EPFL

Chapter 1: Concept of similarity

Similarity and Transport Phenomena in Fluid Dynamics

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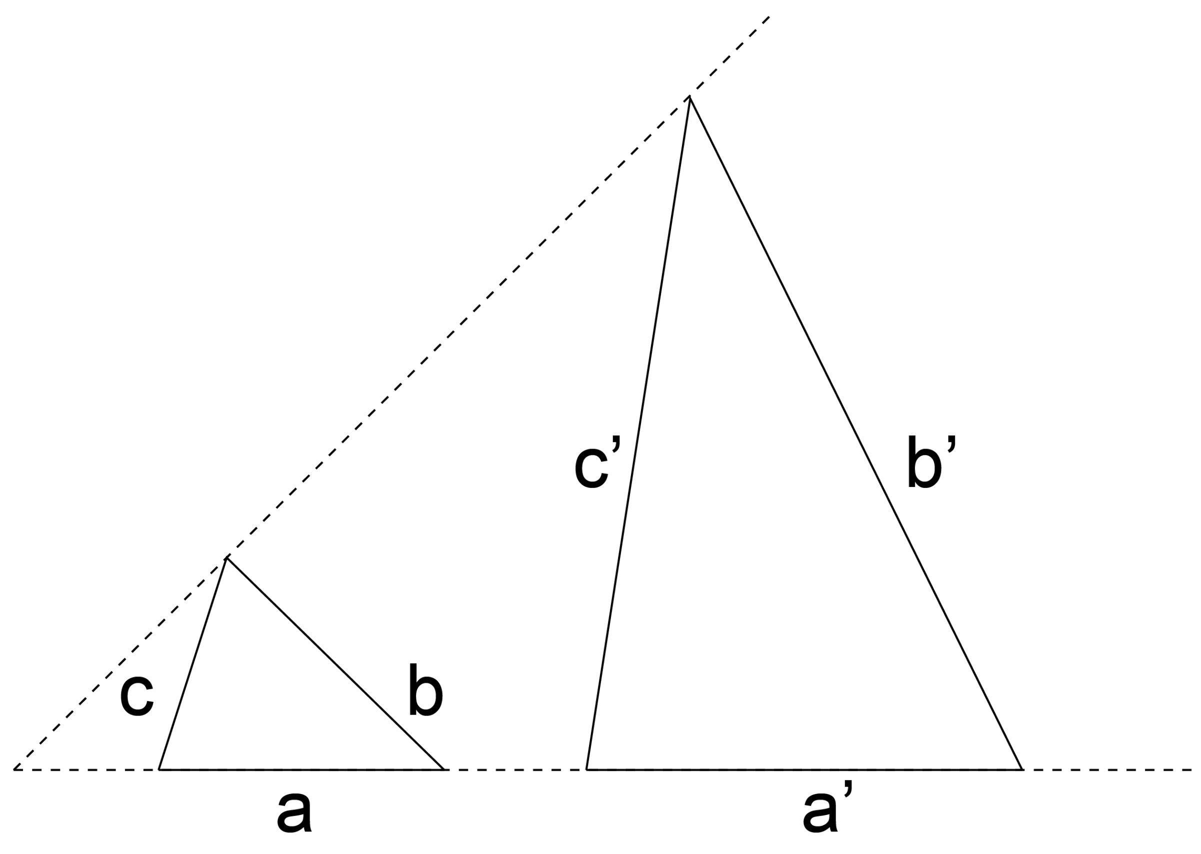
Chapter 1: Concept of similarity

- A few definitions
  - Geometrical similarity
  - Fractal similarity
  - Physical Similarity
- Scaling law
- Complete similarity
- Incomplete similarity
- Historical background

Geometrical similarity

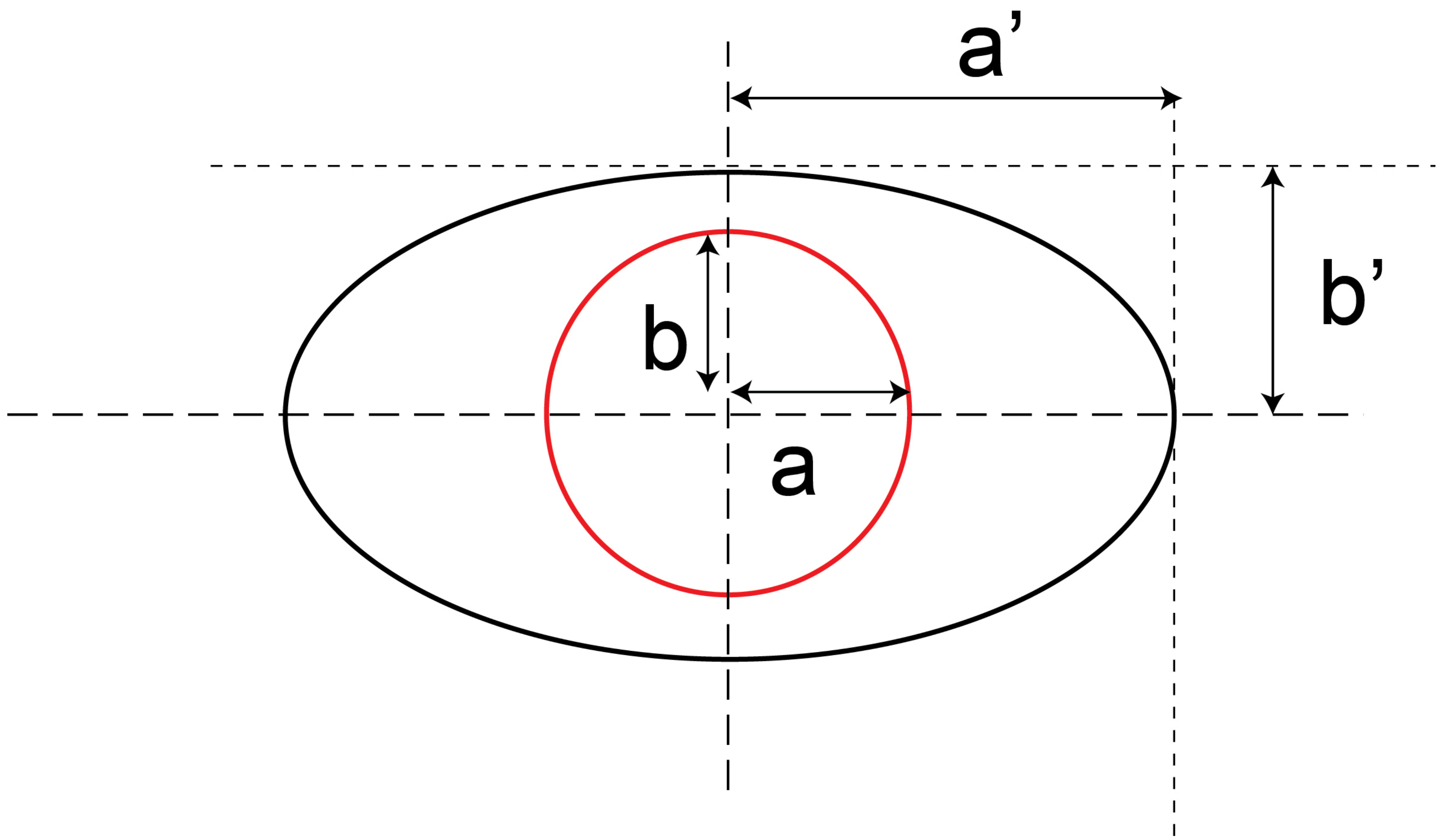


Triangles are geometrically *similar* if

$$\lambda = \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c},$$

with λ the *similarity ratio* (also called the *scale factor* or *scale*). This is an example of *isomorphic transformation*.

Geometrical similarity

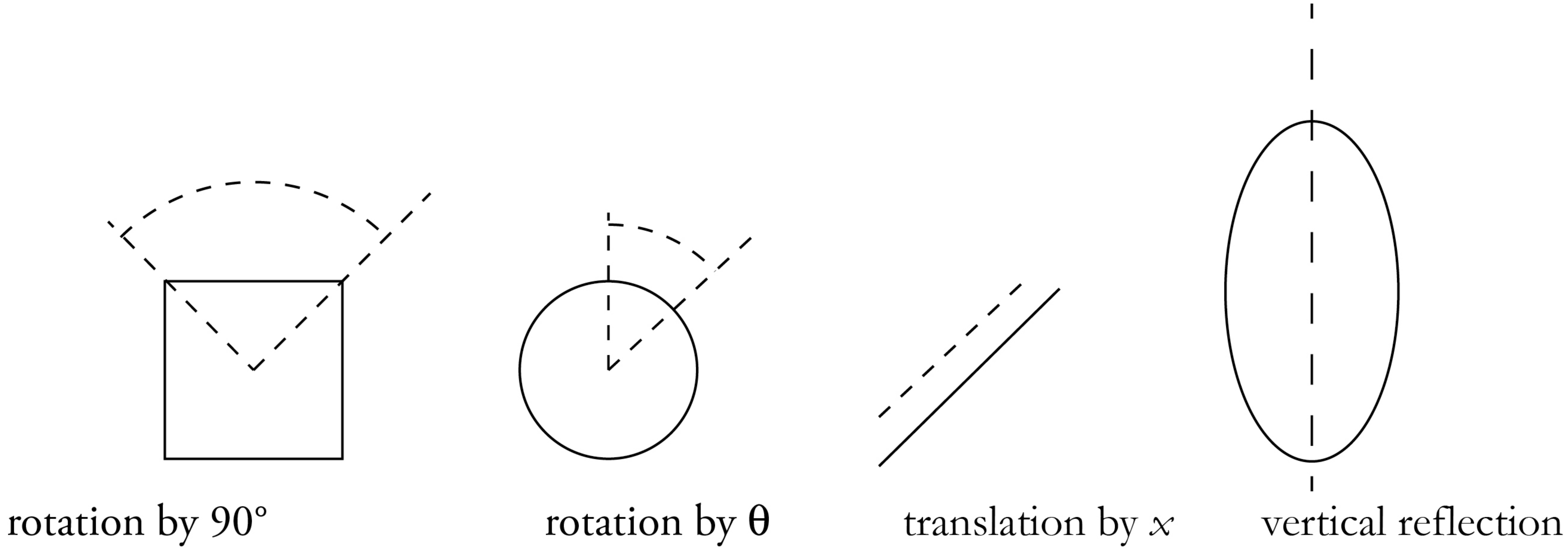


Generalisation: an affine transformation conserves length ratios, with axis-dependent ratios

$$\lambda_x = \frac{a'}{a} \text{ et } \lambda_y = \frac{b'}{b},$$

with λ_x et λ_y the ratios in the horizontal and vertical direction.

Geometrical similarity



Transformations that leave an object invariant

- Discrete transformation, e.g., rotation by $\pi/2$ for a square
- Continuous transformation, eg., rotation by an angle θ for a circle, translation by ϵ for a straight line

Infinitesimal continuous transformations are key to constructing exact solutions.

Geometrical similarity

Some continuous transformations $\Gamma(\theta)$ are one-dimensional Lie groups

- $\Gamma(0)$ is identity Id
- $\Gamma(\theta_1) \circ \Gamma(\theta_2) = \Gamma(\theta_1 + \theta_2)$ (rule of closure)
- the inverse of $\Gamma(\theta)$ is $\Gamma(-\theta)$, and $\Gamma(\theta) \circ \Gamma(-\theta) = Id$
- $(\Gamma(\theta_1) \circ \Gamma(\theta_2)) \circ \Gamma(\theta_3) = \Gamma(\theta_1) \circ (\Gamma(\theta_2) \circ \Gamma(\theta_3))$ (associativity)

Additional property: each image may be represented as a Taylor series in θ (for $\theta \to 0$)

$$\hat{x} = x + \theta \partial_x \Gamma + O(\theta^2).$$

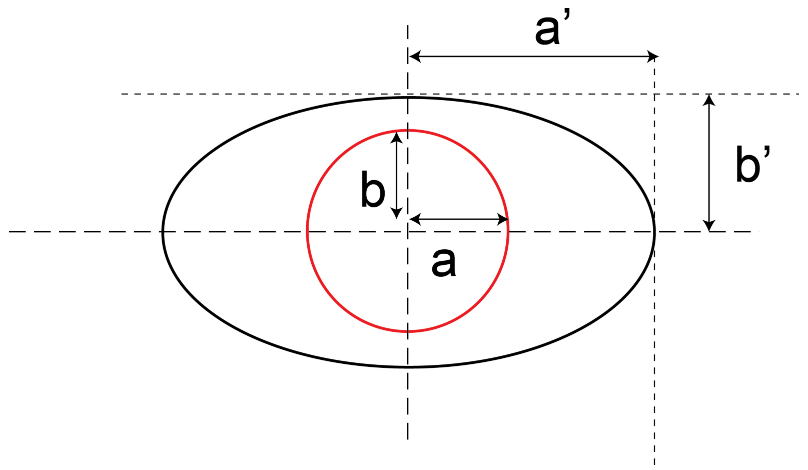
Exercise 1

Can you find symmetries for this equation?

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

Solve this equation and comment on.

Geometrical similarity: invariant



When applying an affine transformation,

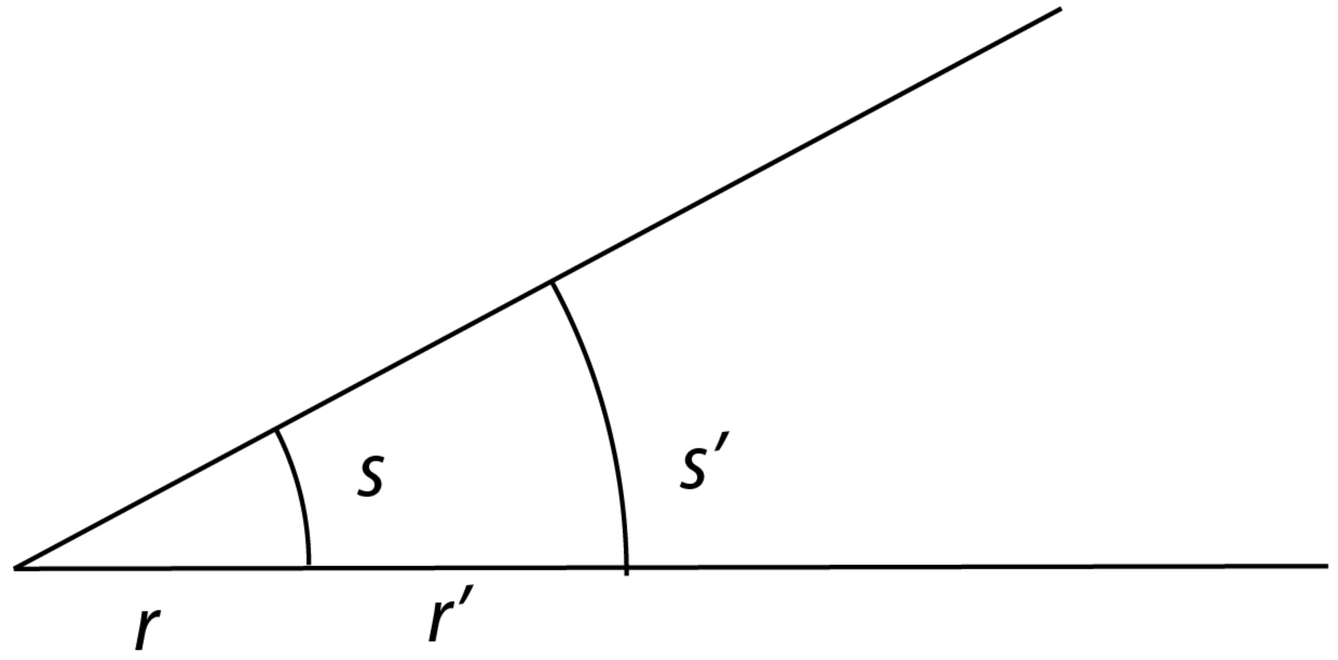
- certain quantities are conserved. They are called *invariants*. For instance, taking the ratio between the surface S and the product of half axes:

$$s = \frac{S}{ab} = \frac{S'}{a'b'} = \pi .$$

- other quantities are not conserved. This is the case of the perimeter

$$P = 4 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \mathrm{d}\theta$$

Scale invariance



An angle is scale invariant

$$\alpha = \frac{s}{r} = \frac{s'}{r'}$$

Scale invariant, but not affine invariant.

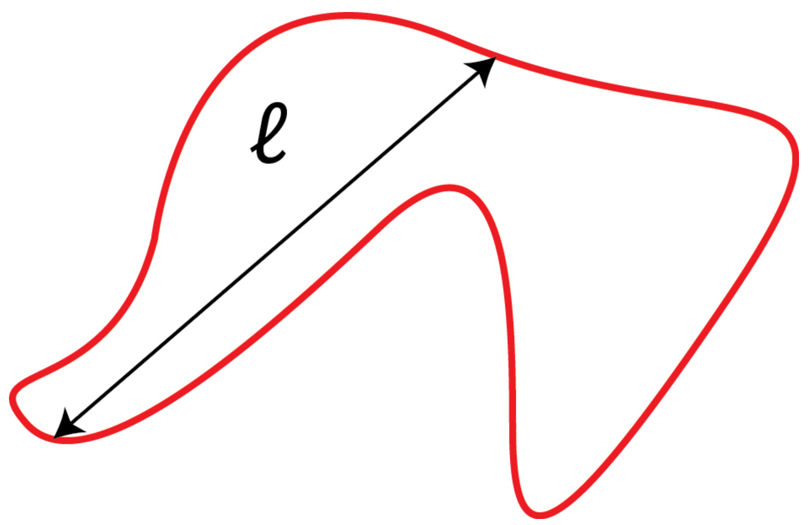
Scaling law

Why are certain quantities conserved, whereas others are not?

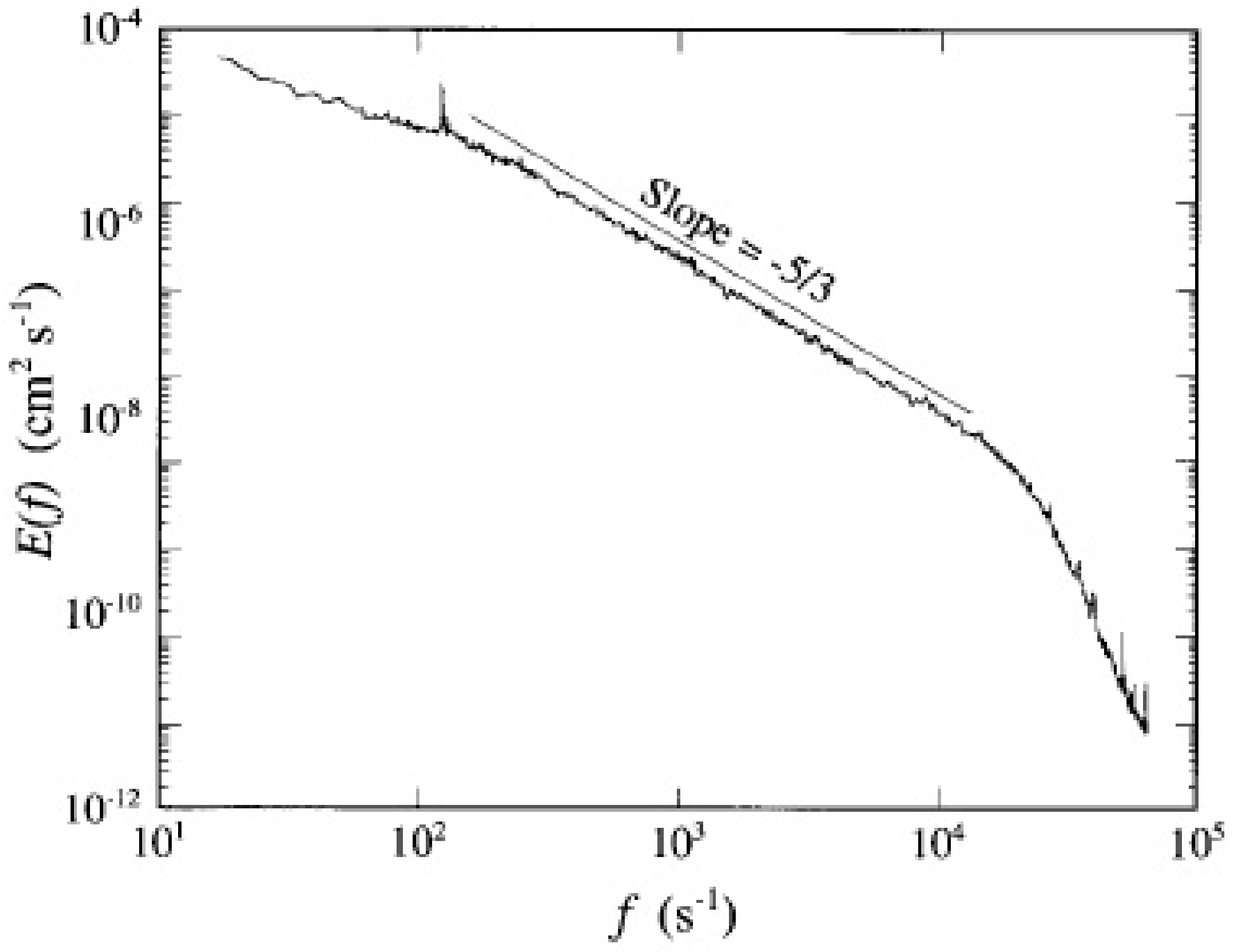
A scaling law describes the proportionality relation between a quantity and the scale(s) of the problem:

- perimeter $P \propto \ell$,
- area $S \propto \ell^2$,
- volume $V \propto \ell^3$,

with ℓ a *length scale* of the object.



Scale invariance

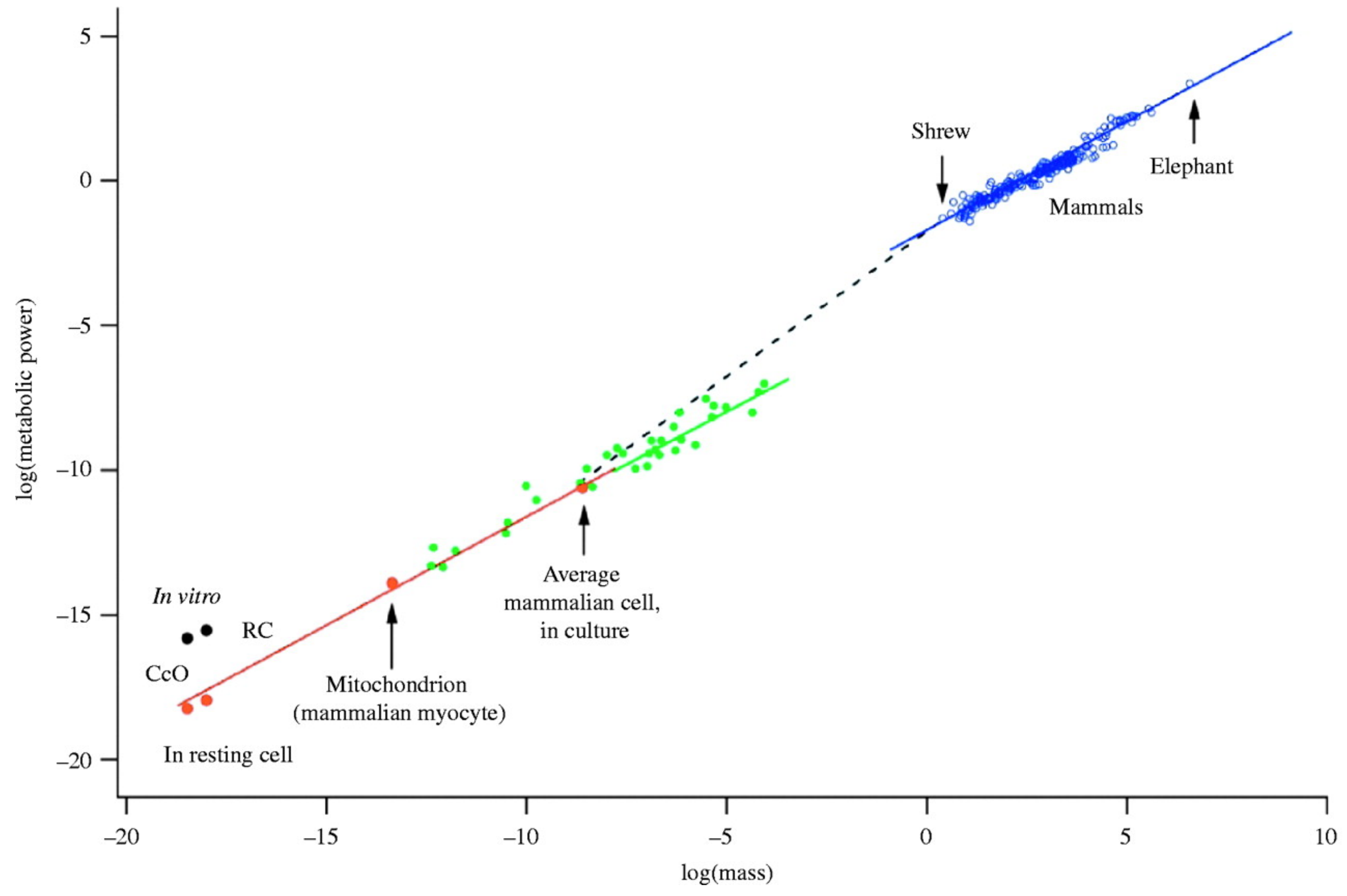


Log-log plot of the energy spectrum in the time domain in helium flow between rotating cylinders

$$E(f) = f^{-5/3}$$

(Frisch, U., Turbulence, Cambridge University Press, Cambridge, 1995; p. 66)

Scale invariance

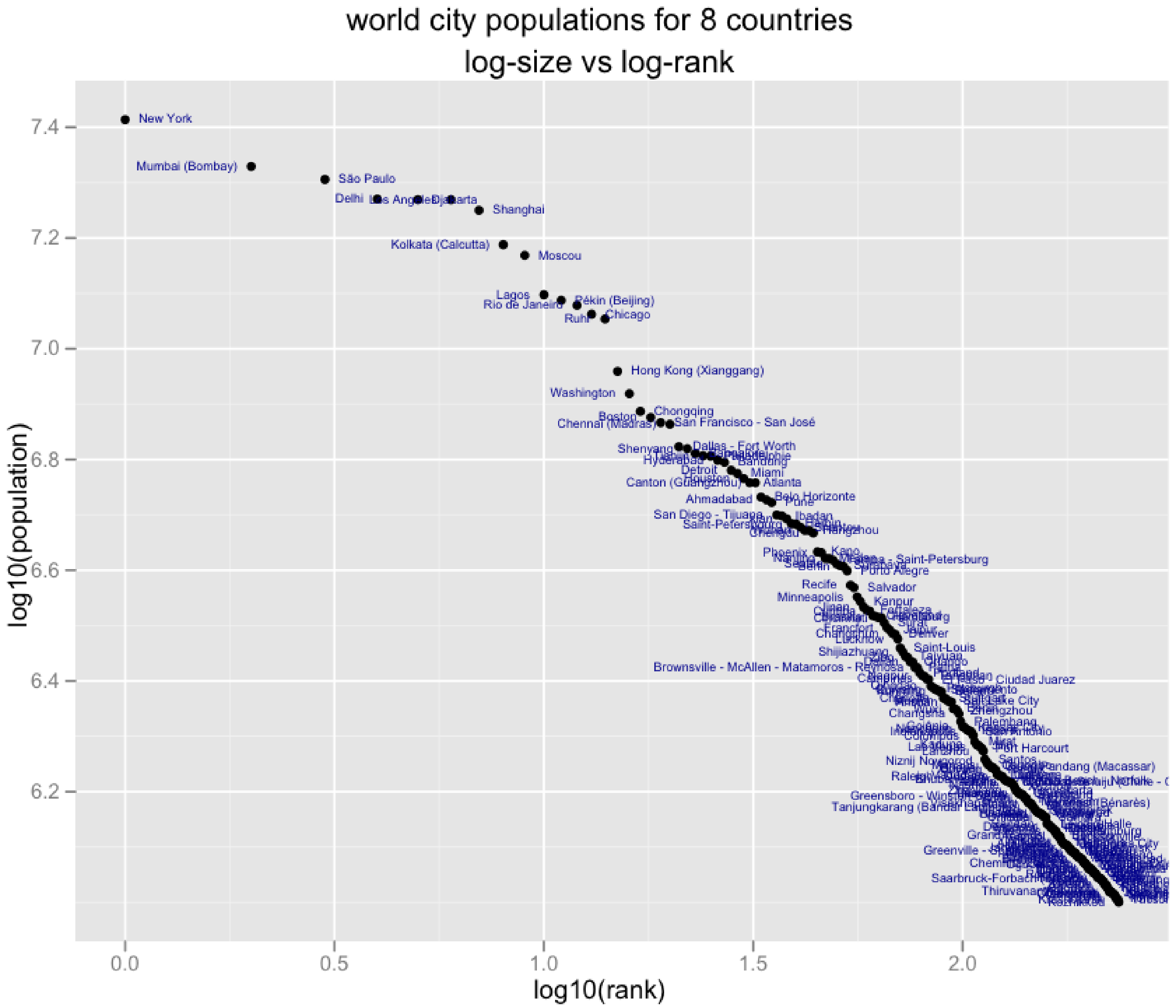


Log-log plot of the basal metabolic rate of mammals and birds (in kcal/day) with mass (M in kg)

$$\mathrm{BMR} \propto M^{-3/4}$$

(West, G.B., and J.H. Brown, The origin of allometric scaling laws in biology from genomes to ecosystems: towards a quantitative unifying theory of biological structure and organization, The Journal of Experimental Biology, 208, 1575-1592, 2005.)

Scale invariance

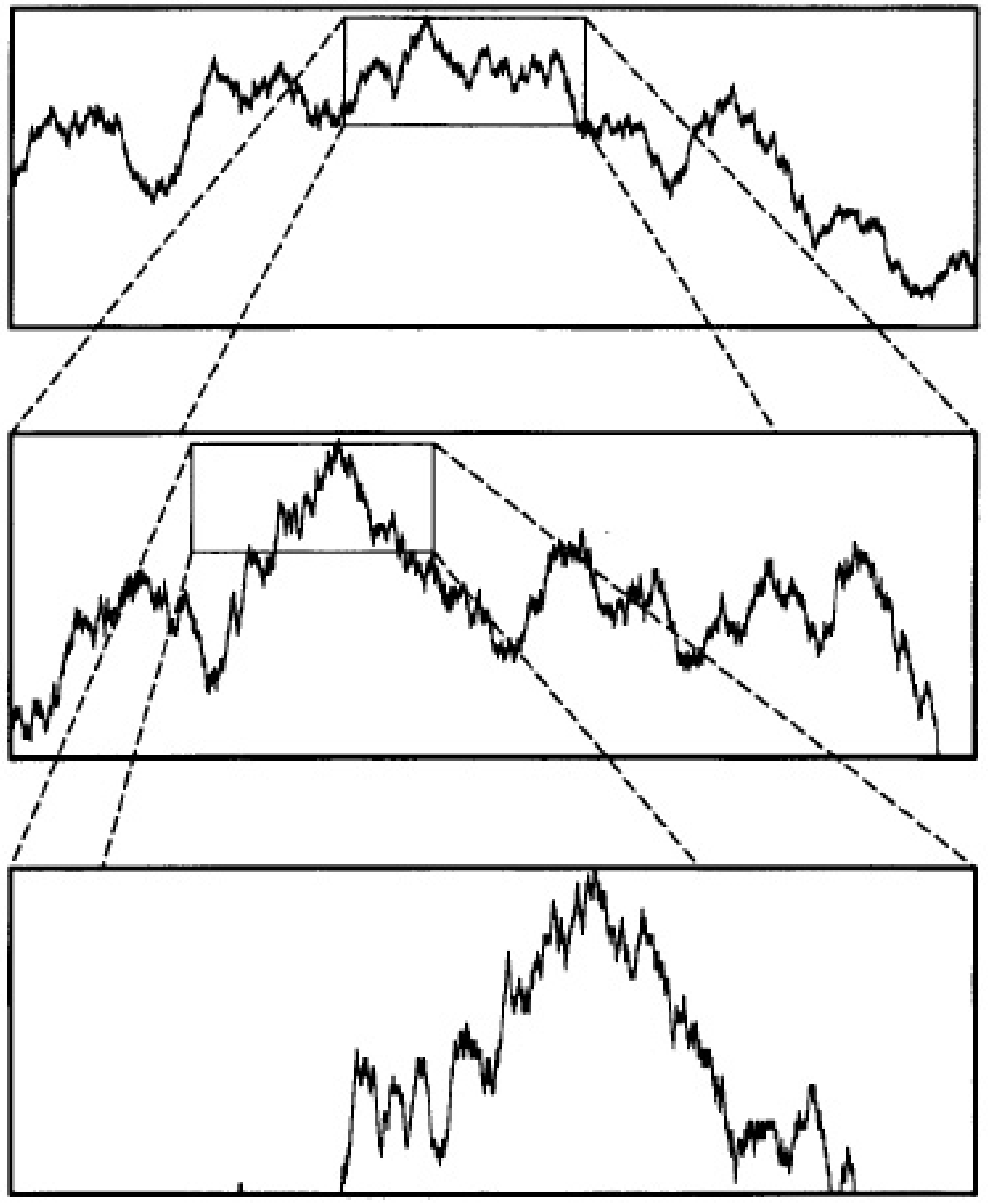


Zipf's law applied to agglomeration size

$$\text{rank} \propto \text{size}^{-0.76}$$

Clauset, A., C.R. Shalizi, and M.E.J. Newman, Power-Law Distributions in Empirical Data, SIAM Review, 51, 661-703, 2009.

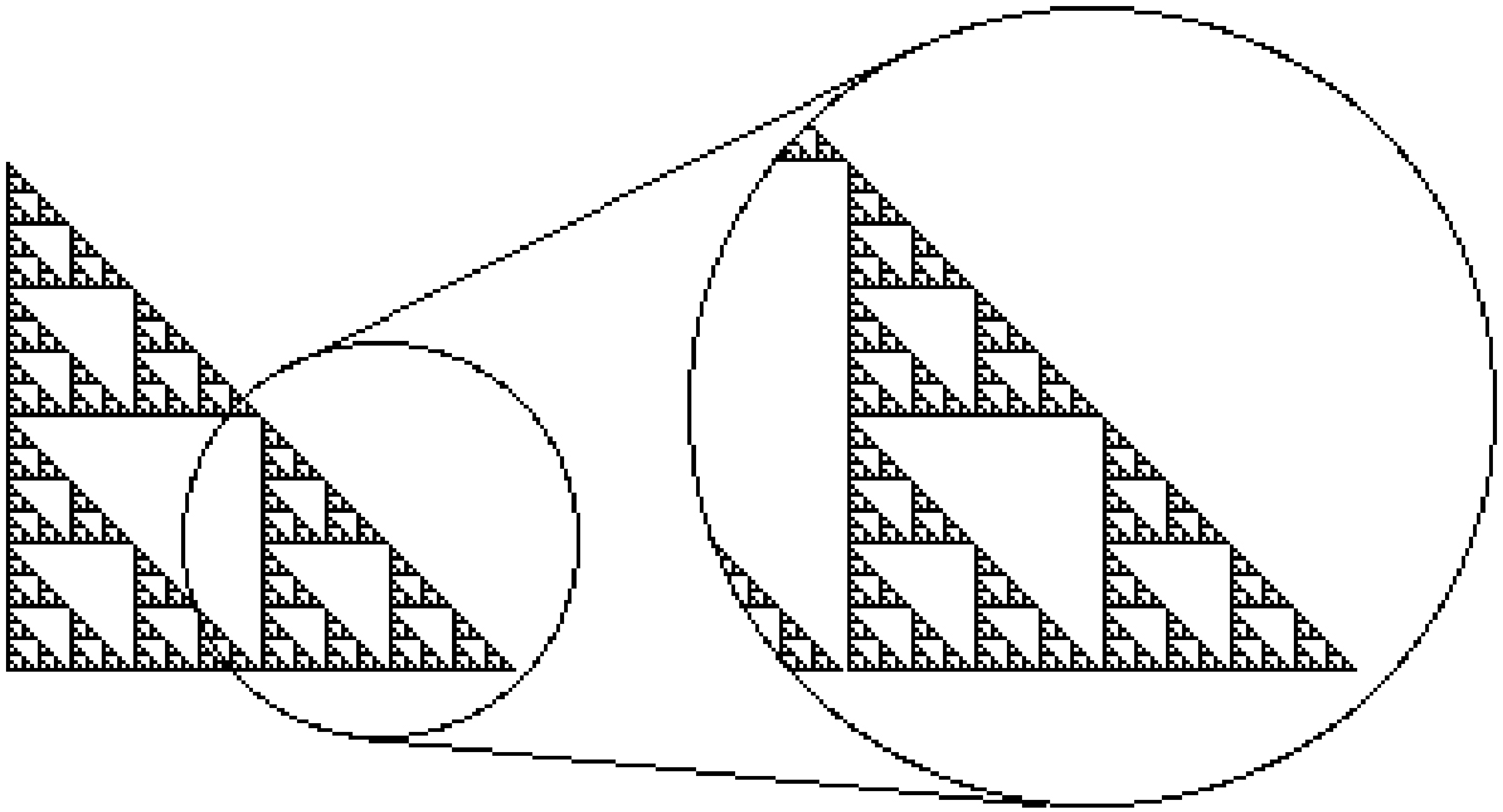
Fractal similarity



(Frisch, U., Turbulence, Cambridge University Press, Cambridge, 1995; p. 66)

Brownian motion: the aspect looks the same in the two enlargements

Fractal similarity



Fractal similarity

Rivers forming tree-like figures in the desert of Baja California, Mexico (© Adriana Franco)

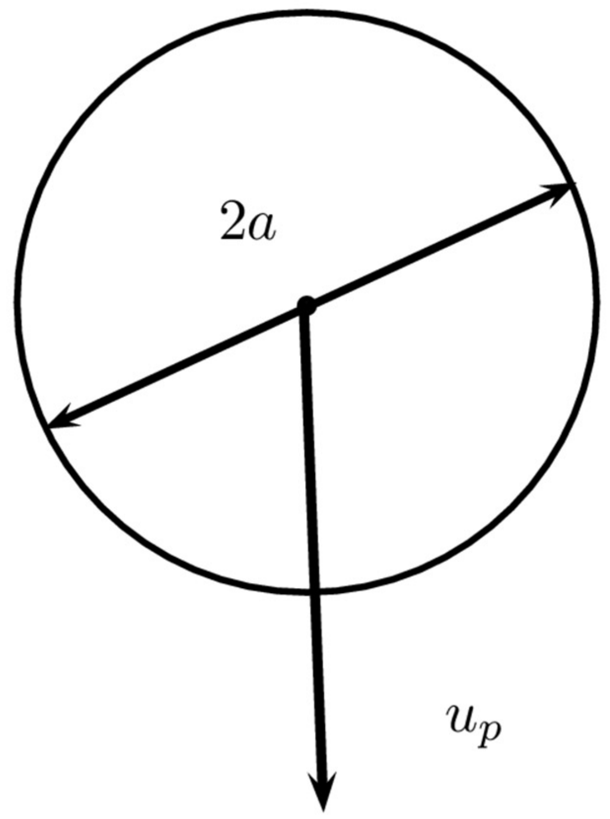
Invariance

Exercises 2 and 3

2. Consider a rectangular triangle whose sides are a, b, c (a is the hypothenuse). Plot the altitude perpendicular to the hypothenuse (In geometry, an altitude of a triangle is a line segment through a vertex and perpendicular to a line containing the base—the opposite side of the triangle). What can you say about of the two triangles split by the altitude? Can you give a proof of Pythagoras' theorem?

3. Calculate the surface of an ellipse (Hint: make use of the affinity between a circle and an ellipse).

Physical similarity



We calculate the drag force F exerted by an incompressible Newtonian fluid on a spherical particles with diameter $2r$ and density ϱ_p.

There are 5 variables: (1) the (unknown) drag force, (2) fluid viscosity μ, (3) fluid density ϱ, (4) particle radius r, and (5) the relative velocity $u = |\mathbf{u}_p - \mathbf{u}_f|$. The Vashy-Buckingham theorem states that we can build $5 - 3 = 2$ dimensionless numbers.

Physical similarity

The general dimensionless expression for the drag force read $\psi(Cd,\ Re) = 0$ or $Cd = \phi(Re)$:

$$C_d = \frac{F}{\frac{1}{2}\pi\varrho r^2 u^2} = \phi(\mathrm{Re}).$$

C_d is the *drag coefficient*. Solving the Navier-Stokes equations in the limit $\mathrm{Re} \ll 1$ yields:

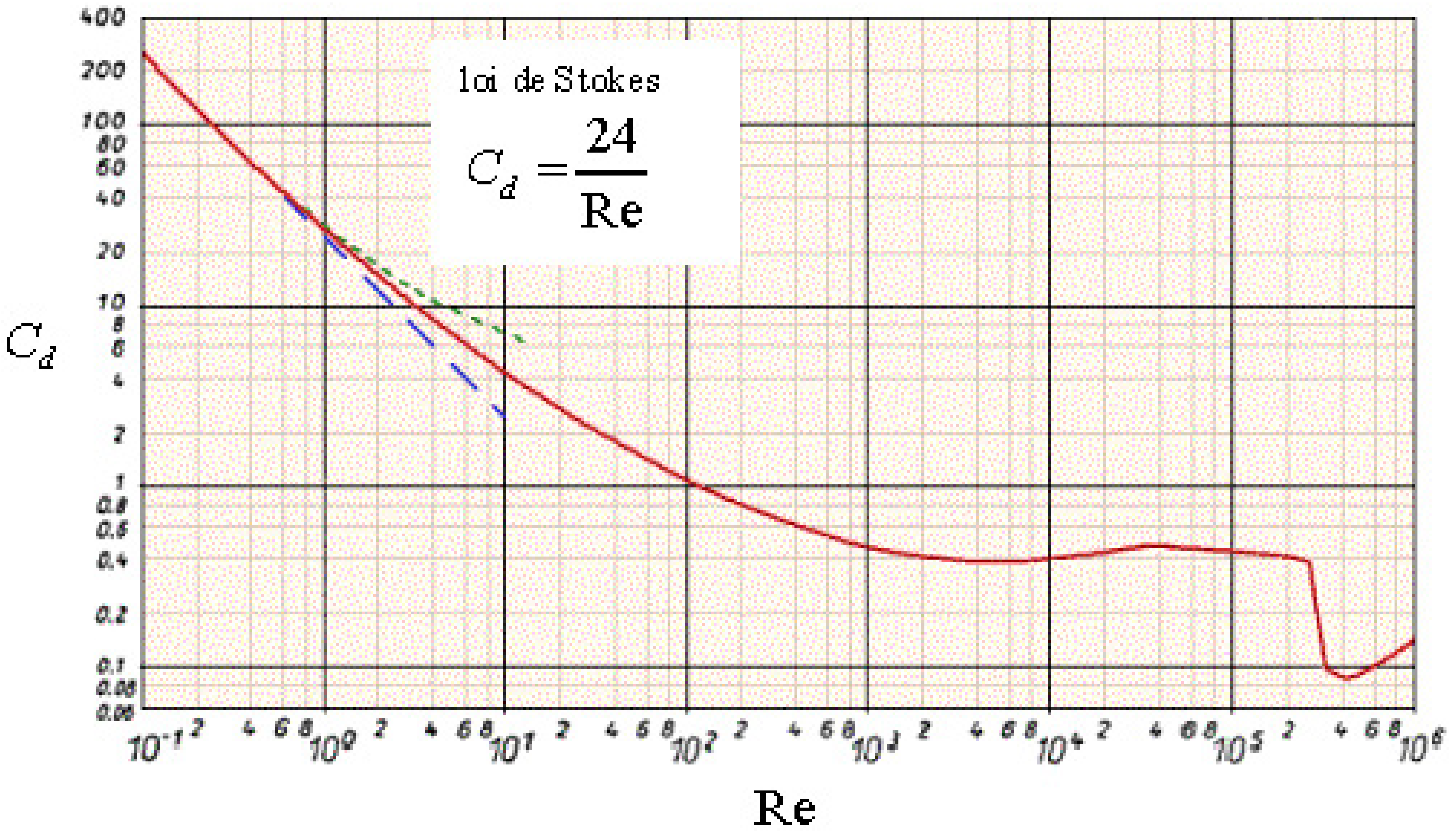
$$C_d = \frac{F}{\frac{1}{2}\pi\varrho r^2 u^2} = \phi(\mathrm{Re}) = \frac{24}{\mathrm{Re}} \text{ quand } \mathrm{Re} \to 0.$$

This is the *Stokes* law. In the limit $\mathrm{Re} \gg 1$, experiments show that:

$$C_d = \frac{F}{\frac{1}{2}\pi\varrho r^2 u^2} = \phi(\mathrm{Re}) \approx 0,4 - 0,5 \text{ when } \mathrm{Re} \to \infty.$$

Physical similarity

Variation of the drag coefficient with the particle Reynolds number: $C_d = \frac{F}{\frac{1}{2}\pi\varrho r^2 u^2}$ and $\mathrm{Re} = \frac{2\varrho r u}{\mu}$:



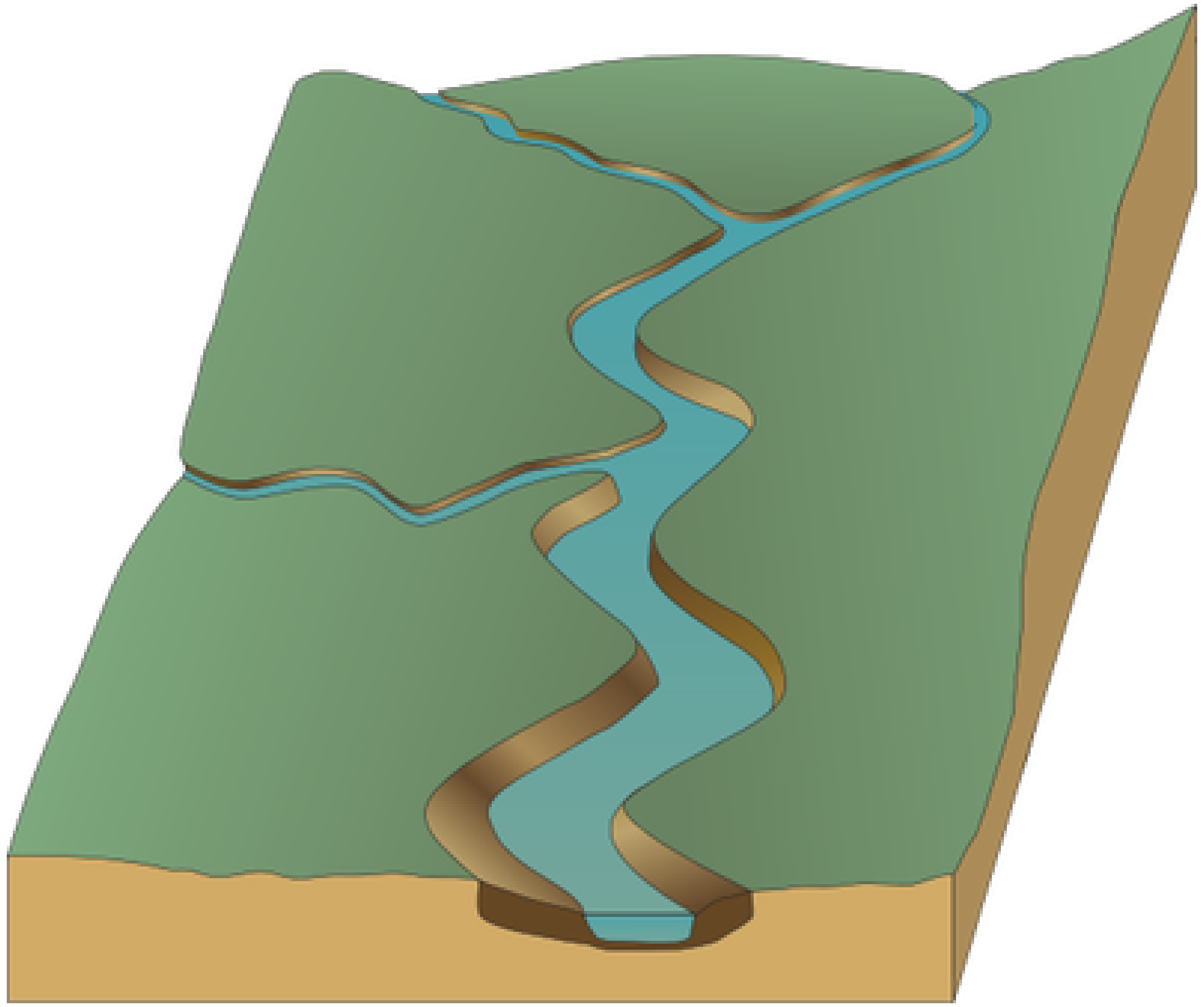
Exercise 4

4. Consider a Newtonian fluid at rest and bounded by an infinite upper boundary. The material is suddenly sheared by moving the boundary at constant velocity. This is the Stokes' first problem. Write the Navier-Stokes equations. How can you simplify them? What are the boundary conditions? Solve the equation and comment on the results. (Hint: Use the Vaschy-Buckingham theorem and transform the NS equation into an ODE).

Exercise 5

5. Consider a Newtonian fluid at rest and bounded by an infinite upper boundary. The material is suddenly experiencing a (constant) body force (the boundary does not move). Write the Navier-Stokes equations. How can you simplify them? What are the boundary conditions? Solve the equation and comment on the results. (Hint: make a change of variable, then transform the PDE into an ODE).

Complete similarity



We are concerned with a water stream, whose variables are: $\bar{u}$ [m/s], h [m], g [m/s^2], and θ [–]. The gravitational acceleration g and θ are put together so that there are only $n = 3$ variables. There are $r = 2$ fundamental units: m and s. Therefore, there is $n - r = 1$ dimensionless group. This is the Froude number $\mathrm{Fr} = \bar{u}/\sqrt{gh\sin\theta}$. The physics of water streams would entirely be embodied in the relation

$$\mathrm{Fr} = cst \Rightarrow \bar{u} \propto \sqrt{gh\sin\theta}.$$

This is the Chézy law.

Incomplete similarity

But what about bed roughness?

Let us introduce the roughness scale k_s [m]. There are $n = 4$ variables and $r = 2$ units. So there are 2 dimensionless groups, e.g., $\Pi_1 = \mathrm{Fr} = \bar{u}/\sqrt{gh\sin\theta}$ and $\Pi_2 = k_s/h$, and among them, there exists a relation in the form:

$$\Pi_1 = f(\Pi_2) \Rightarrow \bar{u} = f(k_s/h)\sqrt{gh\sin\theta}.$$

Since flow depth is usually much larger than k_s, we expect $k_s/h \to 0$ and $f(k_s/h)$ should tend to a constant. If so, this is barely different from the Chézy law!

Incomplete similarity

Another possibility is to assume that f behaves like a power function

$$f(\zeta) = \alpha\zeta^n,$$

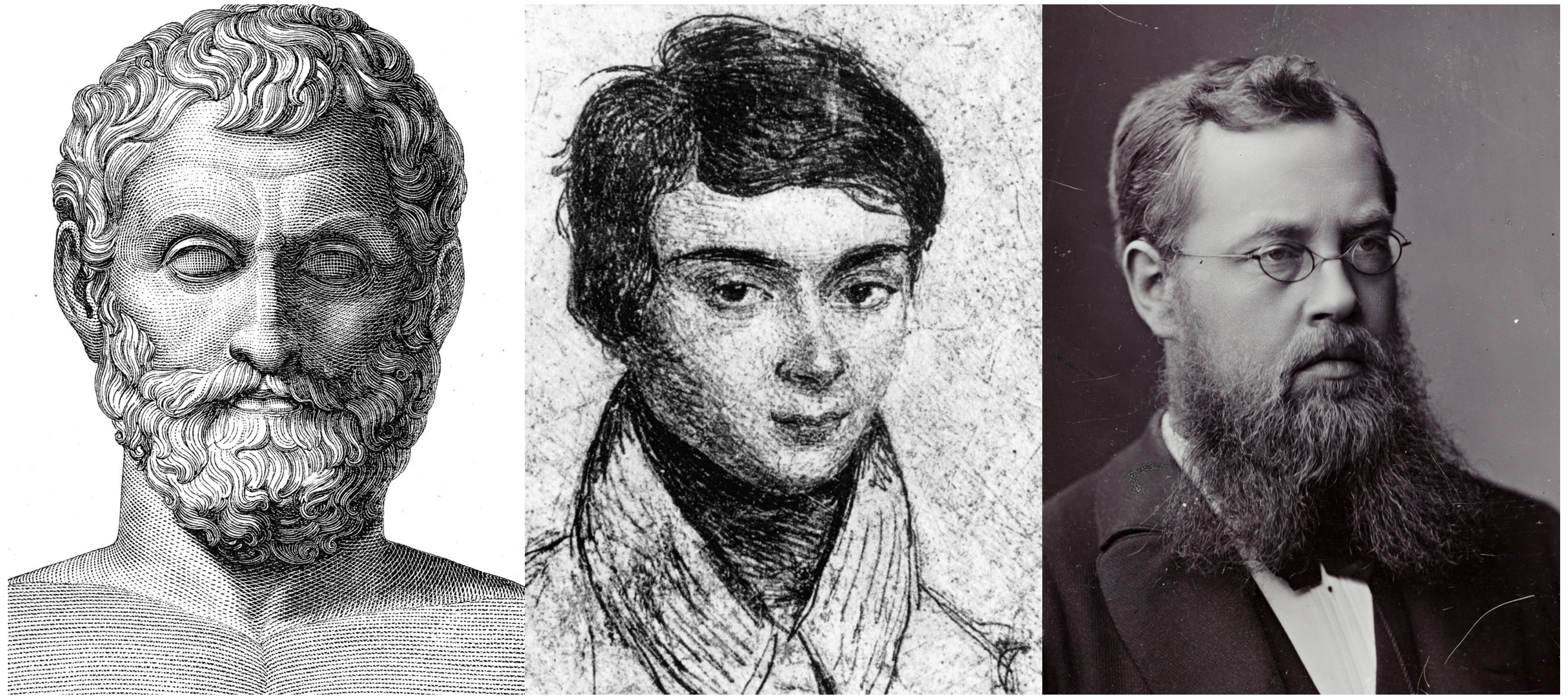
with $\zeta = k_s/h$, α a dimensionless number, and n an exponent. We refer to this behaviour as *incomplete similarity*. Under this assumption, we end up with

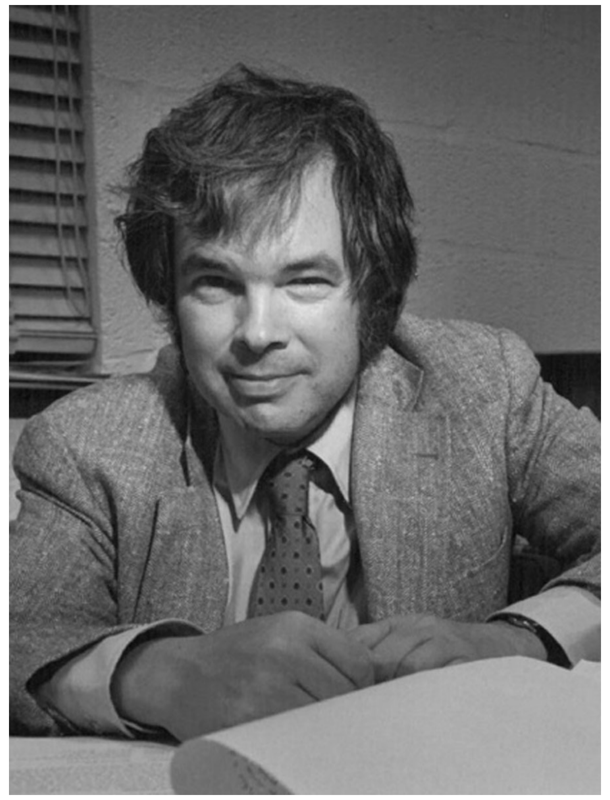
$$\Pi_1 = \alpha\Pi_2^n \Rightarrow \bar{u} = \alpha k_s^n h^{1/2-n}\sqrt{g\sin\theta}.$$

Taking $n = -1/6$ (from experimental observation) leads to the Manning-Strickler flow resistance equation. Therefore the Strickler coefficient K is linked with k_s:

$$K = \alpha\sqrt{g}k_s^{-1/6}.$$

Historical background





- 6th century BC: Thales of Miletus
- ~ 1830 Galois' group theory
- 1870–1900: Sophus Lie and Felix Klein (Backlünd, Noether, etc.)
- mid 20th century: Soviet physical school (Zeldovitch, Barenblatt, Sedov, etc.)
- mid 20th century: American mathematical school (Bluman, Anco, Olver)
- follow-on: renormalization group theory (Kadanoff, Wilson)