
Notes

Wolf Transform

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MICRO-421

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1 Context

When we image 3D objects with 2D sensors we are inevitably forced to lose part of the information. We should therefore introduce new degrees of freedom to compensate this mismatch. Tomography (from the ancient greek *tomos* "slice") is a technique to recover slice by slice the 3D morphology of the sample under observation. However, in presence of diffraction, the common algorithm used in computer tomography, as Radon transform which are based on geometrical optics, fail and a more accurate treatment is required.

Fig. 1 presents the problem we want to study in a 2D word for the sake of simplicity. Usually when we image an object $n(\mathbf{r})$, we illuminate it with a beam u_0 from an emission plane (the dashed line on the left, in this case one-dimensional) and we measure its response on a detection plane (the dashed line on the right). The quantity that we measure with a camera is typically the intensity of the total field $|u_0 + u_s|$ where u_s is the scattered field from the object. With a single illumination we will record on the detector the 1D projection of our 2D object. However, to retrieve its full distribution we could introduce as an additional degree of freedom the illumination angle θ_0 . If we turn around the object to illuminate it from different θ_0 and collect for each of them the complex valued scattered field in all the directions θ (meaning having a detector along the solid circular line) we could fully reconstruct $n(\mathbf{r})$. Starting from the wave equation we will show here below the principles of optical diffraction tomography.

2 Wave equation

The optical scattering problem in Fig. 1 is described by Helmholtz equation:

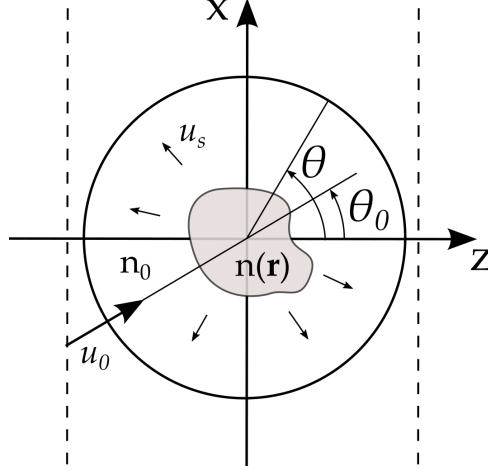


Figure 1: Optical system for optical diffraction tomography

$$(\nabla^2 + k_0^2 n^2(\mathbf{r})) u_t(\mathbf{r}) = 0 \quad (1)$$

where $k_0 = \omega/c = 2\pi/\lambda$ is the vacuum wavenumber and $u_t = u_0 + u_s$. Similarly, if we remove the object, we can write a similar equation for the incident field u_0 :

$$(\nabla^2 + k_0^2 n_0^2) u_0(\mathbf{r}) = 0 \quad (2)$$

By subtracting the two we have the equation for the scattered field:

$$(\nabla^2 + k_0^2 n_0^2) u_s(\mathbf{r}) = -4\pi F(\mathbf{r}) u_t(\mathbf{r}) \quad (3)$$

where $F(\mathbf{r}) = \frac{k_0^2}{4\pi} (n^2(\mathbf{r}) - n_0^2)$ is called the scattering potential and contains all the information on the refractive index of the object. Instead of having a source at infinity and a localized object, solving (3) equals to computing the electromagnetic field u_s in a uniform medium n_0 in the presence of an equivalent localized source $-4\pi F(\mathbf{r}) u_t(\mathbf{r})$. In turn, the solution can be found resorting to the Green function of the Helmholtz operator in the background medium where $k = k_0 n_0$

$$(\nabla^2 + k^2) G(\mathbf{r} - \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}') \quad (4)$$

and in turn find the scattered field

$$u_s(\mathbf{r}) = \int_{\Omega} F(\mathbf{r}') G(\mathbf{r} - \mathbf{r}') u_t(\mathbf{r}') d\mathbf{r}' \quad (5)$$

where Ω is a domain enclosing the scattering object. Equation (5) is called the Lippmann-Schwinger formulation of the scattering problem. Also notice that although Ω is in general a domain enclosing the object, $F(\mathbf{r})$ differs from zero only inside the object volume.

3 Fourier diffraction theorem

The inversion of (5) is not a trivial task and requires some consideration and approximation. We show hereafter the inversion for the 2D case presented in Fig. 1 but the same conclusion can be straightforwardly extended to the 3D case.

As a first step let's introduce the Green function in a homogeneous medium which in 2D is:

$$G(\mathbf{r} - \mathbf{r}') = \frac{i}{\pi} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|) \quad (6)$$

where $H_0^{(1)}$ is the zero-order Hankel function of the first kind (pay attention, in 3D the solution of (4) is different). As it will be clear in few steps, it is more convenient to use the Weyl expansion of the Green function:

$$G(\mathbf{r} - \mathbf{r}') = \frac{i}{4\pi} \int ds_x \frac{1}{s_z} \exp \{ik [s_x(x - x') + s_z(z - z')]\} \quad (7)$$

Where for the sake of notation we wrote the transverse wavevector $k_x = ks_x$ and the longitudinal one $k_z = ks_z$, so that $\mathbf{s} = (s_x, s_z)$ is a unitary vector. If we now plug (7) in (5) we get:

$$\begin{aligned} u_s(\mathbf{r}) &= \frac{i}{4\pi s_z} \int_{\Omega} d\mathbf{r}' \int ds_x F(\mathbf{r}') u_t(\mathbf{r}') \exp \{ik [s_x(x - x') + s_z(z - z')]\} = \\ &= \frac{ie^{iks_z z}}{4\pi s_z} \int ds_x \int_{\Omega} d\mathbf{r}' F(\mathbf{r}') u_t(\mathbf{r}') e^{-iks_x x'} e^{iks_x x} \end{aligned} \quad (8)$$

If now we take the angular spectrum formulation of u_s :

$$u_s(\mathbf{r}) = \frac{k}{2\pi} \int ds_x \tilde{u}_s(k s_x, z) e^{iks_x x} \quad (9)$$

we can extract from (8):

$$\tilde{u}_s(k s_x, z) = \frac{ie^{iks_z z}}{2k s_z} \int_{\Omega} d\mathbf{r}' F(\mathbf{r}') u_t(\mathbf{r}') e^{-iks_x x'} \quad (10)$$

3.1 Born approximation

Equation (10) provided an implicit formula to compute the scattered field. However, this requires to know the total field, which is itself related to scattered one, and it is not easy to invert (remember in our hypothetical experiment we would measure u_s and we want to extract F). To further simplify this formula we have to introduce some approximations. For weakly scattering objects, we can say that the total field is very similar to the incident one $u_t = u_0 + u_s \approx u_0$. This assumption is called Born approximation. If we consider a plane wave excitation $u_0 = \exp iks_0 \cdot \mathbf{r} = \exp ik(s_{0x}x + s_{0z}z)$ and to measure the scattered field on a detector placed at a distance z_0 from the rotation axis, (10) becomes

$$\tilde{u}_s(k s_x, z_0) = \frac{ie^{iks_z z_0}}{2k s_z} \int_{\Omega} d\mathbf{r}' F(\mathbf{r}') e^{-ik(\mathbf{s} - \mathbf{s}_0) \cdot \mathbf{r}'} \quad (11)$$

The right-hand-side now is nothing but the shifted Fourier transform of the scattering potential, which finally leads to:

$$\tilde{F}[k(\mathbf{s} - \mathbf{s}_0)] = -2ie^{iks_z z_0} k s_z \tilde{u}_s(k s_x, z_0) \quad (12)$$

This final equation, commonly referred to as Wolf Transform or Fourier Diffraction Theorem, states that the 1D Fourier transform of the scattered field measured by the detector at a distance z_0 from the rotation axis provides information about a 1D path in the 2D Fourier transform of the scattering potential. Or equivalently, when the object is illuminated in a direction $\mathbf{s}_0$ and the scattered field measured at all the possible directions $\mathbf{s}$ gives information on the Fourier components $\tilde{F}(\mathbf{K})$ whose points lie on a sphere of radius k centered in $-\mathbf{k}s_0$ (cfr. Principles of

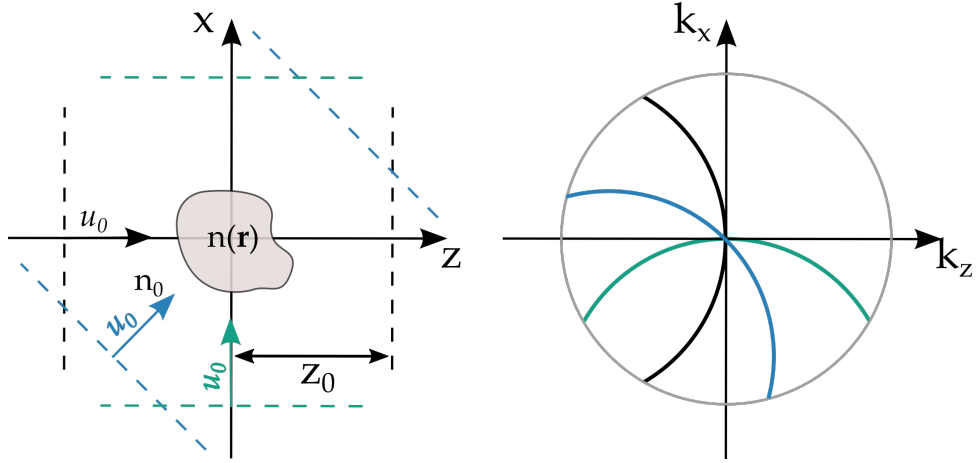


Figure 2: Working principle of optical diffraction tomography. For illumination each angle (colors on the left panel), we fill a portion of the Fourier transform of the object (right panel).

Optics Chapter XIII by E.Wolf and M.Born), see Fig. 2. We commonly refer to this sphere as Ewald's sphere.

Practically, the Fourier diffraction theorem provides us an efficient algorithm to reconstruct the refractive distribution $n(\mathbf{r})$

Algorithm 1: Wolf Transform

Set incident illumination direction $\mathbf{s}_0$
Measure u_t and u_i at $z = z_0$
Compute scattered field $u_s = u_t - u_i$ and its FT $\tilde{u}_s$
Use (12) and fill $\tilde{F}$ along the arc $k(\mathbf{s} - \mathbf{s}_0)$
Repeat for different directions Take the inverse FT of $\tilde{F}$

It is finally worth noticing the similarity of (12) with the Fourier slice theorem. In Radon transform every projection give information about a slice in the Fourier plane of the object. Here in a similar way, every projection of the scattered field gives information on an arc in the Fourier domain.