine Theorem:**

In the first step we will show the relation between $I(\tau)$ and $C(\tau)$:

$$|E(t) + E(t-\tau)|^2 = E(t) \cdot E^*(t) + E(t-\tau) \cdot E^*(t-\tau) + E(t) \cdot E^*(t-\tau) + E(t-\tau) \cdot E^*(t) \tag{1}$$

$$= 2|E|^2 + E(t) \cdot E^*(t-\tau) + E(t-\tau) \cdot E^*(t) \quad \text{(E is a real signal)} \tag{2}$$

$$= 2|E|^2 + 2E(t) \cdot E^*(t-\tau) \tag{3}$$

Therefore :

$$I(\tau) = 2\left\langle |E|^2 \right\rangle + 2\left\langle E(t)\cdot E^*(t-\tau)\right\rangle = 2\bar{P}_{avr} + 2C(\tau)$$

However we should note that the E used in Eq. (13) is the electric field in each arm which is $E_{\text{arm}} = \frac{1}{\sqrt{2}}E_{\text{laser}}$ because in the beam splitter you divide the power equally between two arm. $\left(P_{\text{arm}} = \frac{1}{2}P_{\text{laser}}\right)$. Therefore the modified equation of (13) for laser itself would be:

$$I(\tau) = \bar{P}_{avr} + C(\tau)$$

The Eq. (14) simply means that $I(\tau)$ is the sum of "Auto Correlation Function of Laser" and "Laser Average Power" (Which is constant). As a result If one measure the the laser average power separately, he can reduce it from Eq. (14) and calculate the Auto Correlation Function of laser light by varying one of the arm lengths respect to the other (Which represents varying τ) The next step is to prove the Wiener-Khintchine and show how one can find spectral density out of Auto Correlation Function:

$$\int C(\tau)e^{-i\omega\tau}d\tau = \int \left\langle E(t)\cdot E^*(t-\tau)\right\rangle e^{-i\omega\tau}d\tau \tag{4}$$

$$= \iint E(t)\cdot E^*(t-\tau)dte^{-i\omega\tau}d\tau \tag{5}$$

$$\tag{6}$$

Introduce a variable change $u = t-\tau$:

$$= \iint E(t)\cdot E^*(u)e^{-i\omega u}e^{-i\omega t}dtdu \tag{7}$$

$$= \int E^*(u)e^{-i\omega u}du \cdot \int E(t)e^{-i\omega t}dt \tag{8}$$

$$= E^*(\omega)\cdot E(\omega) = |E(\omega)|^2 = S(\omega) \tag{9}$$

The meaning of Eq. (15) is that one can simply calculate the spectral density from Auto Correlation Function with simply taking a "Fourier Transform" from $C(\tau)$. The next step is to compute a multimode laser auto correlation function (for simplification one can imagine a laser with two modes ω_1 and ω_2) then the phasor of electrical field would be: $E(t) = E_1e^{-i\omega_1 t} + E_2e^{-i\omega_2 t}$. Therefore we would have:

$$\begin{aligned} C(\tau) &= \left\langle \left(E_1e^{-i\omega_1 t} + E_2e^{-i\omega_2 t}\right)\left(E_1^*e^{+i\omega_1(t-\tau)} + E_2^*e^{+i\omega_2(t-\tau)}\right)\right\rangle_t \\ &= \left\langle |E_1|^2 e^{-i\omega_1\tau} + |E_2|^2 e^{-i\omega_2\tau} + E_1E_2^*e^{-i(\omega_2-\omega_1)t}e^{-i\omega_2\tau} + E_2E_1^*e^{+i(\omega_2-\omega_1)t}e^{-i\omega_1\tau}\right\rangle_t \\ &= |E_1|^2 e^{-i\omega_1\tau} + |E_2|^2 e^{-i\omega_2\tau} \end{aligned}$$

and the spectral density would be:

$$S(\omega)=\left|E_1\right|^2 \delta\left(\omega-\omega_1\right)+\left|E_2\right|^2 \delta\left(\omega-\omega_2\right)$$

which means two peaks one in ω_1 and one in ω_2 And in the very last step we calculate the spectral density of this signal: $C(\tau)=E_0^2 \cos\left(\omega_0 \tau\right) e^{-\tau / \tau_c}$ Recall that the Fourier Transform of $e^{-a|\tau|}$ is:

$$\mathcal{F}\left\{e^{-a|\tau|}\right\}=\frac{a}{a^2+\omega^2}$$

Then for calculating the spectral density we should keep in mind that the $\cos\left(\omega_0 \tau\right)$ in $C(\tau)$ will only cause a frequency shift in Fourier Domain and would centralize the Eq. (18) around ω_0 instead of 0 . Therefore the spectral density of $C(\tau)$ would be:

$$S(\omega)=\mathcal{F}\{C(\tau)\}=\left|E_0\right|^2 \frac{1 / \tau_c}{\left(1 / \tau_c\right)^2+\left(\omega-\omega_0\right)^2}$$

which is Lorenzian line shape with linewidth of $\delta \omega=1 / \tau_c$

4. **Allan Deviation:**

In the first step we calculate the average frequency in the n^{th} time interval $(\bar{v}_n)$ In the first step we calculate the average frequency in the n^{th} time interval $(\bar{v}_n)$

$$\bar{v}_n=\langle v(t)\rangle_t=v_0+\frac{1}{\tau} \int_{n \tau}^{(n+1) \tau} \text{ at } d t=v_0+a \tau\left(n+\frac{1}{2}\right)$$

Therefore the variation between to time interval would be:

$$\bar{v}_{n+1}-\bar{v}_n=a \tau$$

As a result, the Allan Deviation of this clock would be:

$$\sigma_v(\tau)=\sqrt{\frac{1}{2}\left\langle\left(\bar{v}_{n+1}-\bar{v}_n\right)^2\right\rangle_n}=\frac{a}{\sqrt{2}} \tau$$