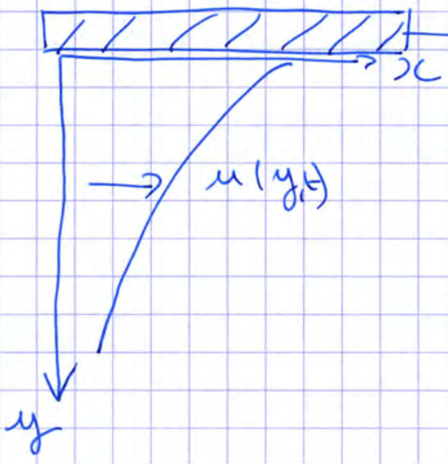
1.4 Stokes' first problem



$u(t) = U$ if $t \geq 0$
$= 0$ if $t < 0$

Boundary conditions

$u(y,t) = U \quad y = 0$
$u(y,t) = 0 \quad y \to \infty$

Navier-Stokes :

$$(1) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$(2) \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

Assumption $u(y,t)$ only

as $v = 0$, (1) $\Rightarrow \frac{\partial u}{\partial x} = 0$

no dependance on x.

$$(2) \Rightarrow \frac{\partial u}{\partial t} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{0} = \underbrace{-\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}}}_{0} + \underbrace{\nu\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \nu\frac{\partial^2 u}{\partial y^2}$$

invariance under x translation

Variables u, t, ν, y, U

if we apply the Π theorem, then we deduce that there are 5 variables and 2 physical units (m and s), and thus $5-2=3$ numbers with no dimension...

Let us assume that the variables are

$\frac{u}{U}$, y, ν, t

4 variables, 2 units $\Rightarrow$ 2 dimensionless numbers.

One is $\frac{u}{U}$, the other is a combination of y, t, and ν

$$[y^\alpha t^\beta \nu^\gamma] = 0 \Rightarrow \xi = \frac{y}{\sqrt{\nu t}}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} = -\frac{1}{2}\frac{\xi}{t}\frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial y} = \frac{1}{\sqrt{\nu t}}\frac{\partial}{\partial \xi} \qquad \frac{\partial^2}{\partial y^2} = \frac{1}{\nu t}\frac{\partial^2}{\partial \xi^2}$$

$$(2) \Rightarrow 2f'' + \xi f' = 0$$

Boundary conditions $f(0)=1$ $f(\infty)=0$

$$f(\xi) = 1 + \mathrm{erf}\left(\frac{\xi}{2}\right)$$

$\boxed{2.1}$ $$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = u^2$$

Characteristic form

$$\frac{dx}{x} = -\frac{dy}{y} = \frac{du}{u^2}$$

$$a' = \ln x + \ln y$$

$$a = xy$$

$$\frac{1}{u} + \ln y = b$$

Implicit solutions: any function

$$F(a, b) = F\left(xy, \frac{1}{u} + \ln y\right) = cst$$

An explicit form

$$G(xy) = \frac{1}{u} + \ln y \Rightarrow u = \frac{1}{G(xy) - \ln y}$$

$\boxed{2.4}$ hyperbolic equation.

- Characteristic form

$$\frac{dx}{t} = -\frac{dt}{x} = \frac{du}{0}$$

$\hookrightarrow \quad a' = x\,dx + t\,dt$

$a = x^2 + t^2$

u is an arbitrary function of a

$$u = F(x^2 + t^2)$$

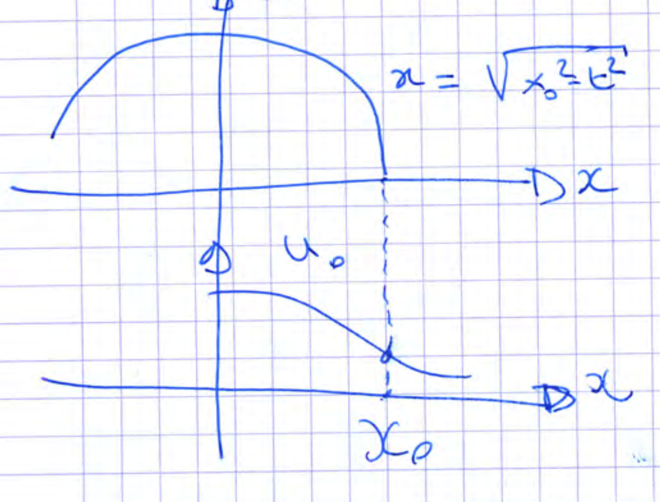
- $$\frac{\partial u}{\partial t} - \frac{t}{x}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \quad \frac{du}{dt} = 0 \text{ along } \frac{dx}{dt} = -\frac{t}{x}$$

$$\Rightarrow \quad x^2 = x_0^2 - t^2$$

where x_0 is a constant.

$x = \sqrt{x_0^2 - t^2} \quad u = \text{cst}.$



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$\boxed{2.7}$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = I \qquad (1)$$

with $u = a h^m$

$(1) \Rightarrow (2)$: $\frac{\partial h}{\partial t} + c(h)\frac{\partial h}{\partial x} = I$

with $c(h) = (m+1) h^m a$

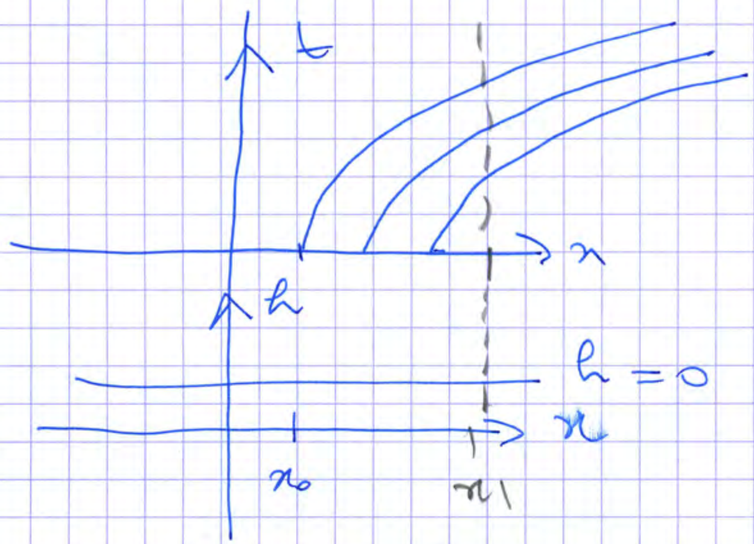
$(2) \Rightarrow (3)$ $\frac{dh}{dt} = I$ along $\frac{dx}{dt} = (m+1) a h^m$

$h = It + cst$

$\underline{IC}$ $t=0$ $h=0 \Rightarrow cst = 0$

$$\frac{dx}{dt} = (m+1)\, a (It)^m$$

$$x = aI^m t^{m+1} + x_0$$



$\boxed{2.7}$

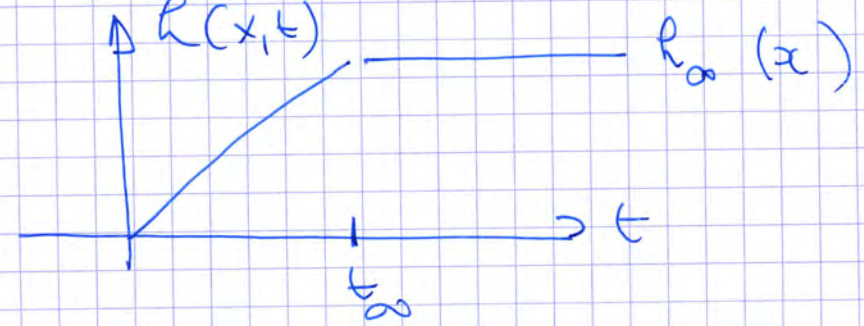
There is an envelope $x = x_0 + a I^n t^{n+1}$

- for a point below this envelope, then

  $h = It$

- for a point above the envelope, then

  $h = cst$



$$t_\infty = \left(\frac{x - x_0}{a I^n}\right)^{1/(n+1)}$$

$$h_\infty(x) = I^{1 - \frac{n}{n+1}} \left(\frac{x - x_0}{a}\right)^{\frac{1}{n+1}} \quad (4)$$

NB if we set $\frac{\partial h}{\partial t} = 0$ in (1)

$$C(h) \frac{\partial h}{\partial x} = I$$

$$a h^{n+1} = I x + cst$$

$$h = \left[\frac{I(x - x_0)}{a}\right]^{\frac{1}{n+1}} \quad (5)$$

(4) and (5) are equivalent.