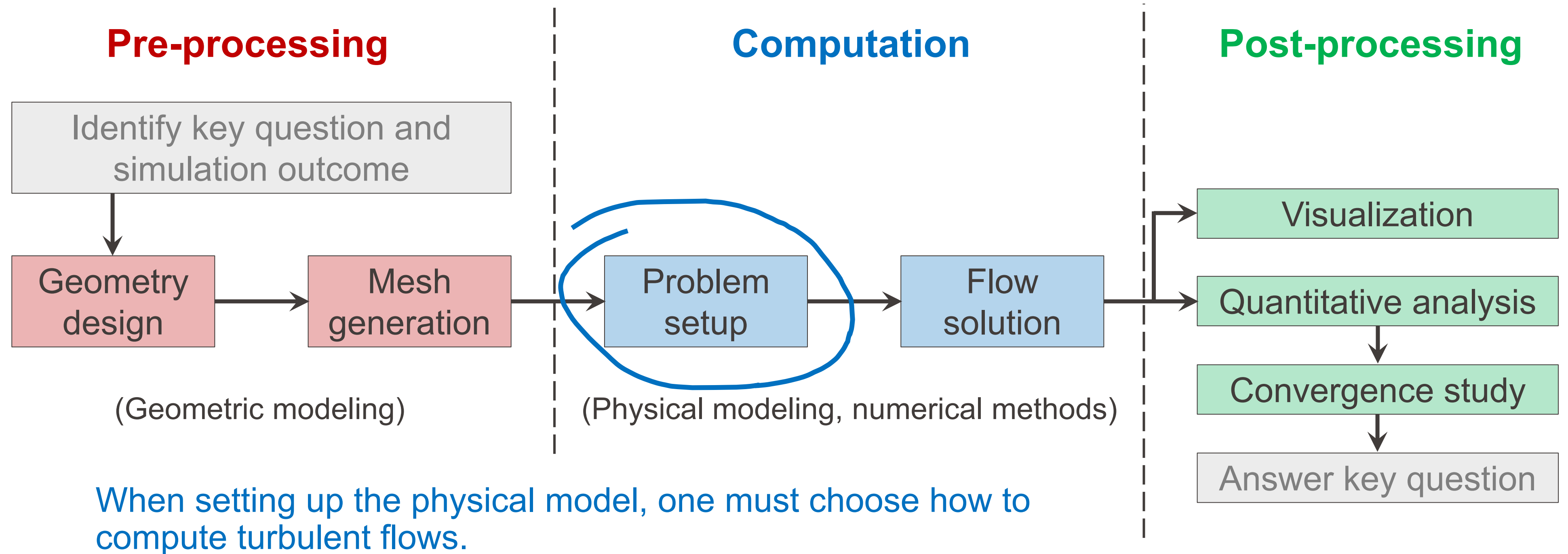


Turbulence modeling

Numerical Flow Simulation

Numerical simulation workflow

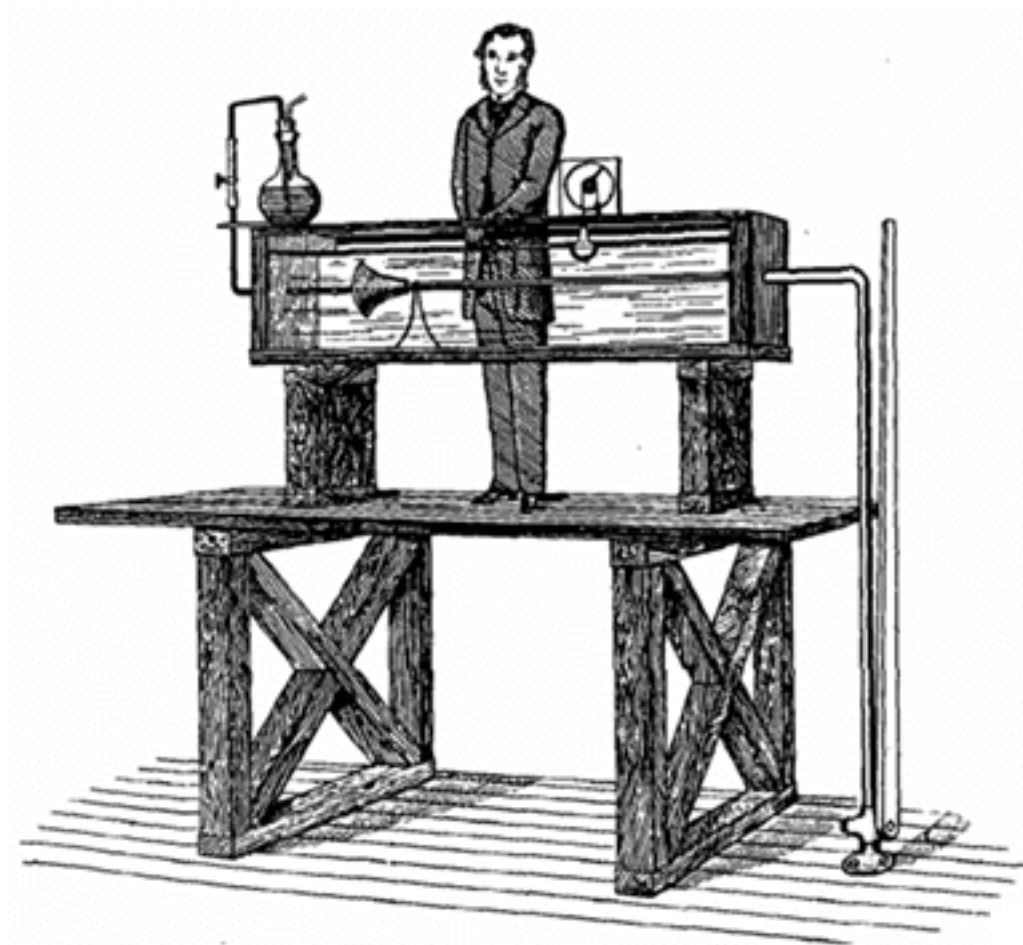


Schematic outline of this lecture:

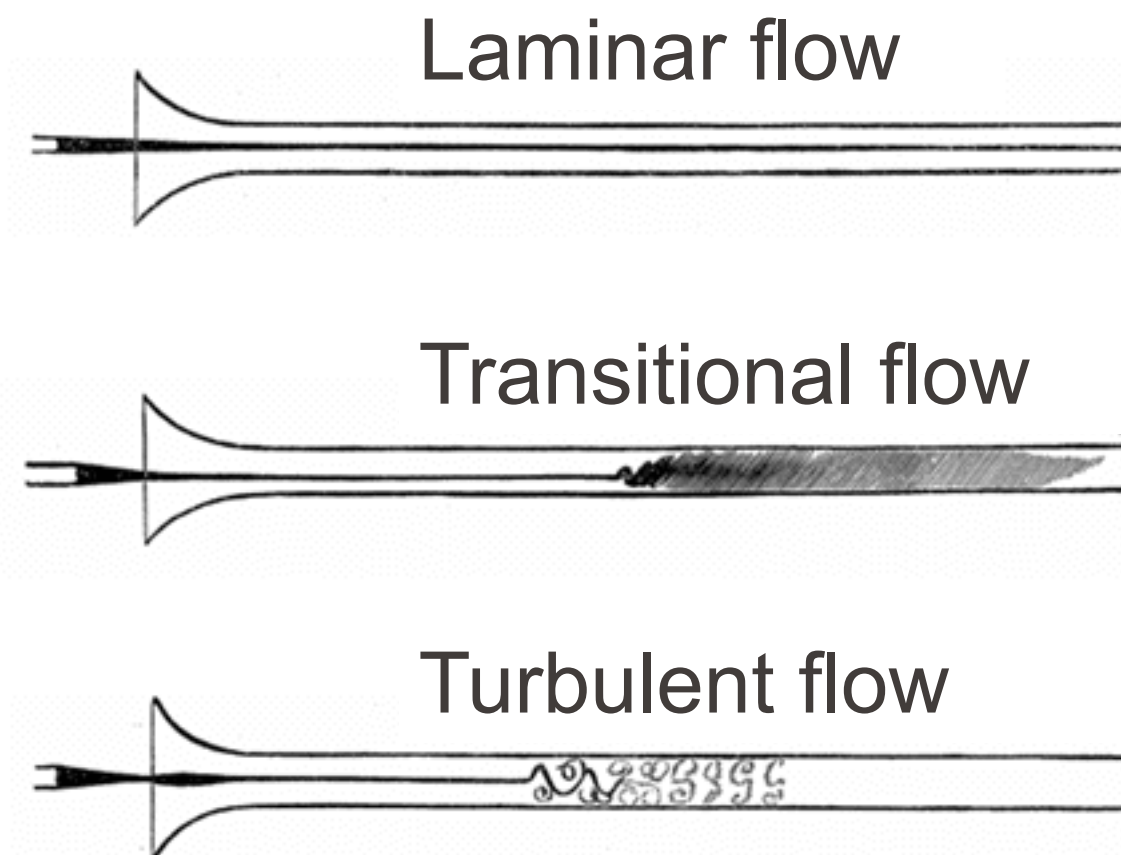
1. What is turbulence? (Brief overview)
2. Numerical simulation of turbulent flows
3. Overview of turbulence models

Basic characteristics of turbulence

- Experimental observation from Osborne Reynolds:



Reynolds 1883



van Dyke 1982

- Flow becomes unstable (and later turbulent) with increasing velocity and size, and with decreasing viscosity. Reynolds understood the role of “his” number:

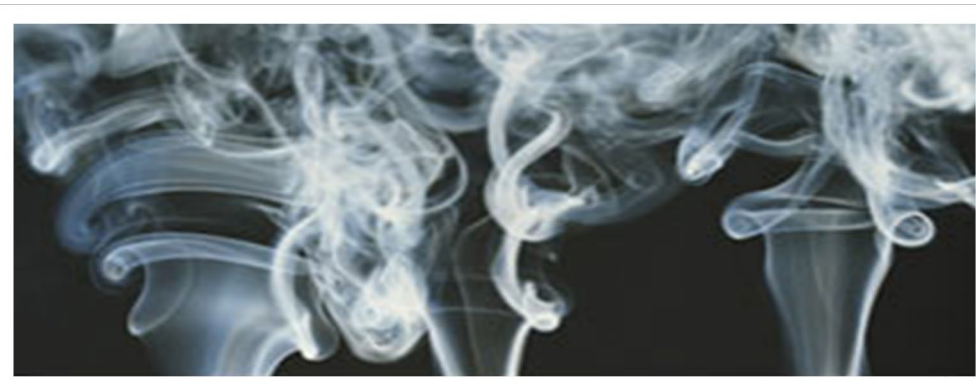
$$Re = \frac{UL}{\nu}$$

Ratio of inertial
to viscous forces:

$$\frac{\text{div}(\rho \mathbf{u} \mathbf{u}) \sim \rho U^2 / L}{\text{div}(\mu \text{grad}(\mathbf{u})) \sim \mu U / L^2}$$

Basic characteristics of turbulence

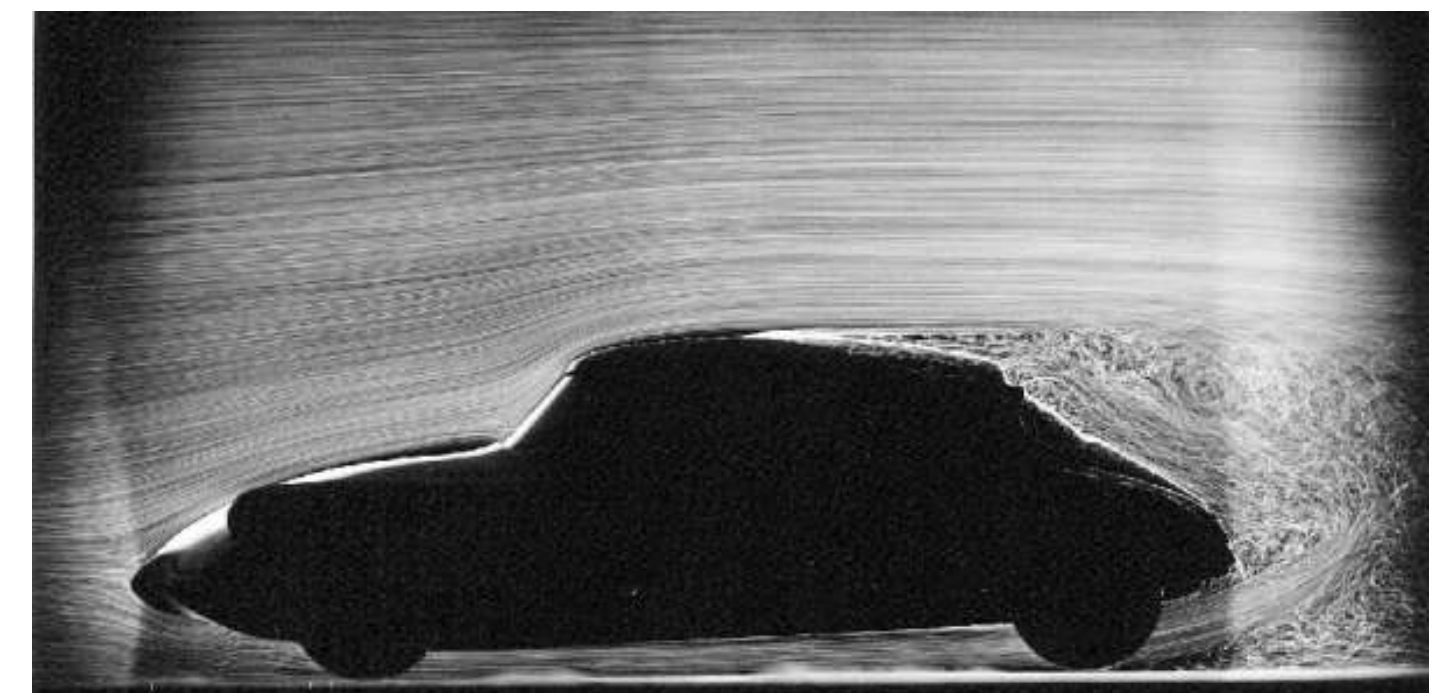
- Flows generally turbulent when Reynolds numbers above $O(10^3-10^4)$.
- Wide variety of turbulent flows in nature and engineering:



da Vinci, 15th c.



Hiroshige, 19th c.



Quiz

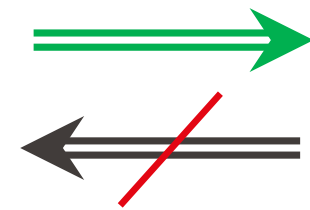


Quiz

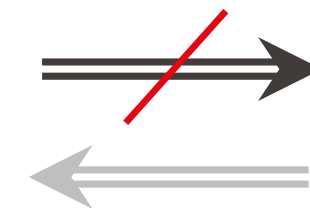
Turbulent flows contain
small-scale vortices.

There is vorticity in
simple shear flows too.

Turbulence



Vorticity



“Rotation” (vortex)

Laminar flows too
can have vorticity.

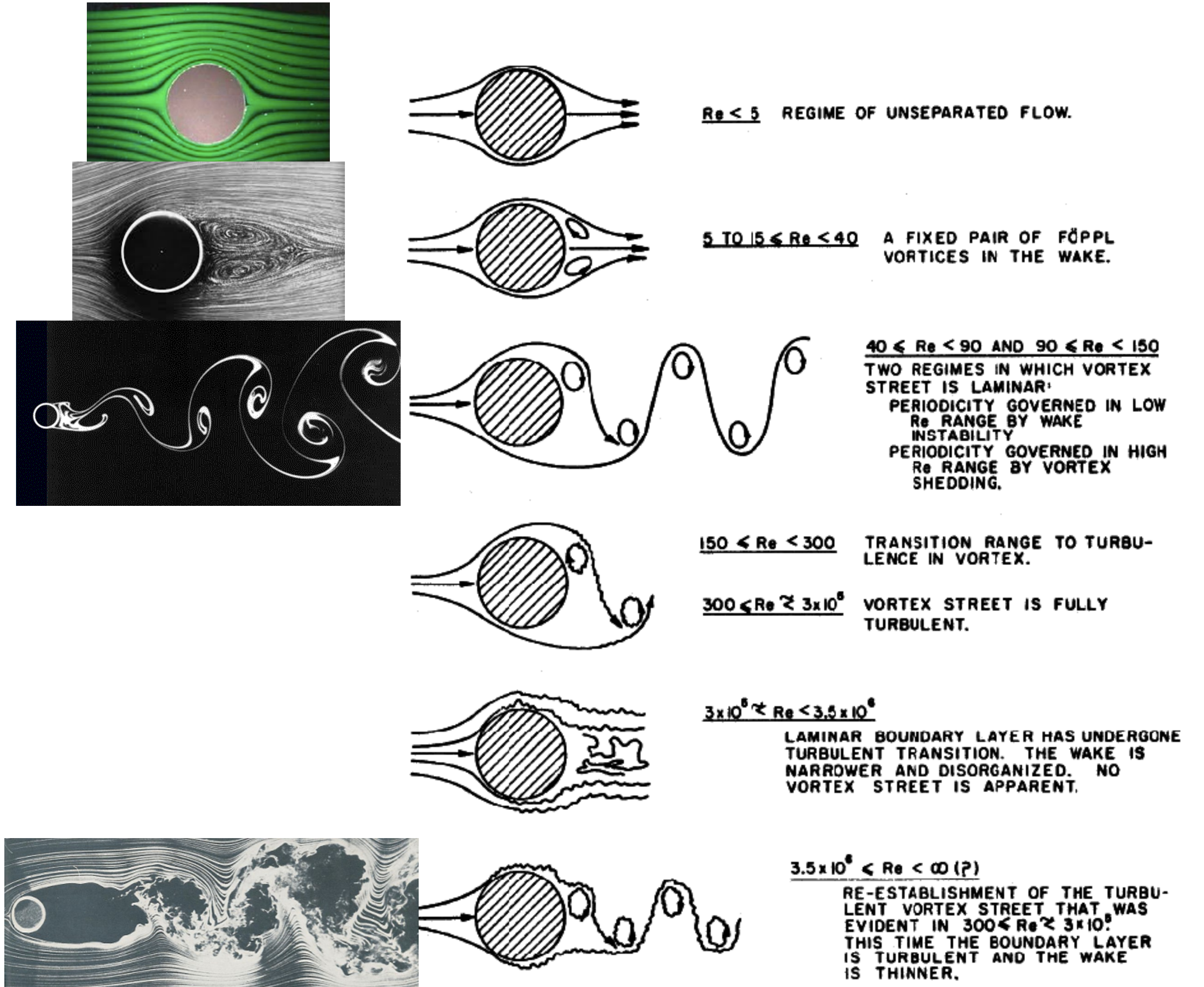
In general vortices have non-
zero vorticity, but there are
irrotational vortices too: for ex.

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1/r \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{\omega} = \mathbf{0}$$

Basic characteristics of turbulence

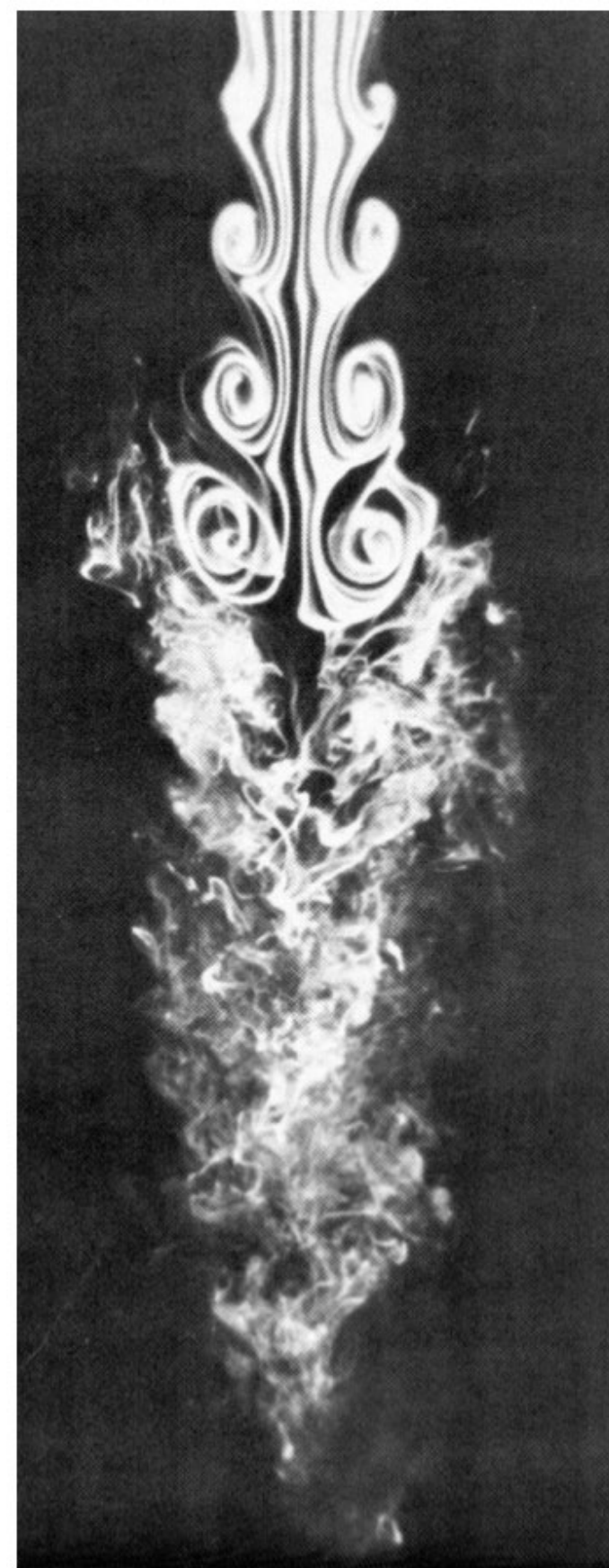
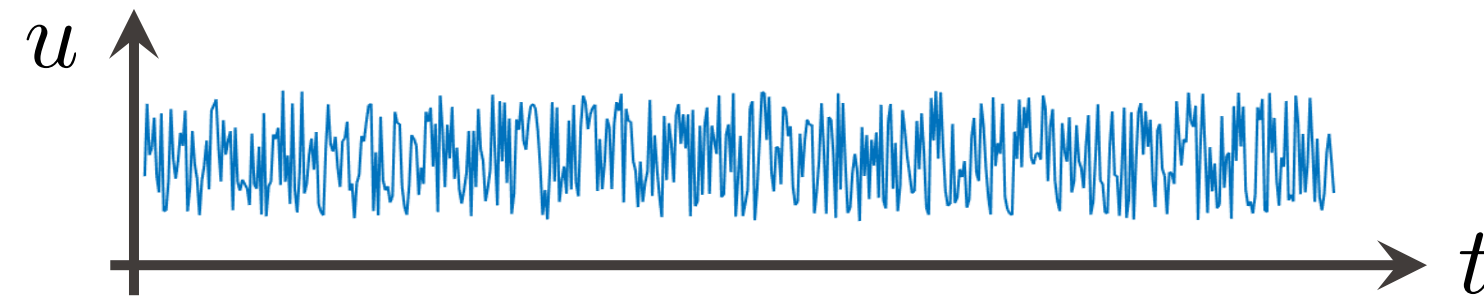
- Turbulent flows contain small-scale vortices, but **vortical flows are not necessarily turbulent.**

Increasing Re



Basic characteristics of turbulence

- No universal definition, but a set of characteristics:
 1. **Always unsteady** (even with steady boundary conditions).



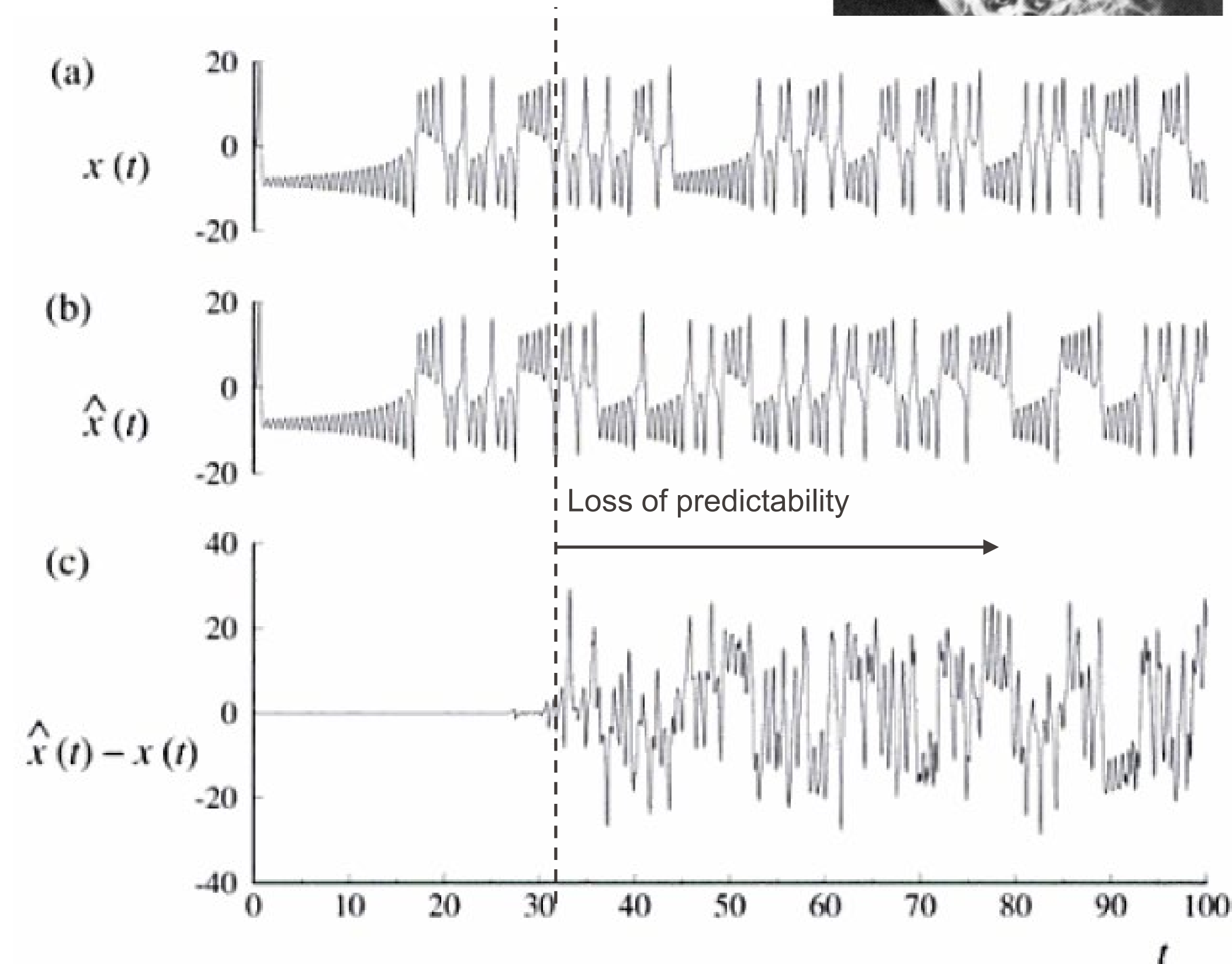
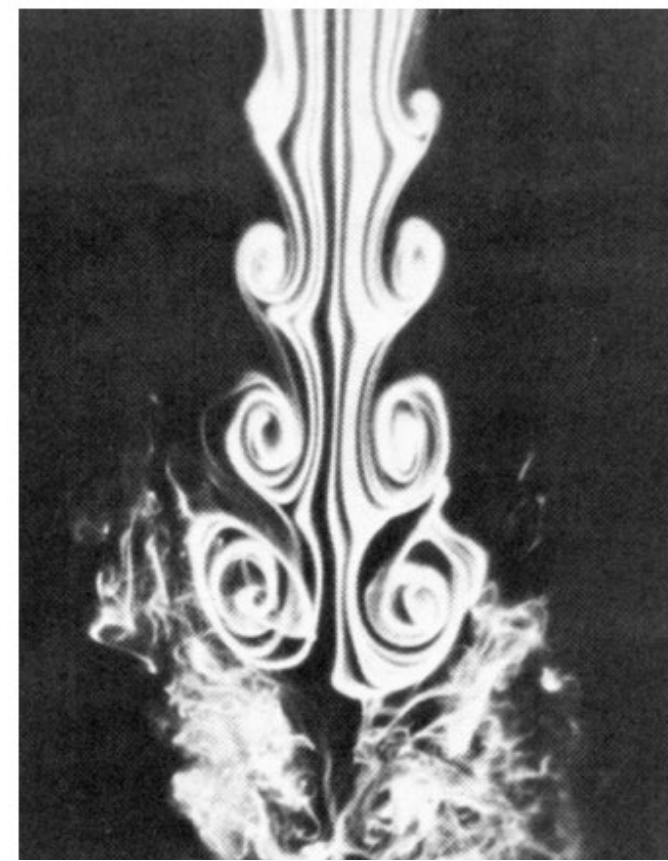
Basic characteristics of turbulence

- No universal definition, but a set of characteristics:

2. Random process.

- Signals seem unpredictable in time and space. Repetitions of the same experiment under same conditions yield different measurements. (But statistical properties *are* predictable.)
- The NS equations are deterministic, but become extremely sensitive to perturbations as Re increases. Similar to chaotic systems (“butterfly effect”).

Example: Lorenz’ system (1963) of ODEs with only 3 degrees of freedom (x,y,z). Two simulations with nearby initial conditions, $x(0)=0.1$ and 0.100001 .



Basic characteristics of turbulence

- No universal definition, but a set of characteristics:

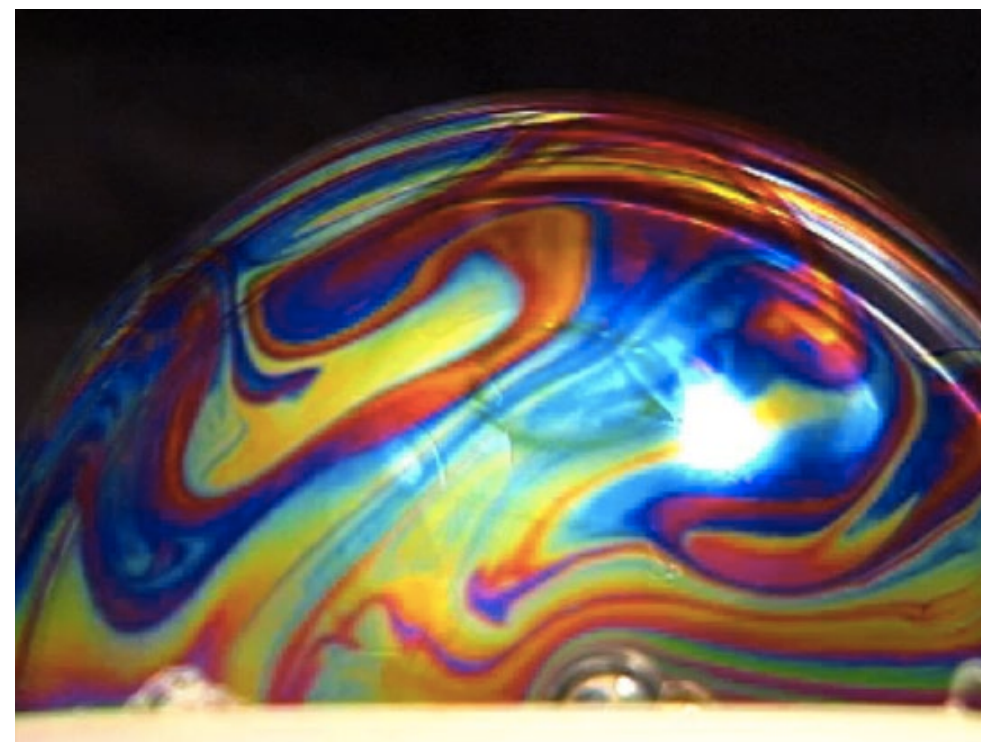
3. Always 3D.

- Fluctuations in all directions, even in geometry invariant in 3rd direction.
- 3D is necessary for vortex stretching (intensification) and tilting (deformation), an essential mechanism in turbulent flows.

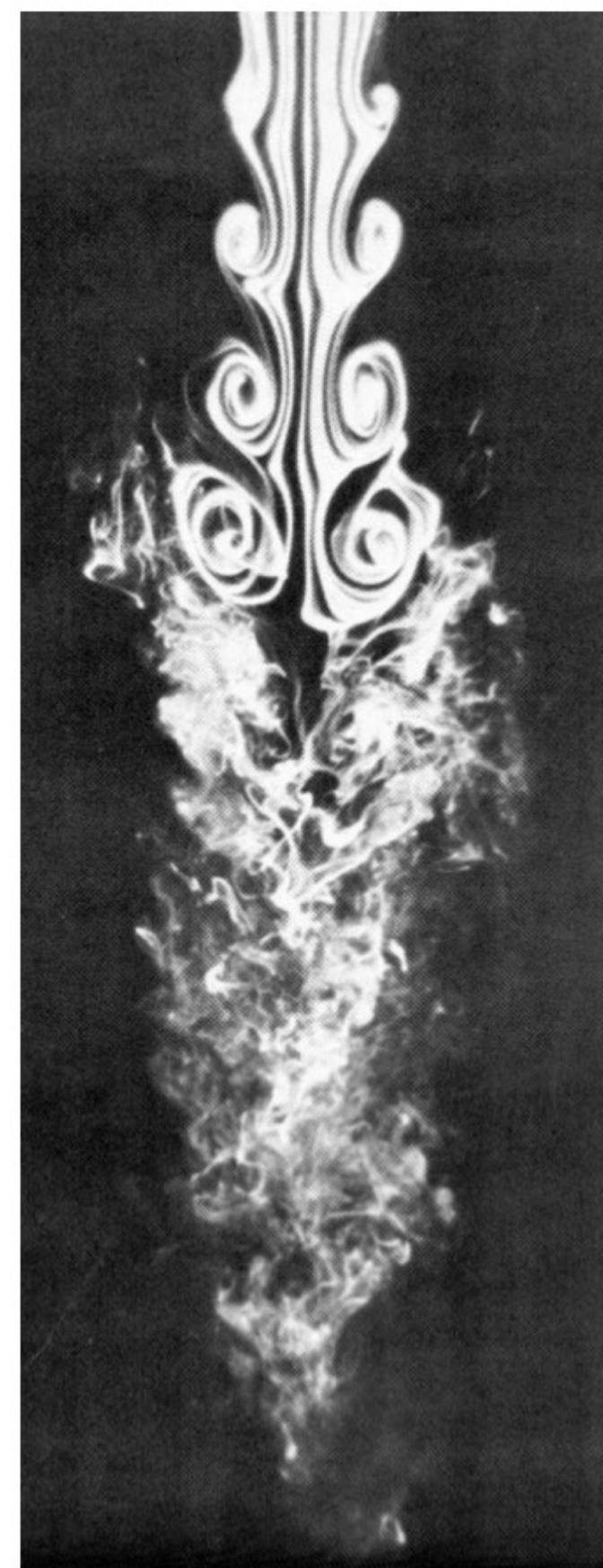
Recall the (incompressible) vorticity equation:

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t}}_{\text{convection}} + \underbrace{(\mathbf{u} \cdot \nabla) \boldsymbol{\omega}}_{\text{diffusion}} = \nu \nabla^2 \boldsymbol{\omega} + \underbrace{(\boldsymbol{\omega} \cdot \nabla) \mathbf{u}}_{\text{vortex stretching/tilting}}$$

- Really 2D only in strongly confined systems (e.g. soap films, geophysical flows).



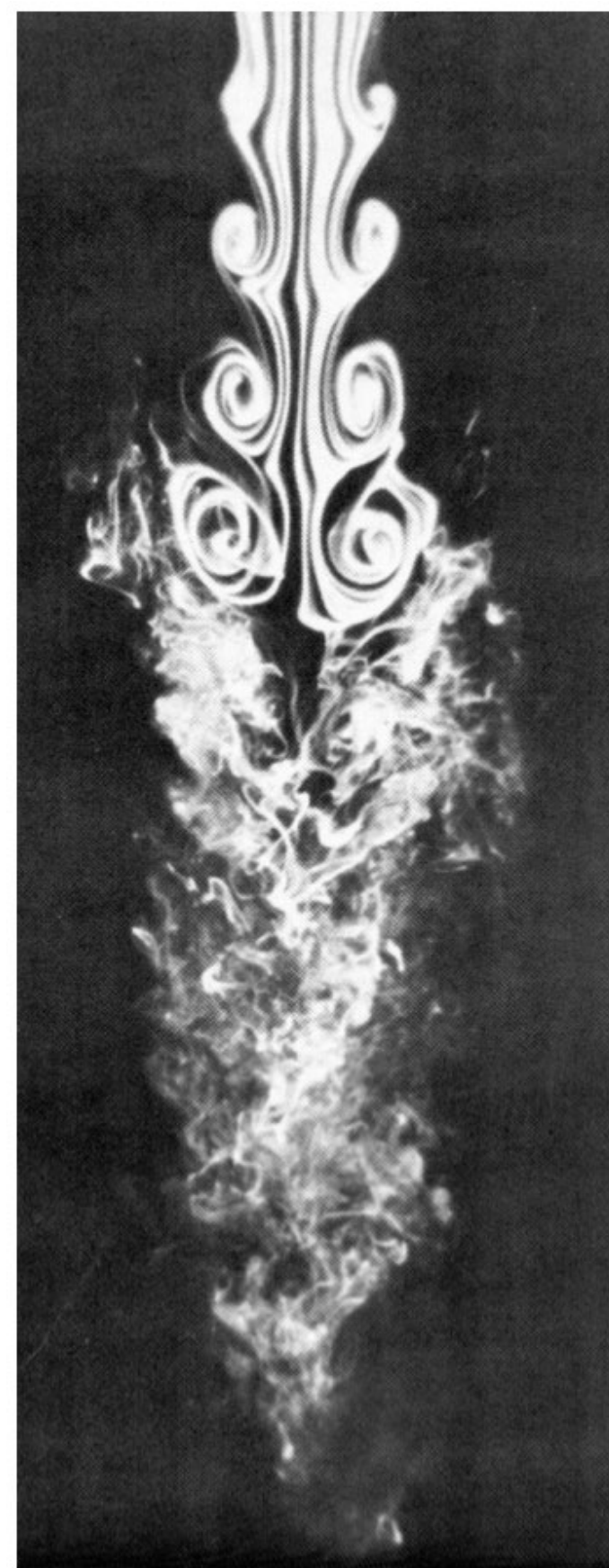
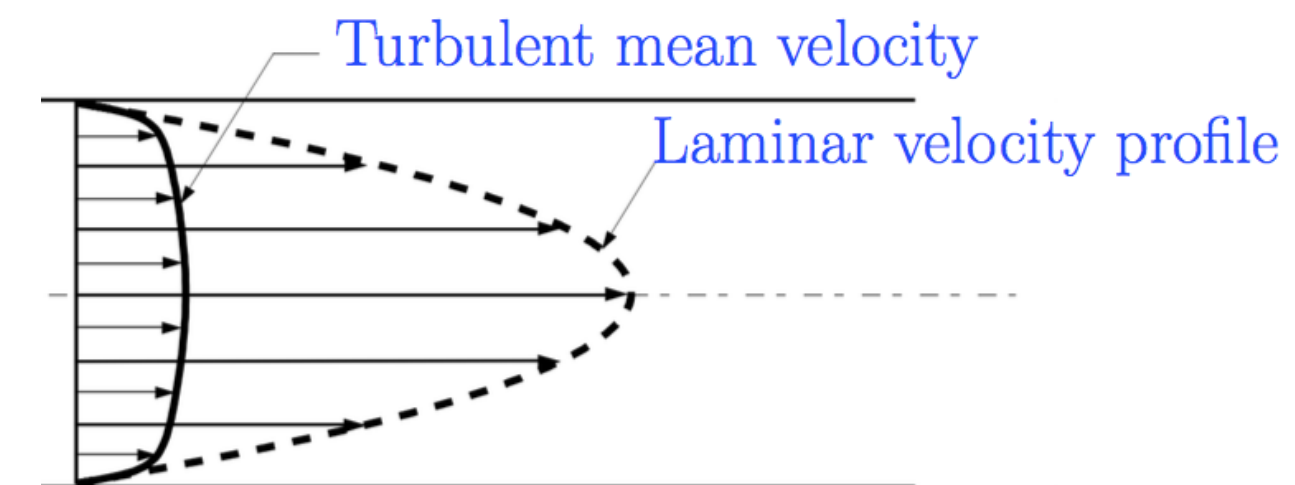
Jupiter



Basic characteristics of turbulence

- No universal definition, but a set of characteristics:
 4. **Turbulence mixes all quantities** (mass, momentum, energy, passive scalars), because fluid particles initially far away can be brought close together by the eddying motions
→ increased mixing and heat transfer, but increased drag/losses too.

Numerical Flow Simulation

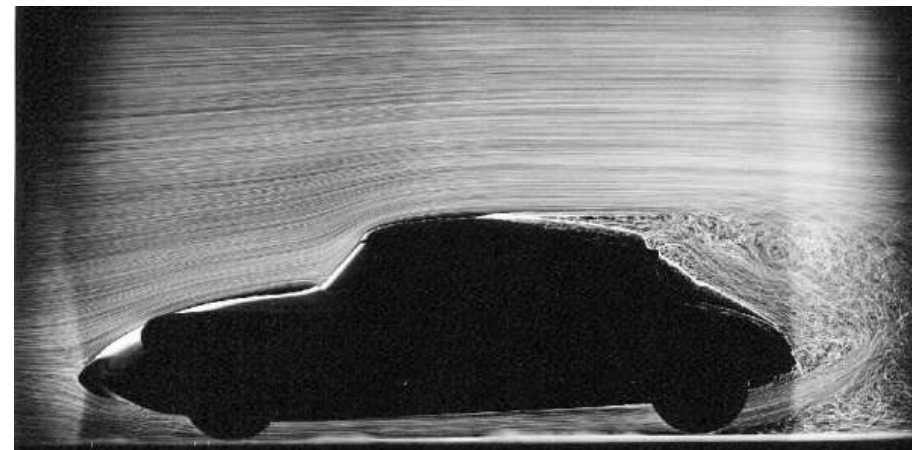


Basic characteristics of turbulence

- No universal definition, but a set of characteristics:
 - 5. Wide range of length and time scales** (broadband spectra). For smallest vortices (“eddies”), Kolmogorov length scale and time scale:

$$\eta \sim \frac{L}{Re^{3/4}} \quad \tau \sim \frac{L/U}{Re^{1/2}}$$

- Examples (air: $\nu \sim 10^{-5} \text{ m}^2/\text{s}$):



$$\begin{aligned} L &\sim 1 \text{ m} \\ U &\sim 10 \text{ m/s} \\ Re = \frac{UL}{\nu} &\sim 10^6 \end{aligned}$$



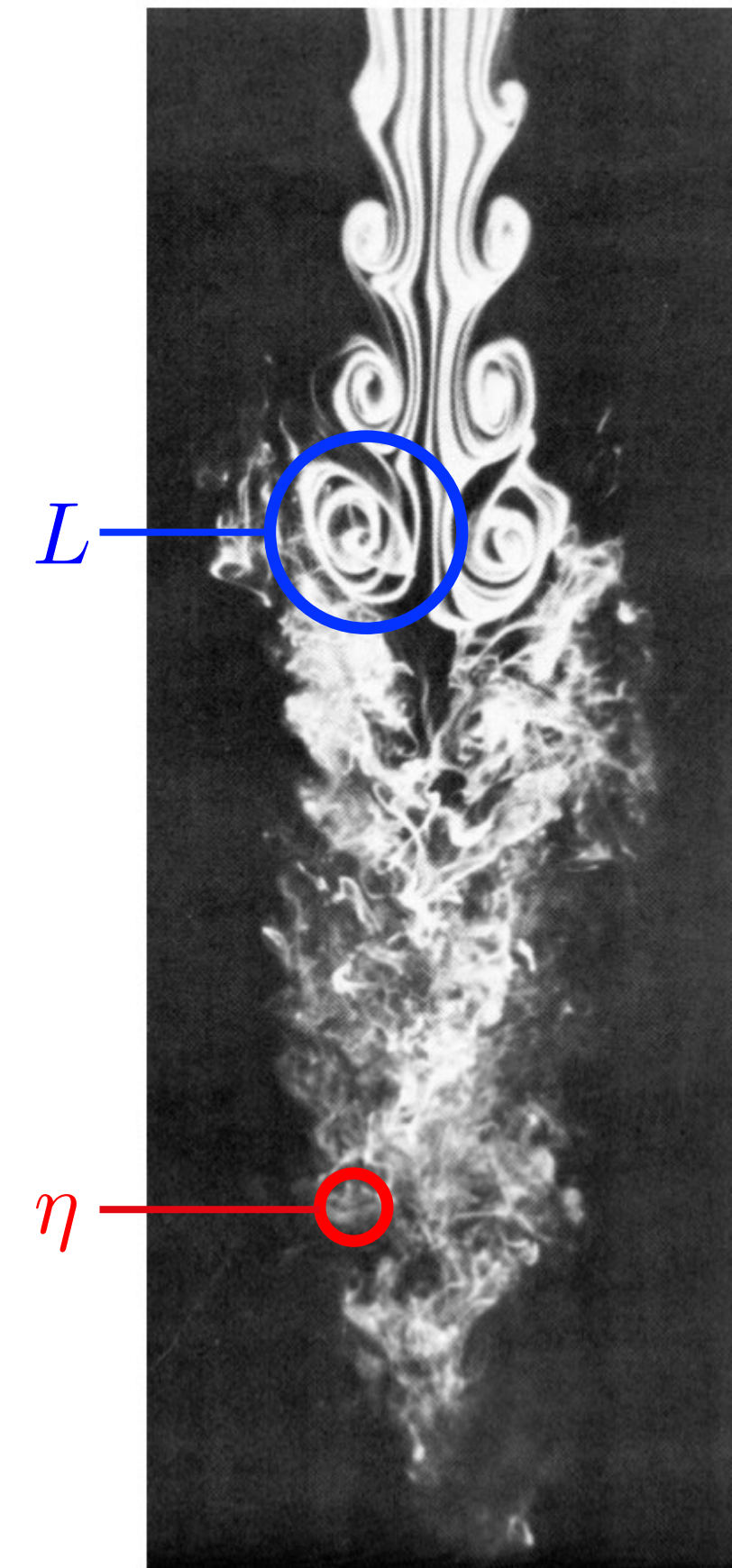
$$\begin{aligned} L &\sim 10^3 \text{ m} \\ U &\sim 10 \text{ m/s} \\ Re = \frac{UL}{\nu} &\sim 10^9 \end{aligned}$$

$$\frac{L}{\eta} \sim \frac{1 \text{ m}}{3 \times 10^{-5} \text{ m}} \sim Re^{3/4} \sim 3 \times 10^4$$

$$\frac{L/U}{\tau} \sim \frac{0.1 \text{ s}}{10^{-4} \text{ s}} \sim Re^{1/2} \sim 10^3$$

$$\frac{L}{\eta} \sim \frac{10^3 \text{ m}}{2 \times 10^{-4} \text{ m}} \sim Re^{3/4} \sim 6 \times 10^6$$

$$\frac{L/U}{\tau} \sim \frac{10^2 \text{ s}}{3 \times 10^{-3} \text{ s}} \sim Re^{1/2} \sim 3 \times 10^4$$



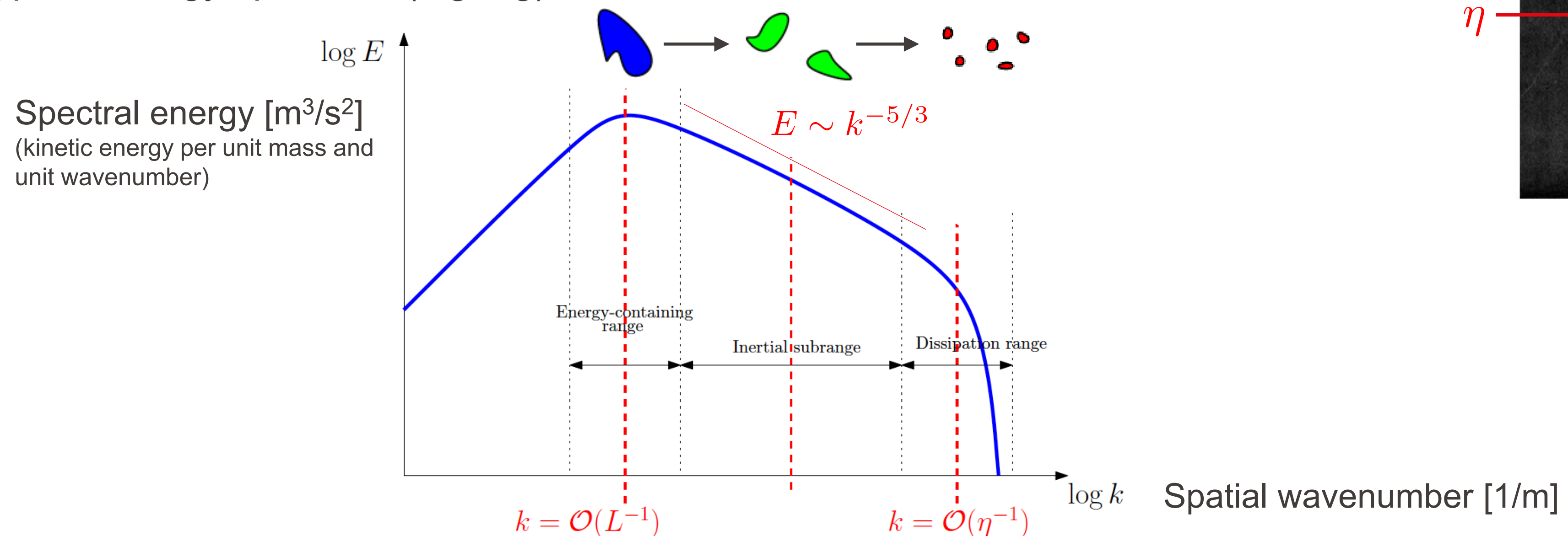
Basic characteristics of turbulence

- No universal definition, but a set of characteristics:

6. Energy cascade:

- Large structures created by hydrodynamic instabilities (**energy injection**),
- Vortex stretching breaks up eddies into smaller ones (**energy transfer**).
- Viscosity dissipates smallest eddies (**energy dissipation**).

Typical energy spectrum (log-log):



Basic characteristics of turbulence

- No universal definition, but a set of characteristics:

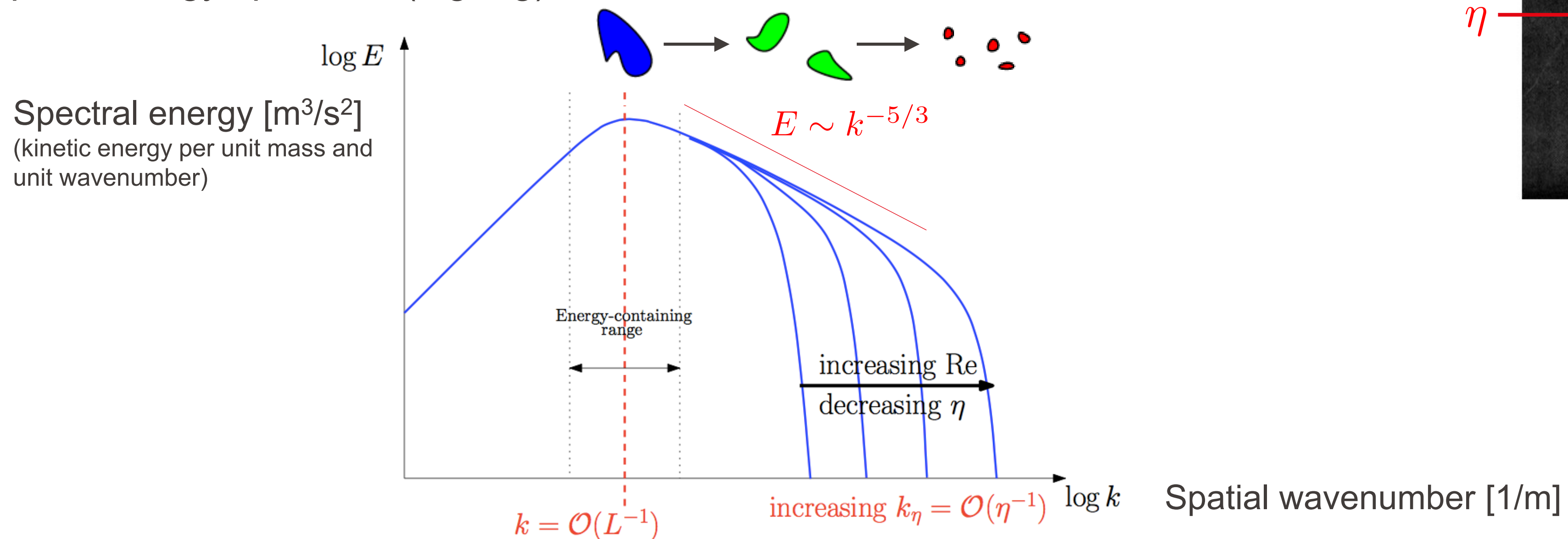
6. Largest eddies:

- Flow & geometry-dependent,
- Most energetic,
- Insensitive to viscosity.

Smallest eddies:

- Homogeneous and isotropic,
- Dissipate energy via viscosity,
- Become smaller as Re increases, but are still much larger than the molecular free path (continuum).

Typical energy spectrum (log-log):



Turbulence is difficult

- Some (verified and unverified) quotations:
 - *“When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first.”* (Heisenberg?)
 - *“I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.”* (Horace Lamb?)
 - *“The study of turbulence is not easy (...) even after various theoretical hypotheses have been absorbed, there are relatively few situations in which we can make positive predictions.”* (P.A. Davidson, “Turbulence: an introduction for scientists and engineers”, Oxford University Press, 2004)

Turbulence is difficult

- Why?
 - Wide range of length and time scales, that increases (dramatically) with Re .
 - Extreme sensitivity (like chaotic systems), random character.
- How can we tackle it?
 - **Analytical approach:** no simple analytical closed-form theory today.
 - **Empirical approach:** describe turbulence statistically.
 - **Numerical approach:** can solve the Navier-Stokes equations directly if enough computational resources to resolve all the scales. Otherwise, need a turbulence model (approximation).

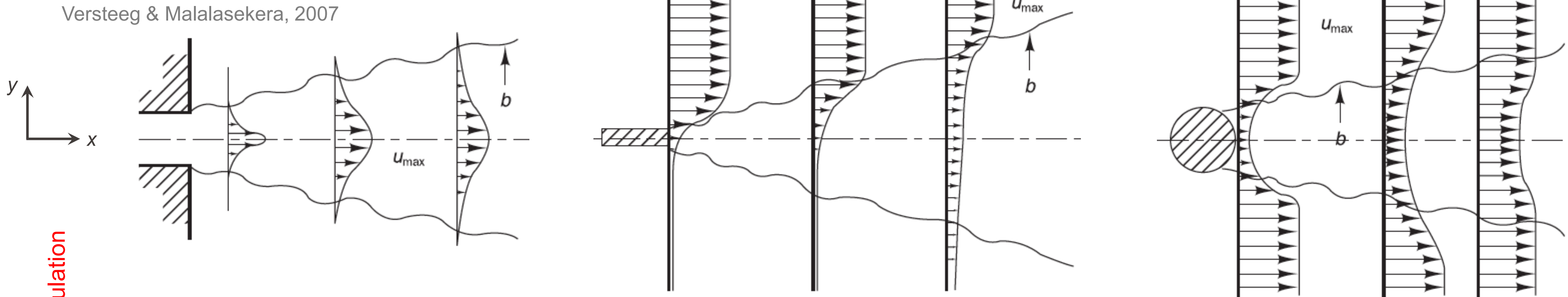
Simple turbulent flows

- Free turbulent flows:

Jet

Mixing layer

Wake



- Mean flow and turbulent structures are **self-similar** (far enough downstream).
For instance, in jets:

Mean streamwise velocity: $\frac{\bar{u}}{\bar{u}_{max}} = f\left(\frac{y}{b}\right)$

Turbulent correlations: $\frac{\overline{u'^2}}{\bar{u}_{max}} = g\left(\frac{y}{b}\right)$ $\frac{\overline{u'v'}}{\bar{u}_{max}} = h\left(\frac{y}{b}\right)$ etc.

Simple turbulent flows

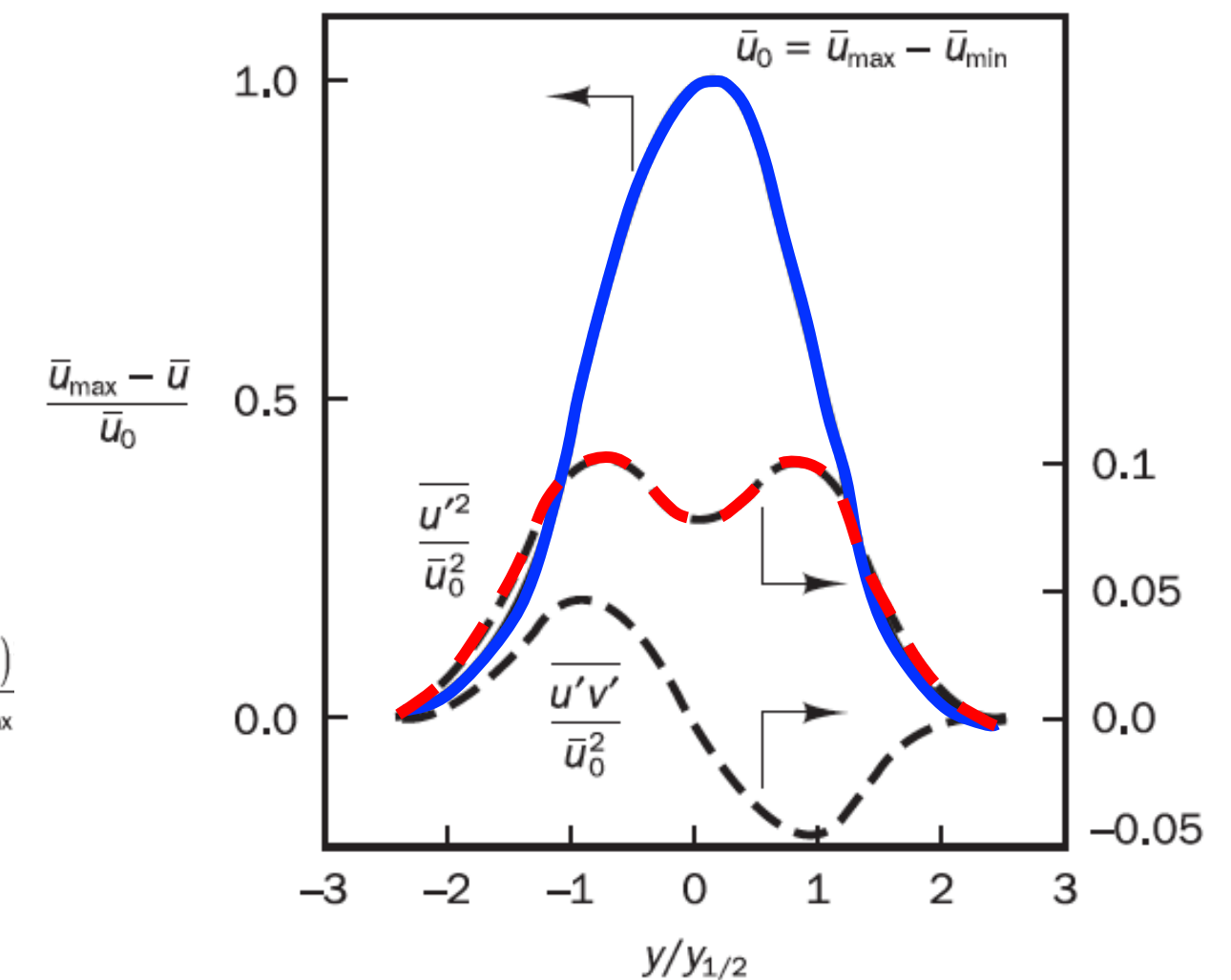
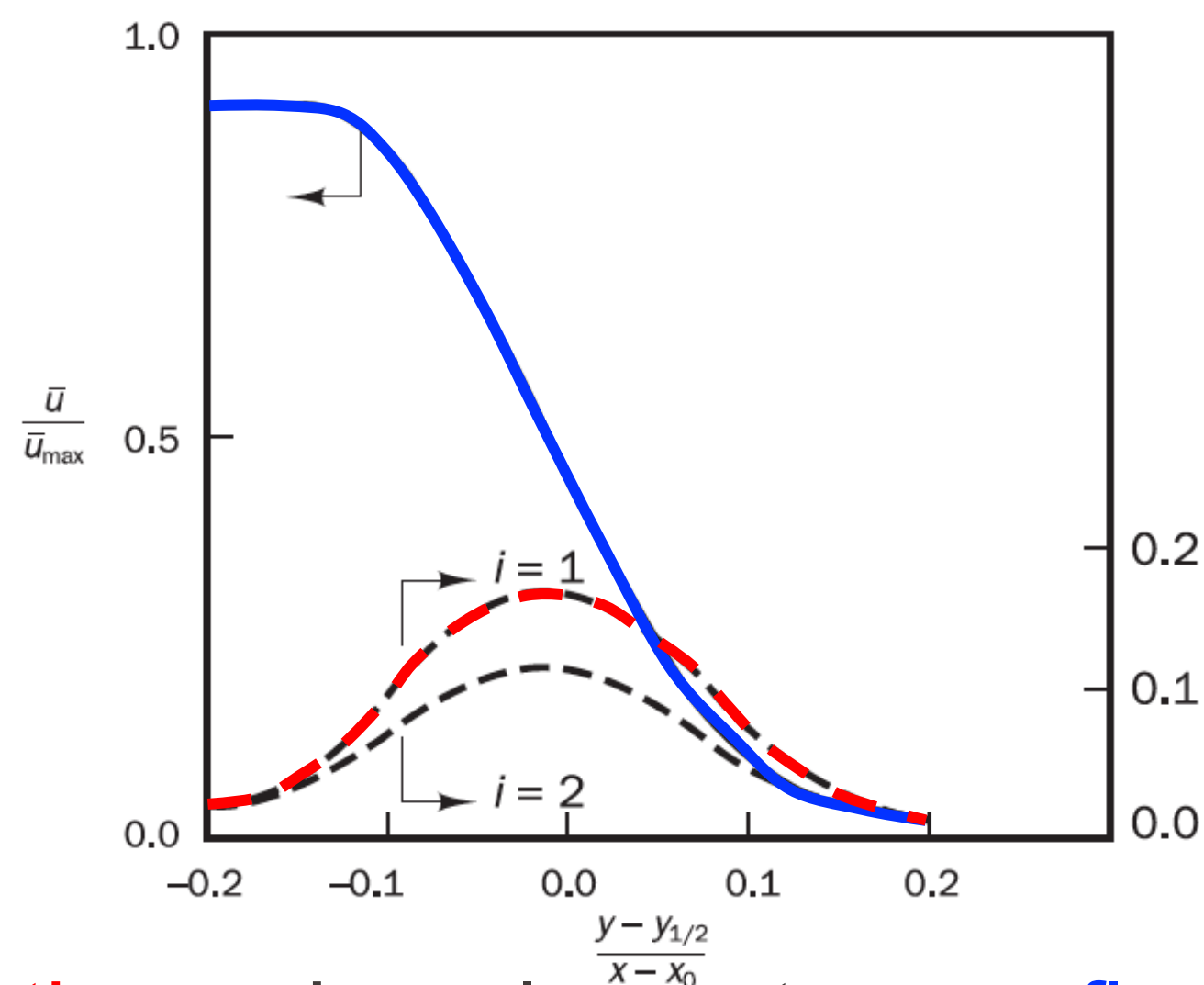
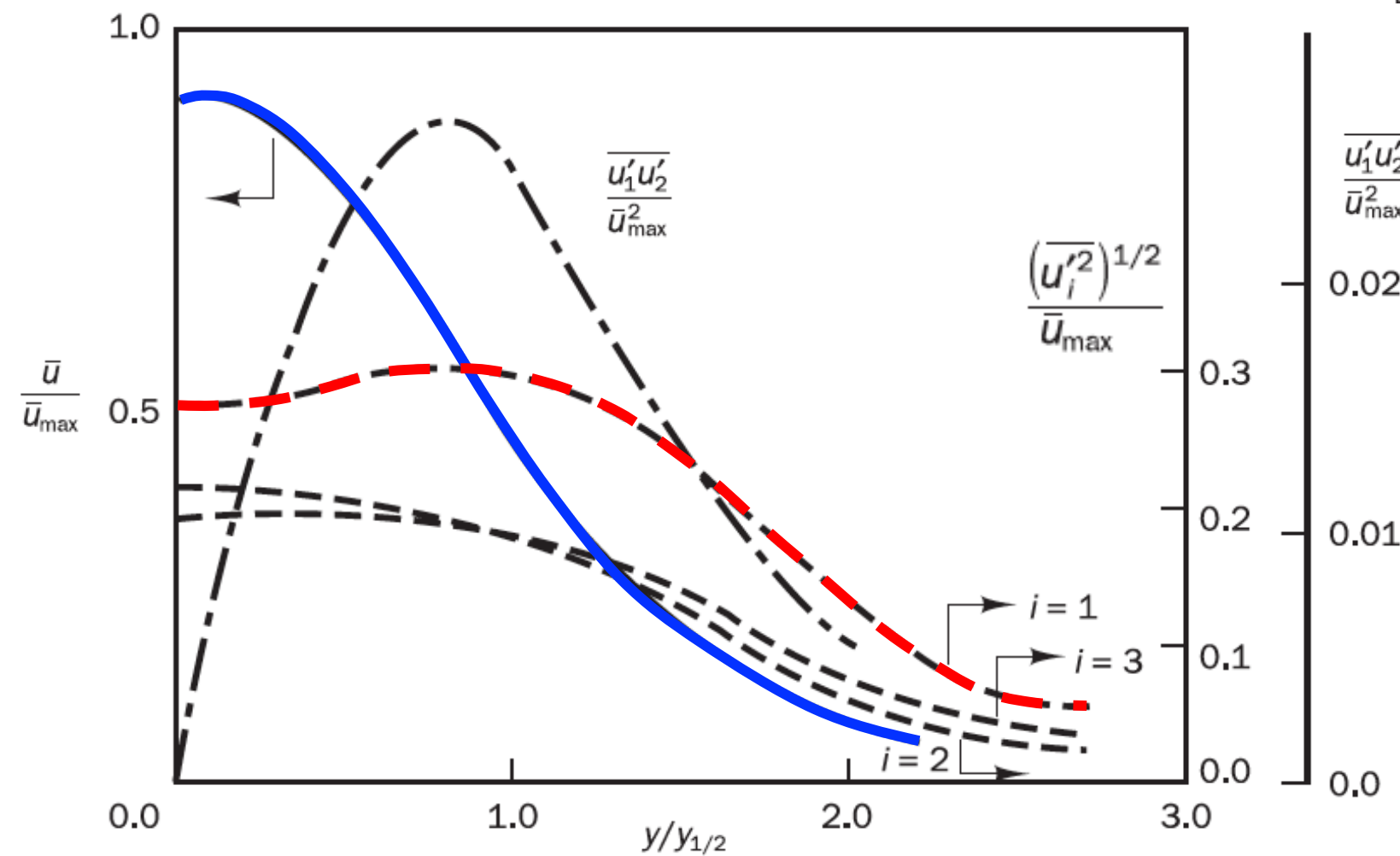
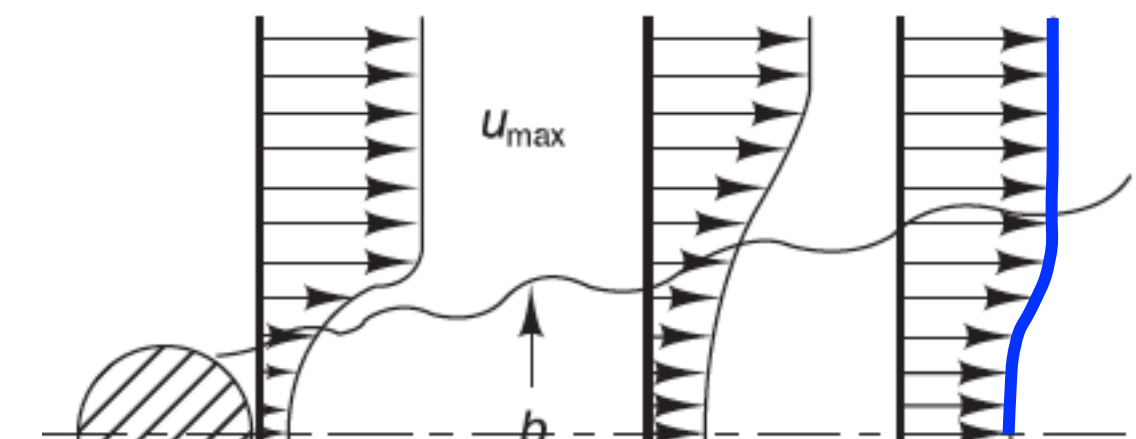
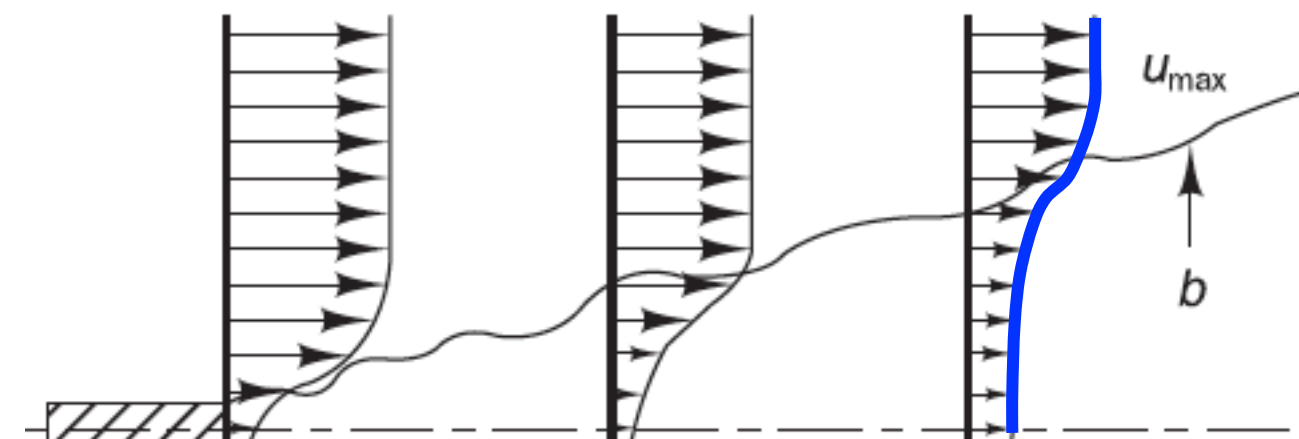
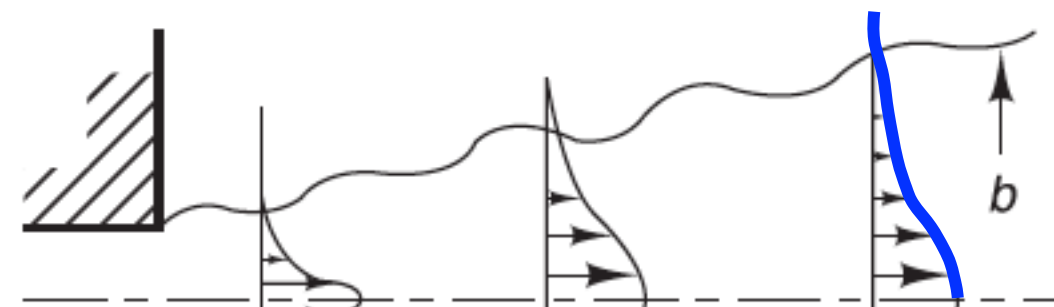
- Free turbulent flows:

Jet

Mixing layer

Wake

Versteeg & Malalasekera, 2007



- Largest **turbulent fluctuations** where largest **mean flow** gradient.
- Turbulence is anisotropic (e.g. $\overline{u'^2} > \overline{v'^2}$) in these flows.

Simple turbulent flows

- Wall-bounded flows: flat plate boundary layer, pipe flow, channel flow...
 - Close to the wall, flow influenced by viscous effects and independent of free-stream parameters → mean flow velocity depends on distance from the wall y and wall shear stress τ_w . From dimensional analysis:

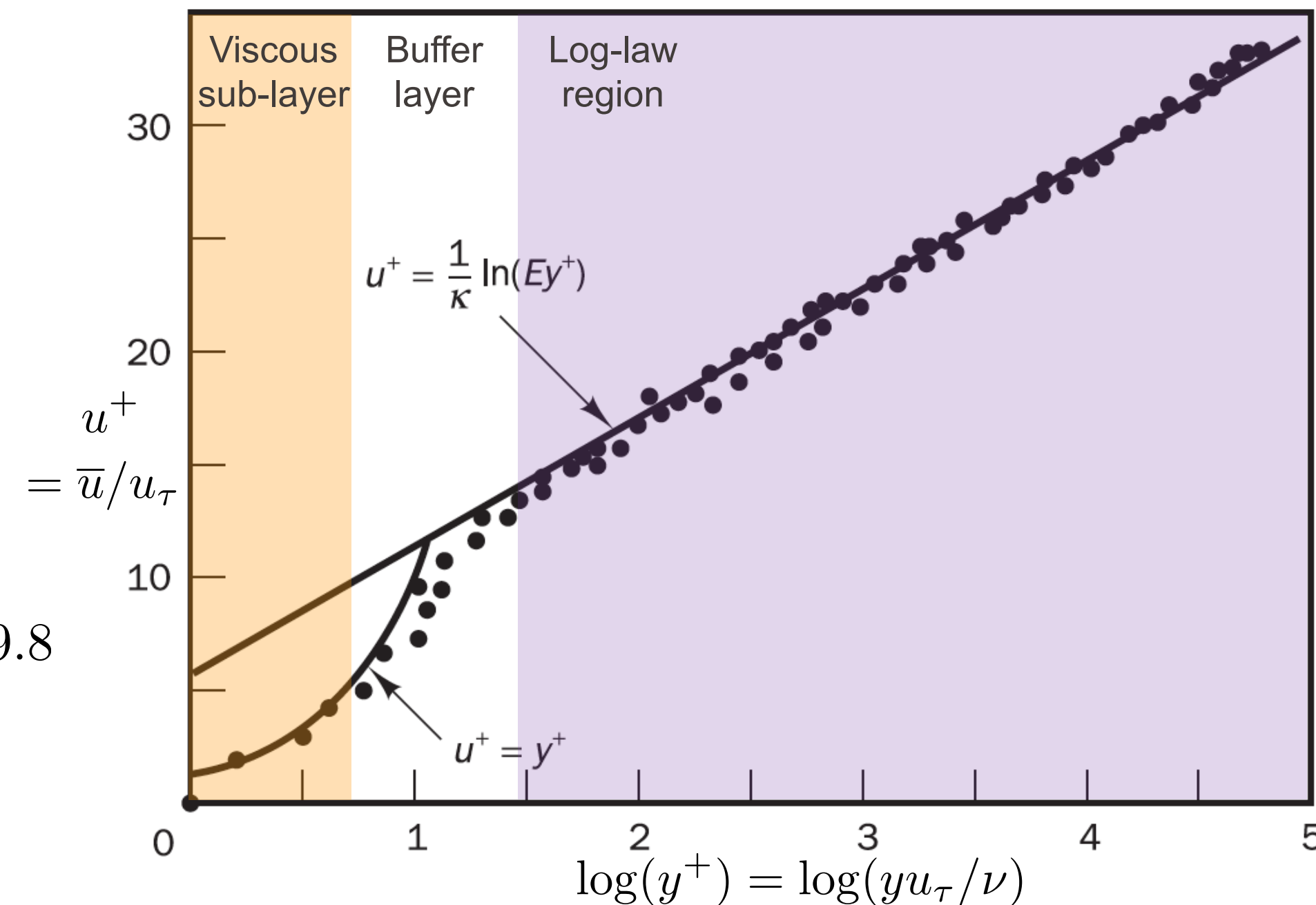
Law of the wall: $\frac{\bar{u}}{u_\tau} = f\left(\frac{yu_\tau}{\nu}\right)$ with velocity scale $u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \frac{\partial u_x}{\partial y}}$ (“friction velocity”)

$$u^+ = f(y^+)$$

- **Viscous sub-layer** ($y^+ < 5$): dominated by viscous effects, very thin, linear relationship

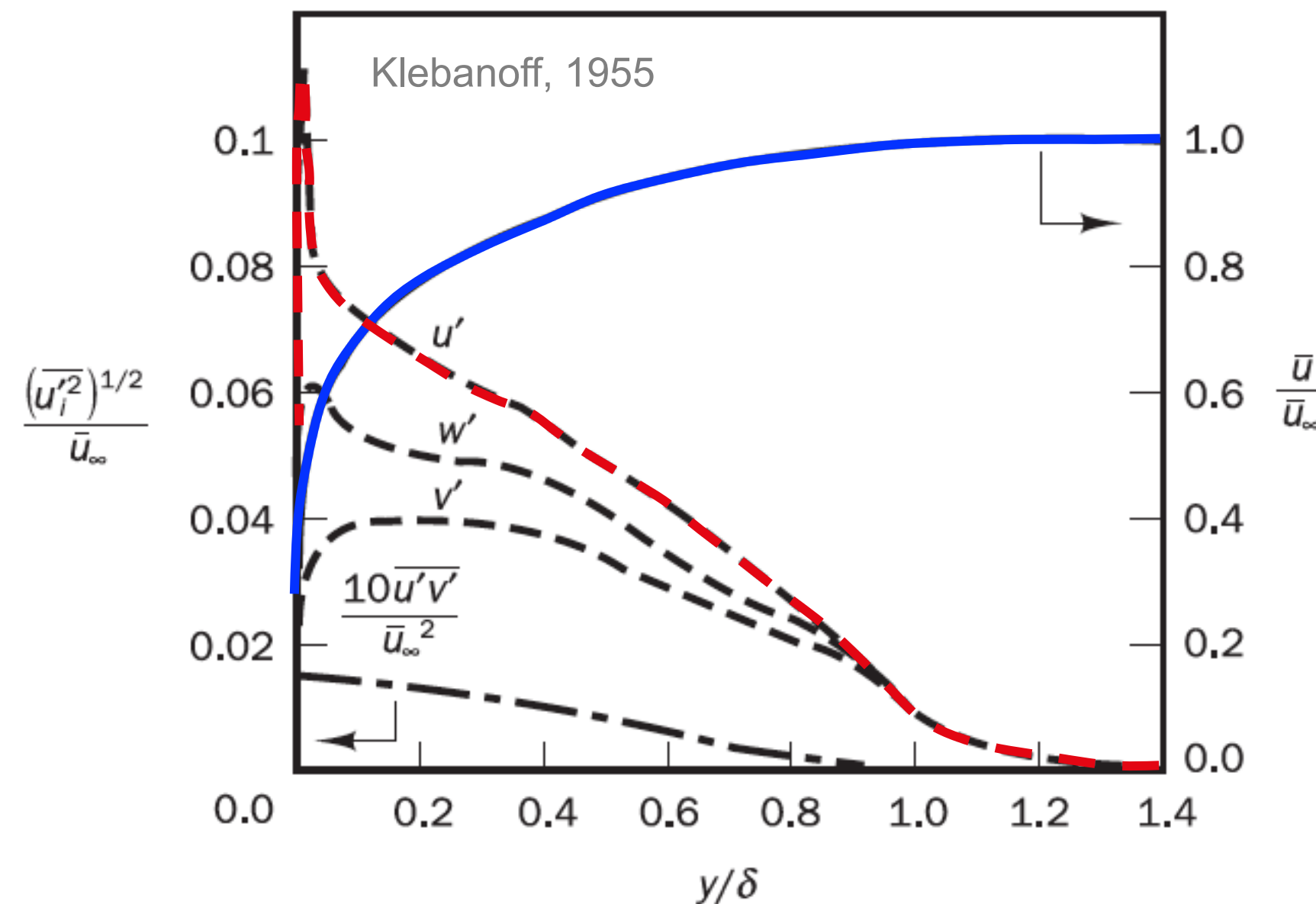
$$u^+ = y^+$$
- **Log-law layer** ($30 < y^+, y/\delta < 0.3$): logarithmic relationship

$$u^+ = \frac{1}{\kappa} \ln(Ey^+) \quad \kappa \approx 0.4, E \approx 9.8$$
- Outer layer ($50 < y^+$): viscous effects negligible, fully turbulent.



Simple turbulent flows

- Wall-bounded flows
 - Example: flat-plate boundary layer (without pressure gradient)



- High **turbulent** production where large **mean flow** gradient.
- Turbulence strongly anisotropic near the wall, becomes isotropic farther away.

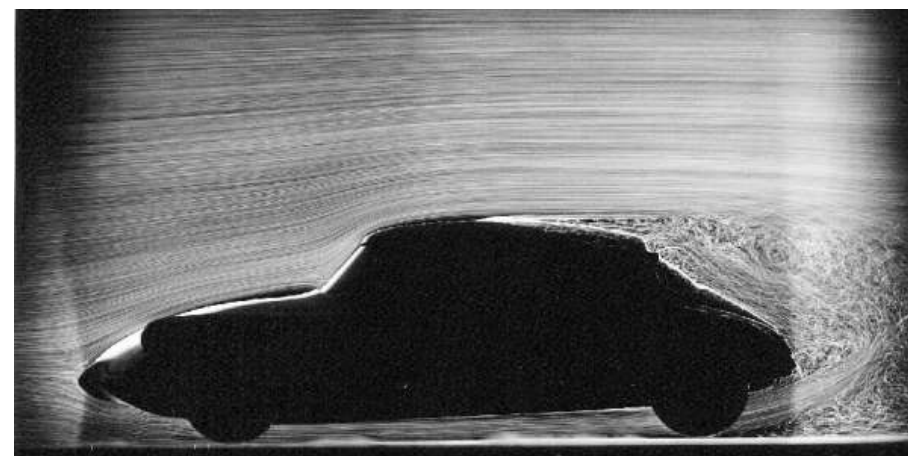
Numerical simulation of turbulent flows

- Challenging because of the wide range of length and time scales

- Recall the ratio of large scales to Kolmogorov scales: $\frac{L}{\eta} \sim Re^{3/4}$
- Number of operations needed to resolve all scales: $\frac{L/U}{\tau} \sim Re^{1/2}$

$$N = N_{\mathbf{x}} N_t \sim \left(\frac{L}{\eta} \right)^3 \left(\frac{L/U}{\tau} \right) \sim Re^{11/4}$$

- Examples:



$$L \sim 1 \text{ m}$$

$$U \sim 10 \text{ m/s}$$

$$Re = \frac{UL}{\nu} \sim 10^6$$

$$N \sim 10^{33/2} \sim 10^{16.5}$$



$$L \sim 10^3 \text{ m}$$

$$U \sim 10 \text{ m/s}$$

$$Re = \frac{UL}{\nu} \sim 10^9$$

$$N \sim 10^{99/4} \sim 10^{25}$$

- Starts to become accessible to fastest supercomputers, but not quite to regular computers used in the industry.

Numerical simulation of turbulent flows

- Three main methods:

- 1. Direct numerical simulation (DNS):**

- Resolve all scales, all the way down to Kolmogorov scale

- 2. Large-eddy simulation (LES):**

- Resolve larger eddies (flow & geometry-dependent) down to a size in the inertial subrange
- Model the effect of smaller eddies (isotropic, universal behavior)

- 3. Reynolds-averaged Navier-Stokes (RANS) eqs. + turbulence models**

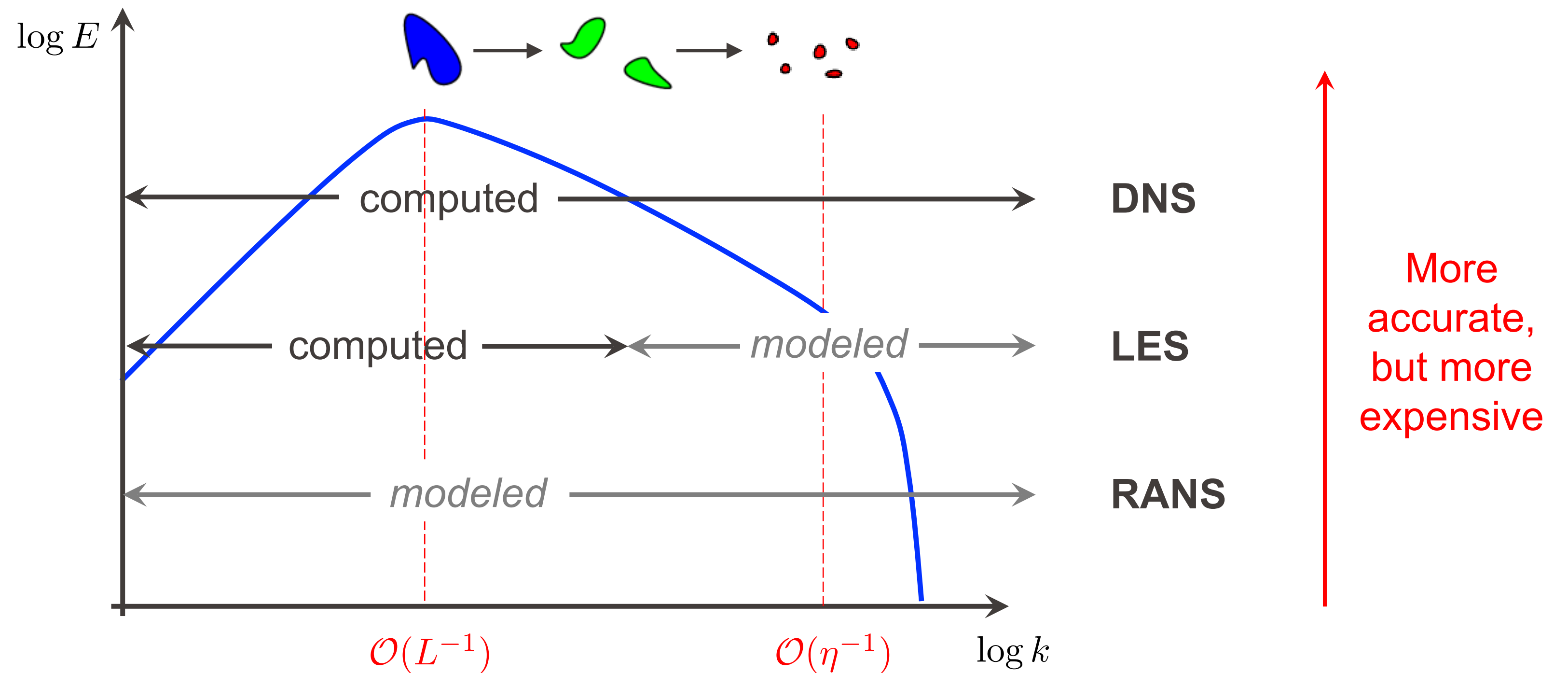
- Solve for mean flow (ensemble average)
- Model fluctuations at all scales



More
accurate,
but more
expensive

Numerical simulation of turbulent flows

- Three main methods:



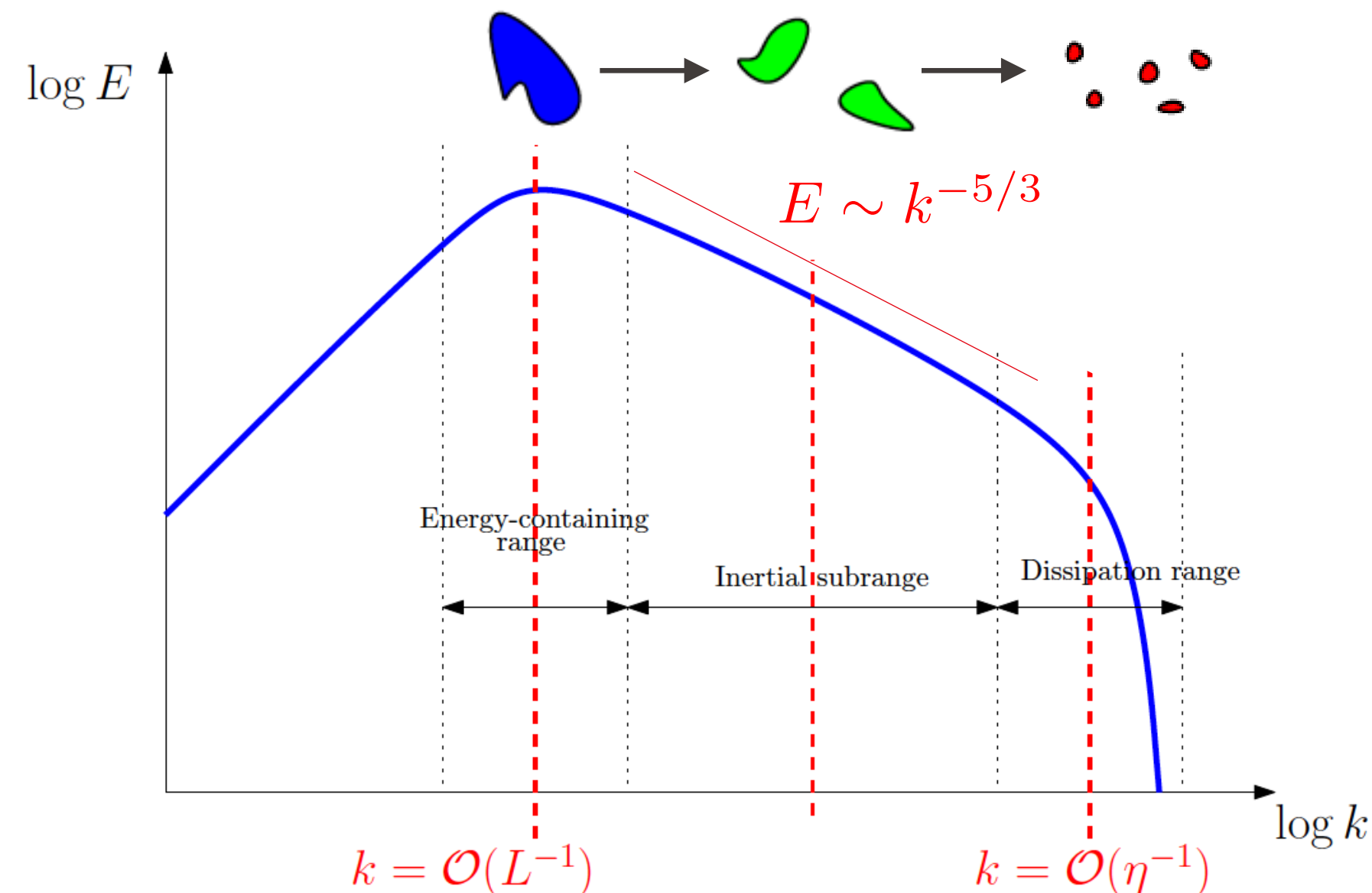
Direct numerical simulation (DNS)

- Resolve all scales, all the way down to Kolmogorov scale. No modeling.
- Assessment:
 - Highest possible level of description (all scales are resolved, all orders of the statistical moments can be computed)
 - Complete by nature (no flow-specific parameter)
 - Prohibitive cost at high Re
 - Range of applicability: theoretically unlimited
 - Accuracy: theoretically unlimited (practically limited by the precision of the numerical method)

Large-eddy simulations (LES)

- Compute explicitly large (energy-containing) eddies. Their dynamics is flow and geometry-dependent.
- Model the effect of small (dissipative) eddies. They have a more isotropic and flow-independent dynamics → easier to find a universal model.
- Large savings compared to DNS (which spends most computational effort on smallest dissipative scales).

Numerical Flow Simulation



Large-eddy simulations (LES)

- Two main steps:

1. Choice of a filtering operation to separate resolved scales and modeled scales
→ spatial filtering of the Navier-Stokes equations.
2. Choice of a closure model: subgrid-scale (SGS) modeling.

1. Spatial filtering:

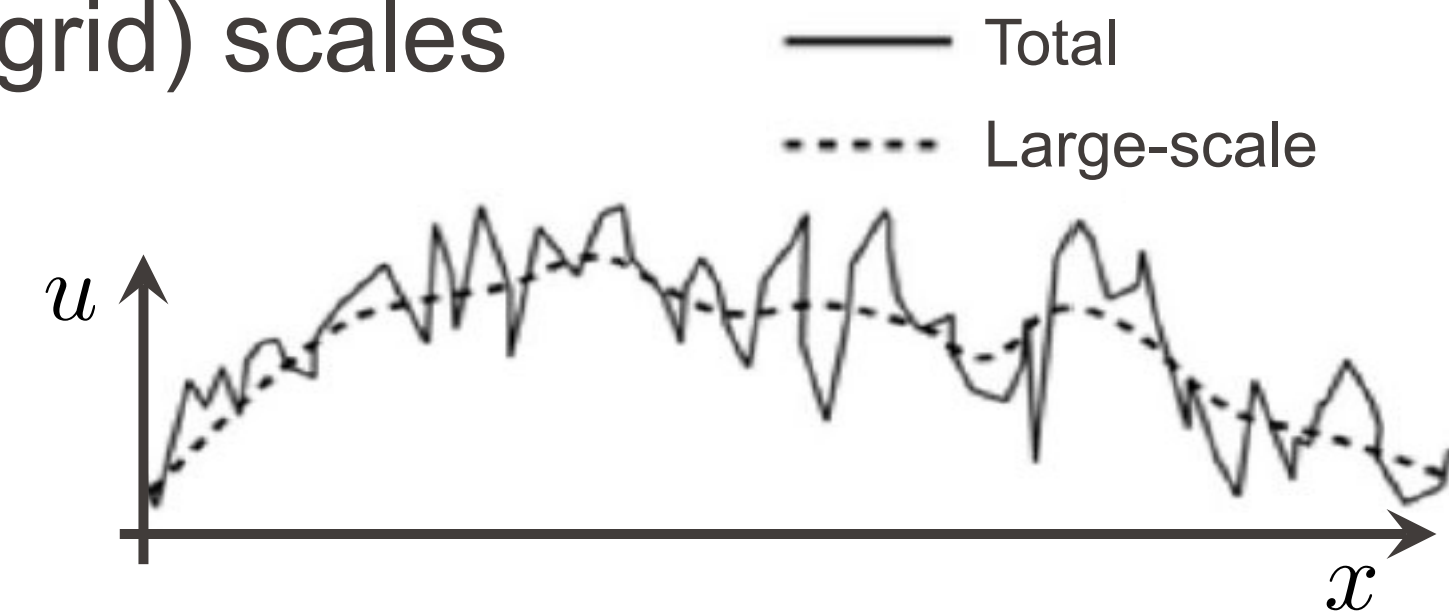
- Total flow = filtered (resolved) scales + residual (subgrid) scales

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}^\dagger(\mathbf{x}, t)$$

$$\bar{\mathbf{u}}(\mathbf{x}, t) = \int \mathbf{u}(\mathbf{x}', t) G(\mathbf{x} - \mathbf{x}') d\mathbf{x}'$$

- Filtering = convolution in physical space
= multiplication in spectral space

G : filter, or convolution kernel. Many types: sharp, Gaussian... Determines the size of the resolved eddies.



Large-eddy simulations (LES)

1. Filtered NS eqs. (incompressible): eqs. for the resolved, large-scale eddies

$$\nabla \cdot \bar{\mathbf{u}} = 0, \quad \frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}} - \boxed{\nabla \cdot \boldsymbol{\tau}}$$

- Additional term: **subgrid-scale (SGS) stress tensor** $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$
- Effect of the small eddies on the dynamics of the resolved, large-scale eddies.
- **Closure problem**: SGS stress unknown (non-resolved terms)! Must be modeled.

2. SGS modeling

- Most common choice: eddy viscosity model. SGS stress proportional to rate of deformation of the resolved flow, and same coefficient of proportionality for all components.

$$\tau_{ij} - \frac{1}{3} \tau_{kk} \delta_{ij} = -\nu_{SGS} (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)_{ij} \quad \nu_{SGS} : \text{eddy viscosity [m}^2/\text{s]}$$

- Many SGS models proposed for eddy viscosity.
For example: Smagorinsky-Lilly, dynamic Smagorinsky-Lilly, wall-adapted local eddy-viscosity (WALE), dynamic kinetic energy subgrid-scale (all available in Fluent).

Reynolds-averaged Navier-Stokes (RANS) equations

- Reynolds decomposition

- Total flow = mean flow (ensemble average) + fluctuations

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t)$$

- Averaging properties:

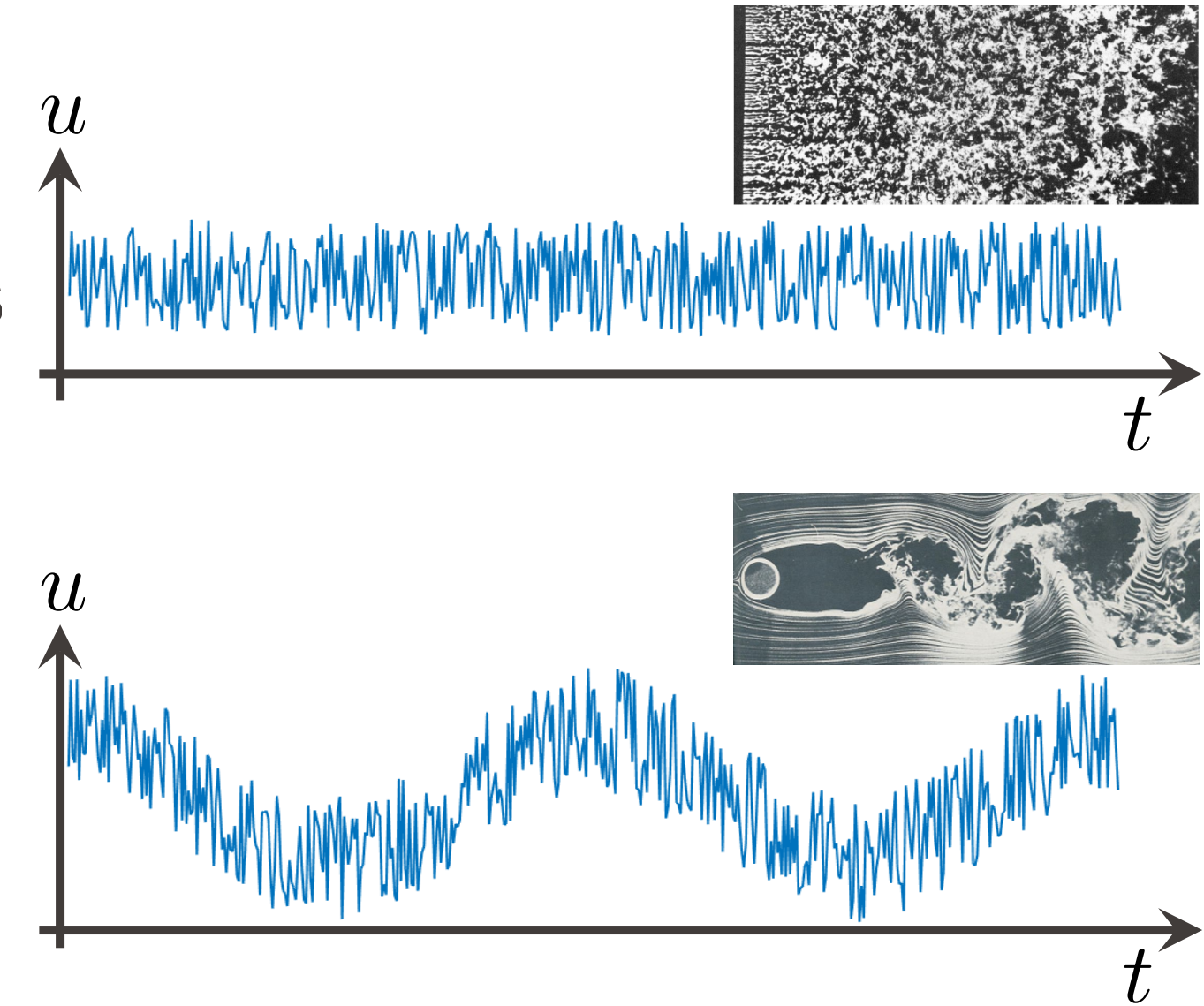
$$\overline{f'} = 0 \qquad \overline{\overline{f}g} = \overline{f}g$$

$$\overline{\overline{f}g'} = 0 \qquad \overline{\frac{\partial f}{\partial x}} = \frac{\partial \overline{f}}{\partial x} \qquad \overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t}$$

- Incompressible NS equations

- Continuity eq.: $\overline{\nabla \cdot \mathbf{u}} = 0 \longrightarrow \nabla \cdot \bar{\mathbf{u}} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u}' = 0$

The mean flow and fluctuations are incompressible.



Reynolds-averaged Navier-Stokes (RANS) equations

- Incompressible NS equations

- Momentum eq.:

$$\overline{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u}} = -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \mathbf{u}$$

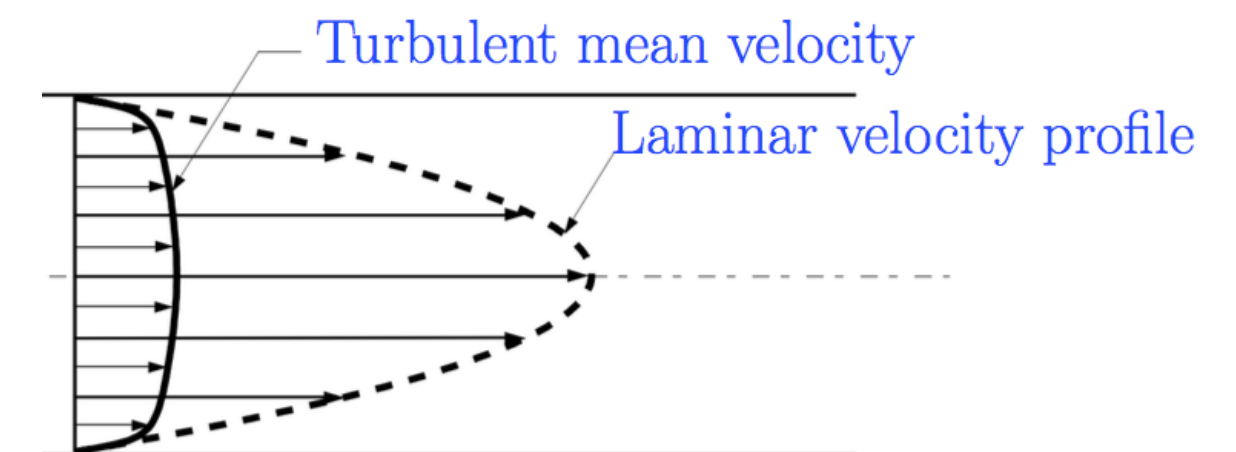
$$\overline{\frac{\partial (\bar{\mathbf{u}} + \mathbf{u}')}{\partial t} + ((\bar{\mathbf{u}} + \mathbf{u}') \cdot \nabla) (\bar{\mathbf{u}} + \mathbf{u}')} = -\frac{1}{\rho} \nabla (\bar{p} + p') + \nu \nabla^2 (\bar{\mathbf{u}} + \mathbf{u}')$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \overline{(\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}}} + \overline{(\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}'} + \overline{(\mathbf{u}' \cdot \nabla) \bar{\mathbf{u}}} + \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} = -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}}$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}} - \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'}$$

$= \nabla \cdot \boldsymbol{\tau}$

- RANS eqs. for the mean flow: almost like NS eqs., but divergence of an additional stress, the **Reynolds stress tensor** $\boldsymbol{\tau} = -\overline{\mathbf{u}' \mathbf{u}'}$ (variance of fluctuations, 6 independent components). Makes all the difference!



- **Closure problem:** Reynolds stress unknown! More unknown than equations. To solve the RANS eqs., must **model** the Reynolds stress.

Reynolds-averaged Navier-Stokes (RANS) equations

- Must model the Reynolds stress: $\tau = -\overline{u'u'} = ?$
- Several types of turbulence models of increasing complexity
 - Turbulent viscosity models:
 - 0-equation models (algebraic models): mixing length models
 - 1-equation models: Spalart-Allmaras, turbulent kinetic energy model
 - 2-equation models: k - ε , k - ω models
 - Reynolds-stress eq. models (RSM): 7 additional transport eqs.

More accurate
or general,
but more
expensive

- “*Essentially, all models are wrong but some are useful.*” (G. Box)
- “*Models of turbulence are inevitably incomplete.*” (S. Pope)

RANS + turbulent viscosity models

- To close the RANS equations, assume that the Reynolds stress is related to the mean flow, i.e. $\tau = -\overline{\mathbf{u}'\mathbf{u}'}$ can be expressed as a function of $\bar{\mathbf{u}}$.
- **Boussinesq hypothesis** (1877): Reynolds stress proportional to mean rate of deformation, and same coefficient of proportionality for all components

$$\tau + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

where $k = \frac{1}{2} (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$ is the turbulent kinetic energy (TKE) [m²/s²],
and ν_t is the **turbulent viscosity** (eddy viscosity) [m²/s].

- Analogy with viscous stress (proportional to rate of deformation) and molecular viscosity:

$$\sigma + p\mathbf{I} = \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

- Central issue: how to determine ν_t ? Specific to each model (mixing length, Spalart-Allmaras, k - ε , k - ω ...)

RANS + turbulent viscosity models

- To close the RANS equations, assume that the Reynolds stress is related to the mean flow, i.e. $\tau = -\overline{\mathbf{u}'\mathbf{u}'}$ can be expressed as a function of $\bar{\mathbf{u}}$.
- **Boussinesq hypothesis** (1877): Reynolds stress proportional to mean rate of deformation, and same coefficient of proportionality for all components

$$\tau + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

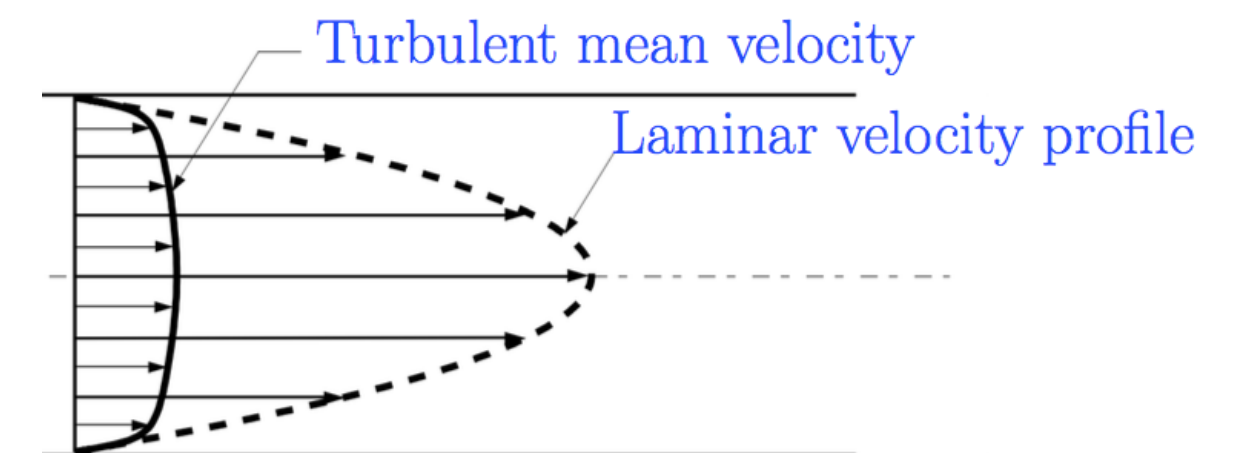
- With this hypothesis, the RANS eqs. for the mean flow

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}} \overset{= \nabla \cdot \tau}{- \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'}}$$

are closed:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = -\frac{1}{\rho} \nabla \left(\bar{p} + \frac{2k}{3} \right) + (\nu + \nu_t) \nabla^2 \bar{\mathbf{u}}$$

- Note: turbulent viscosity must depend on space, $\nu_t(\mathbf{x})$ (otherwise, RANS eqs. would simply be equal to NS eqs. with a different viscosity/different Re , which is not true).



$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

RANS + turbulent viscosity models

- Zero-equation models (algebraic models): mixing length models
 - From dimensional analysis, turbulent viscosity [m²/s] = length * velocity: $\nu_t \sim l_t u_t$
 - Assume strong connection between mean flow and behavior of largest (energy-containing) eddies → build velocity scale on mean gradients:

$$l_t = l_m$$

Length scale = “mixing length”,
to be specified (flow-dependent)

$$u_t = l_m \|\nabla \bar{\mathbf{u}}\| = l_m \left(\sum_{i,j} \left(\frac{\partial \bar{u}_i}{\partial x_j} \right)^2 \right)^{1/2} \longrightarrow \nu_t = l_m^2 \|\nabla \bar{\mathbf{u}}\|$$

Velocity scale

- Simple 2D flows: only one significant mean gradient: $u_t = l_m \left| \frac{\partial \bar{u}}{\partial y} \right| \longrightarrow \nu_t = l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$
+ only one significant Reynolds stress:

$$\tau_{xy} = \tau_{yx} = -\overline{u'v'} = \nu_t \frac{\partial \bar{u}}{\partial y} = l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \frac{\partial \bar{u}}{\partial y}$$

RANS + turbulent viscosity models

Boussinesq hypothesis:

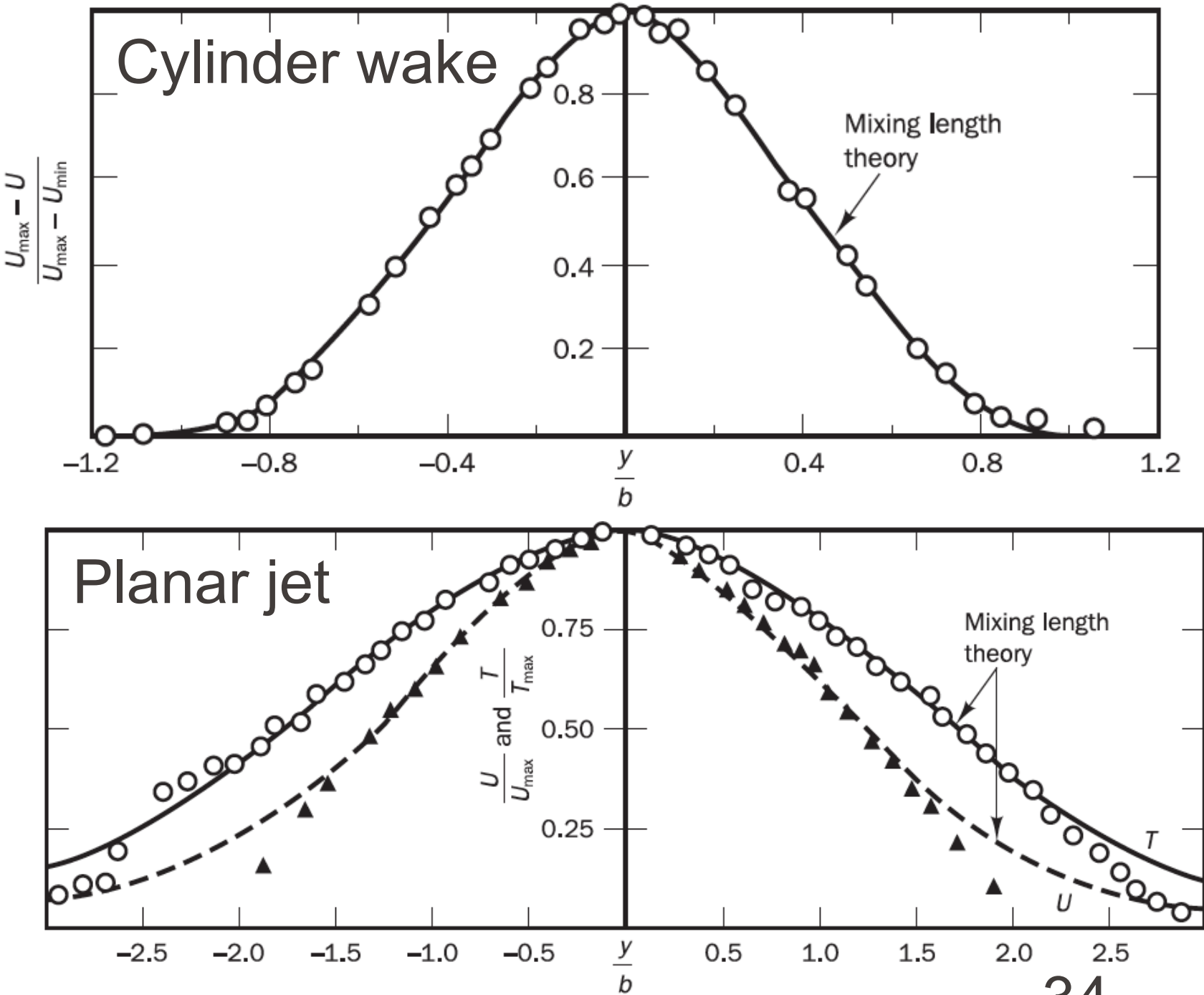
$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right)$$

- Zero-equation models (algebraic models): mixing length models
 - Easy to implement, and computationally cheap.
 - Works well only in simple 2D flows: mixing layers, wakes, jets, boundary layers... Cannot describe flows with separation/recirculation.

Numerical Flow Simulation

Flow	Mixing length ℓ_m	L
Mixing layer	$0.07L$	Layer width
Jet	$0.09L$	Jet half width
Wake	$0.16L$	Wake half width
Axisymmetric jet	$0.075L$	Jet half width
Boundary layer ($\partial p / \partial x = 0$)		
viscous sub-layer and	$\kappa y [1 - \exp(-y^+ / 26)]$	Boundary layer
log-law layer ($y / L \leq 0.22$)		thickness
outer layer ($y / L \geq 0.22$)	$0.09L$	Pipe radius or
Pipes and channels		channel half width
(fully developed flow)	$L [0.14 - 0.08(1 - y / L)^2 - 0.06(1 - y / L)^4]$	

Rodi, 1980



Schlichting, 1979

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

RANS + turbulent viscosity models

- One-equation models: Spalart-Allmaras model
 - Specifically designed for external aerodynamics (boundary layers with adverse pressure gradients)
 - Solve one transport equation for an “eddy viscosity parameter” $\tilde{\nu}$ [m²/s]

$$\underbrace{\frac{\partial(\rho\tilde{\nu})}{\partial t}}_{\text{rate of change}} + \underbrace{\nabla \cdot (\rho\tilde{\nu}\bar{\mathbf{u}})}_{\text{convection}} = \underbrace{\frac{1}{\sigma_\nu} \left\{ \nabla \cdot [\rho(\nu + \tilde{\nu})\nabla\tilde{\nu}] + C_{b2}\rho(\nabla\tilde{\nu})^2 \right\}}_{\text{molecular and turbulent diffusion}} + \underbrace{C_{b1}\rho\tilde{\nu}\tilde{\Omega}}_{\text{production}} - \underbrace{C_{w1}\rho\left(\frac{\tilde{\nu}}{\kappa y}\right)^2}_{\text{dissipation}} f_w$$

rate of change convection molecular and turbulent diffusion production dissipation

y : distance to closest wall

$\sigma_\nu, C_{b2}, C_{b1}, C_{w1}, \kappa$: empirical constants (flow-independent)

$\tilde{\Omega}, f_w$: functions containing other empirical constants

- Then, turbulent viscosity obtained as: $\nu_t = \tilde{\nu} f_{\nu 1}$

$f_{\nu 1}$: function that tends to
1 for large $\tilde{\nu}/\nu$, and
0 for small $\tilde{\nu}/\nu$.

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

RANS + turbulent viscosity models

- One-equation models: turbulent kinetic energy model
 - Improvement on mixing length model
 - Turbulent viscosity: still $\nu_t \sim l_t u_t$, and $l_t = l_m$, but build velocity scale as $u_t = c\sqrt{k}$ with c a constant and k the turbulent kinetic energy [m^2/s^2]

$$\longrightarrow \nu_t = cl_m \sqrt{k}$$

- An equation for k can be derived rigorously from NS (same method as for RANS eqs.):

$$\underbrace{\frac{\partial(\rho k)}{\partial t}}_{\text{rate of change}} + \underbrace{\nabla \cdot (\rho k \bar{\mathbf{u}})}_{\text{convection}} = \underbrace{\nabla \cdot \left[-\overline{p' \mathbf{u}'} - \rho \overline{\mathbf{u}'^2 \mathbf{u}'} \right]}_{\text{diffusion}} + \underbrace{\rho \boldsymbol{\tau} : \nabla \bar{\mathbf{u}}}_{\text{production}} - \underbrace{\rho \nu \overline{\|\nabla \mathbf{u}'\|^2}}_{\text{dissipation}}$$

Note (see later): $\epsilon = \nu \overline{\|\nabla \mathbf{u}'\|^2}$ is the rate of dissipation of TKE per unit mass [m^2/s^3]

- Again, **closure problem**: unknown diffusion and dissipation terms.

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

- One-equation models: turbulent kinetic energy model

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \bar{\mathbf{u}}) = \nabla \cdot \left[-\overline{p' \mathbf{u}'} - \overline{\rho \mathbf{u}'^2 \mathbf{u}'} + \rho \nu \nabla k \right] + \rho \boldsymbol{\tau} : \nabla \bar{\mathbf{u}} - \rho \nu \overline{\|\nabla \mathbf{u}'\|^2}$$

- Solve a model transport equation for k :
 - On dimensional ground, write $\epsilon \sim U^3/L$ and choose $\epsilon = C_D \frac{k^{3/2}}{l_m}$ with C_D a constant.
 - Model diffusive flux as $\rho \frac{\nu_t}{\sigma_k} \nabla k$, with σ_k a constant.

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \bar{\mathbf{u}}) = \nabla \cdot \left[\rho \frac{\nu_t}{\sigma_k} \nabla k \right] + \rho \boldsymbol{\tau} : \nabla \bar{\mathbf{u}} - \rho C_D \frac{k^{3/2}}{l_m}$$

- Major drawback: still need to specify mixing length l_m .

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

- Two-equation models: k - ϵ model (standard)
 - Improvement on one-equation TKE model (no need to specify a mixing length)
 - Turbulent viscosity: on dimensional ground $\nu_t \sim k^2/\epsilon \longrightarrow \text{let } \nu_t = C_\mu \frac{k^2}{\epsilon}$
 - Solve 2 model transport equations for k (same as TKE: theoretical eq. + closure approx.) and for ϵ (empirical, but based on the analysis of homogeneous turbulence, turbulent free-shear flows, and wall-bounded turbulence):

$$\begin{array}{ccccccc} \frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \bar{\mathbf{u}}) = \nabla \cdot \left[\rho \frac{\nu_t}{\sigma_k} \nabla k \right] + \rho \boldsymbol{\tau} : \nabla \bar{\mathbf{u}} - \rho \epsilon & & & & & & \sigma_k, \sigma_\epsilon, C_{1\epsilon}, C_{2\epsilon} \\ \frac{\partial(\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \bar{\mathbf{u}}) = \nabla \cdot \left[\rho \frac{\nu_t}{\sigma_\epsilon} \nabla \epsilon \right] + \rho C_{1\epsilon} \boldsymbol{\tau} : \nabla \bar{\mathbf{u}} \frac{\epsilon}{k} - \rho C_{2\epsilon} \frac{\epsilon^2}{k} & & & & & & \text{empirical constants} \\ \text{rate of change} & \text{convection} & \text{diffusion} & \text{production} & \text{dissipation} & & \text{(flow-independent)} \end{array}$$

- Need special treatment at the wall (see later).
- Well established, widely validated, well calibrated for internal flows. However, poor performance in unconfined flows, swirling/rotating flows, curved boundary layers.
- Three versions in Fluent: standard (Launder & Spalding, 1974), RNG, realizable.

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

- Two-equation models: k - ω model (standard)
 - Instead of ε , use the specific turbulent dissipation rate $\omega = \varepsilon/k$ [1/s] (not to be confused with vorticity) as 2nd variable.
 - Turbulent viscosity: on dimensional ground, $\nu_t \sim k/\omega$
 - Solve 2 model transport equations for k (similar to that of k - ε) and for ω .
 - Unlike k - ε , no need for special treatment at the wall.
 - More widely applicable than k - ε . But problematic in some complex external flows.
 - Several versions in Fluent: standard (Wilcox, 1998), SST (Menter, 1992; combines k - ε in the freestream and standard k - ω near the wall), GEKO, BSL.

RANS + Reynolds-stress model (RSM)

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$$

■ Reynolds-stress model

- Don't use Boussinesq hypothesis (isotropic turbulent viscosity). Instead, solve explicitly transport eqs. for the 6 Reynolds stresses (+1 eq. for ε).
- Can derive a rigorous eq. for $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'}$:

$$\underbrace{\frac{\partial \overline{u'_i u'_j}}{\partial t}}_{\text{rate of change}} + \underbrace{\overline{u_k} \partial_k \overline{u'_i u'_j}}_{\text{convection}} = \underbrace{- \left(\overline{u'_i u'_k} \partial_k \overline{u_j} + \overline{u'_j u'_k} \partial_k \overline{u_i} \right)}_{\text{production}} \underbrace{- \partial_k \overline{u'_i u'_j u'_k}}_{\text{turbulent and molecular diffusion}} + \nu \partial_{kk} \overline{u'_i u'_j} - 2\nu \underbrace{\overline{\partial_k u'_i \partial_k u'_j}}_{\text{dissipation}} - \frac{1}{\rho} \underbrace{\left(\overline{u'_i \partial_j p'} + \overline{u'_j \partial_i p'} \right)}_{\text{pressure-strain interaction}}$$

- Again, **closure problem**: must model the unknown correlations. The diffusion and pressure-strain terms are modelled as a function of $\boldsymbol{\tau}$, the dissipation term as a function of ε .
- Allow for anisotropic turbulence. Accurate in simple and complex flows.
- However, less validated than k - ε .

RANS: summary

- The Reynolds stress tensor $\tau = -\overline{\mathbf{u}'\mathbf{u}'}$ is unknown.
- Turbulent viscosity models: Boussinesq hypothesis $\tau + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)$
 - Mixing length model: $\nu_t = l_m^2 ||\nabla\bar{\mathbf{u}}||$ where the mixing length l_m is flow-dependent.
No additional equation.
 - Spalart-Allmaras model: $\nu_t = \tilde{\nu} f_{\nu 1}$ *1 transport equation for $\tilde{\nu}$*
 - TKE model: $\nu_t \sim l_m \sqrt{k}$ where the mixing length l_m is flow-dependent.
1 transport equation for k
 - k - ε models: $\nu_t \sim k^2 / \varepsilon$ *2 transport equations for k and ε*
 - k - ω models: $\nu_t \sim k / \omega$ *2 transport equations for k and ω*
- Reynolds-stress eq. models (RSM): solve directly for τ .
7 transport eqs. for the 6 independent components of τ and for ε .

RANS: summary

- The Reynolds stress tensor $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'}$ is unknown.
- Turbulent viscosity models: Boussinesq hypothesis $\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$

- Mixing length model: $\nu_t = l_m^2 ||\nabla \bar{\mathbf{u}}||$

- Spalart-Allmaras model: $\nu_t = \tilde{\nu} f_{\nu 1}$

- TKE model: $\nu_t \sim l_m \sqrt{k}$

- k - ϵ models: $\nu_t \sim k^2 / \epsilon$

- k - ω models: $\nu_t \sim k / \omega$

Note: in flows where turbulent regions and laminar regions coexist, *in principle* all turbulence models can predict laminar regions (i.e. they can yield $\nu_t = 0$ and $\boldsymbol{\tau} = \mathbf{0}$, and the RANS eqs. reduce to the standard NS eqs.), except the mixing length model (ν_t is non-zero wherever the mean velocity gradient is non-zero). However, depending on the model, BCs etc., spurious production or insufficient dissipation of k (and ν_t) is possible.

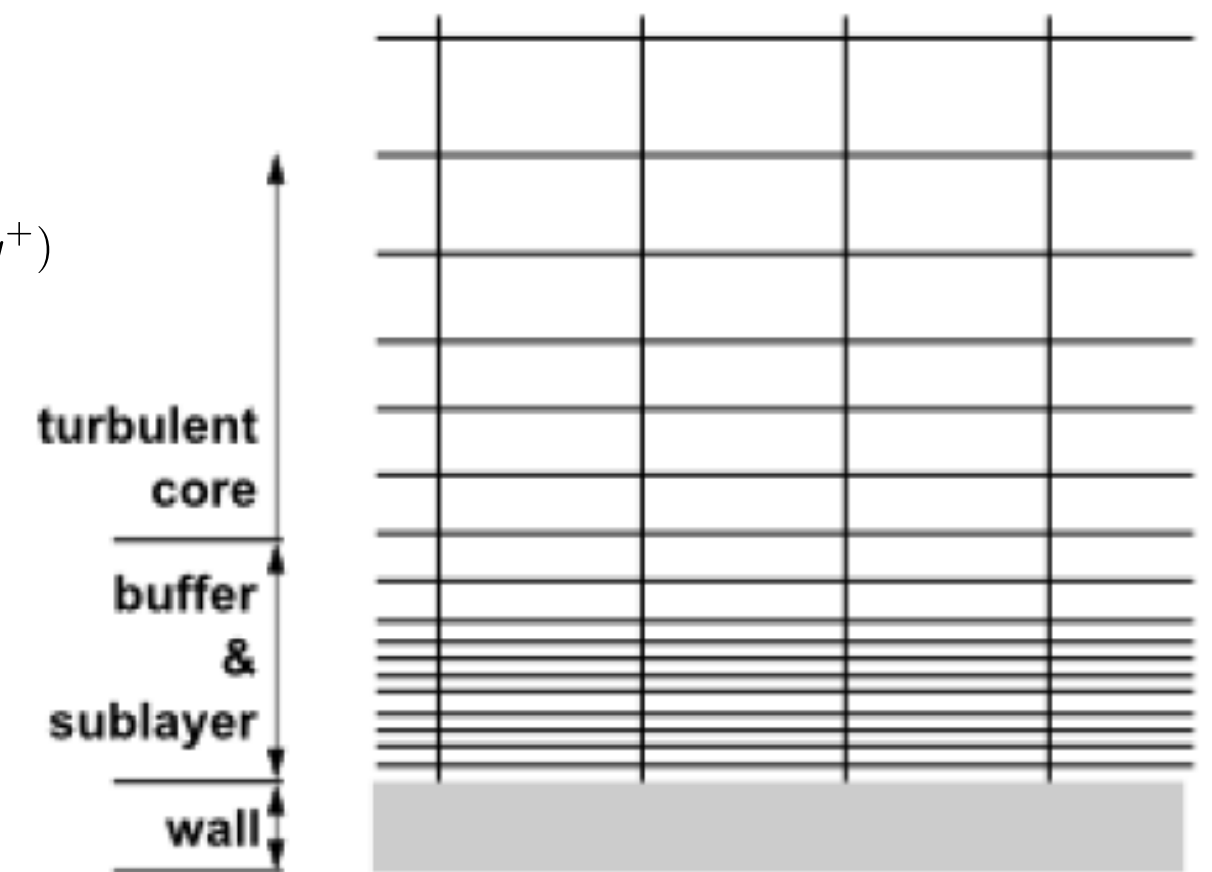
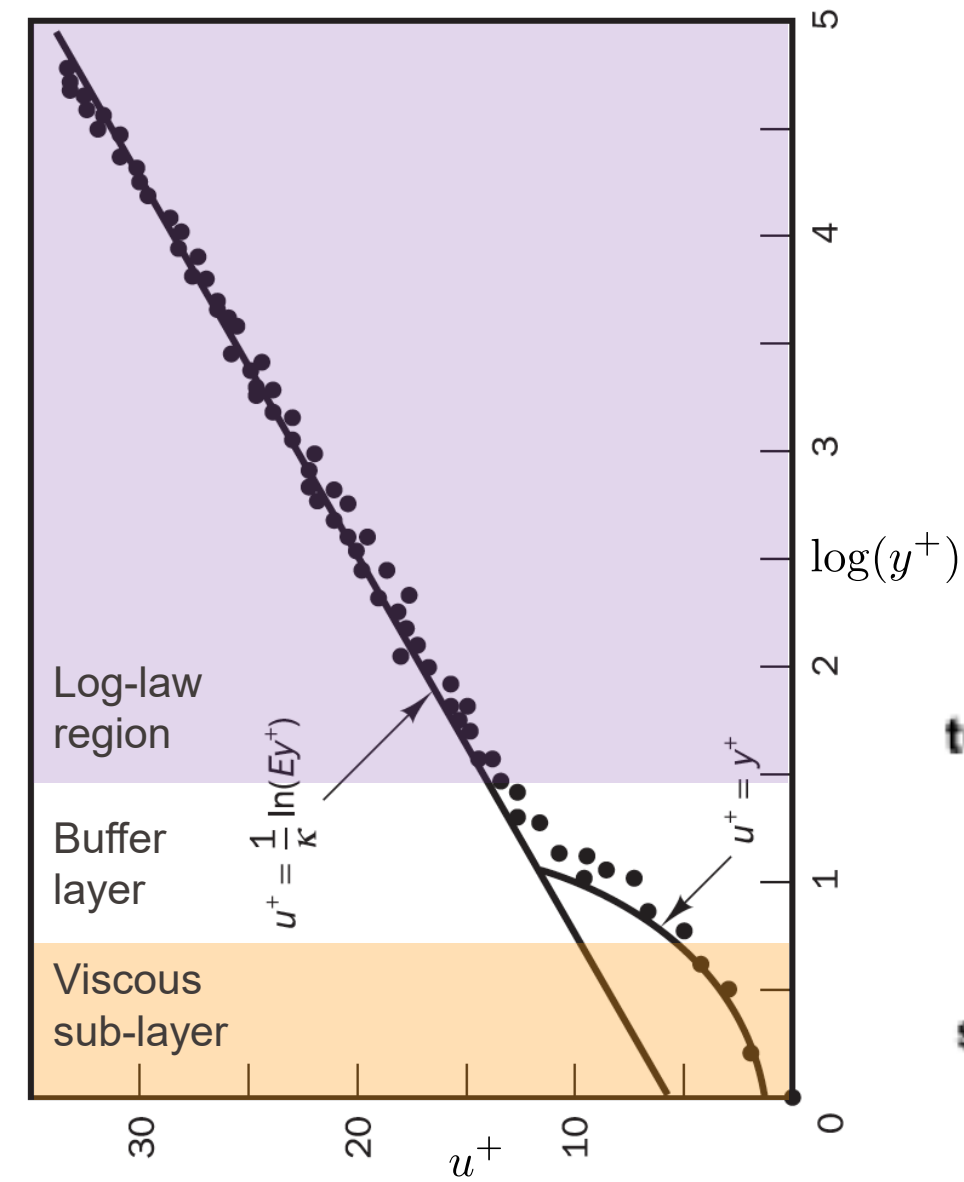
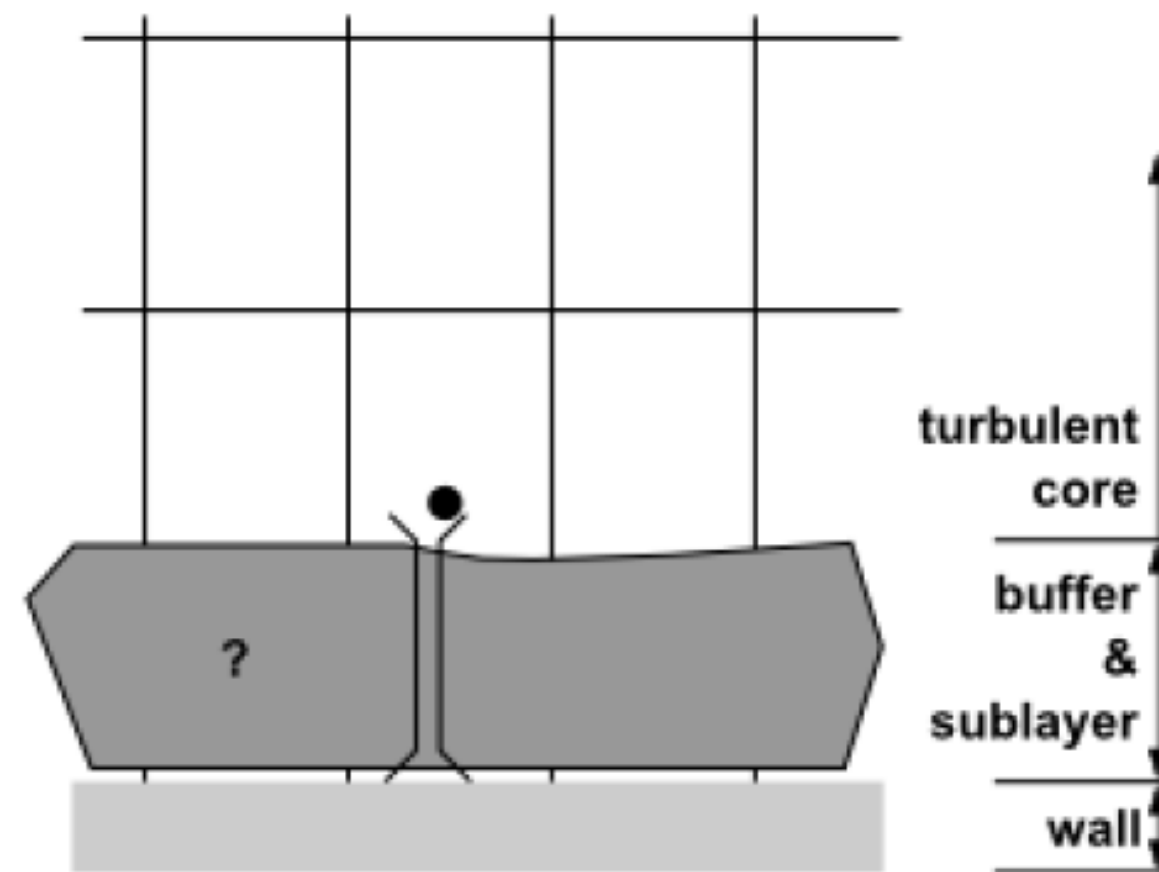
- Reynolds-stress eq. models (RSM): solve directly for $\boldsymbol{\tau}$.

Near-wall treatment

- Walls are the main source of mean vorticity and turbulence → strong impact on turbulence.
- Some hypotheses used to develop turbulence models are not valid near the walls (e.g., the hypothesis that viscous dissipation is negligible compared to other terms is valid far from walls because Re is large, but not close to walls because the flow is much slower, so the effective Re is much smaller) → near-wall modeling significantly affects the accuracy of turbulent flow simulations.

Near-wall treatment

- Numerically, two approaches:



Wall functions:

- Viscosity-affected region (viscous sub-layer + buffer layer) is not resolved, but bridged with semi-empirical “wall functions”.
- Outside this region, turbulent models can be used “as is”.
- Cheaper, but less accurate.

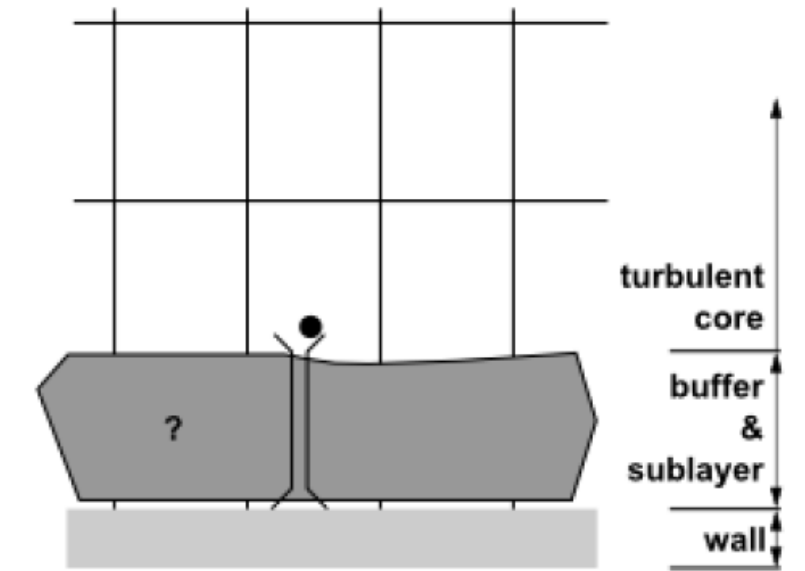
Near-wall models:

- Near-wall region is resolved all the way down to the wall. Requires a sufficiently fine mesh.
- Some turbulence models must be modified to account for the presence of the wall
- More accurate, but more expensive.

Near-wall treatment

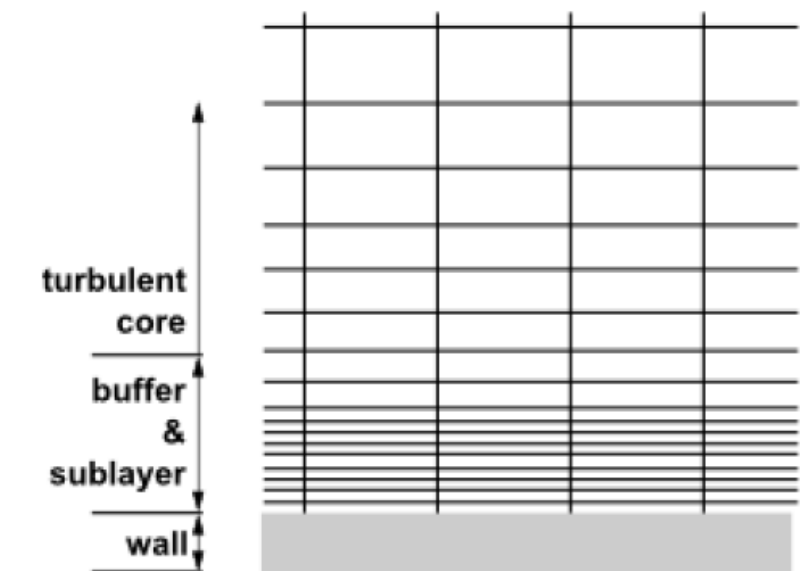
- **Wall function approach:**

- For high- Re flows, this approach is economical, robust, reasonably accurate → practical option for industrial flows.
- Inadequate where low- Re effects are important:
 - Hypotheses of wall functions breaks down (actual profile $\neq$ turbulent law of the wall).
 - Must use near-wall model approach instead.
- First mesh cell center should be in the log layer $y^+ > 30$ (less true with more recent advanced wall functions; see Fluent Theory Guide).



- **Near-wall model approach:**

- First mesh cell should be at $y^+ \sim 1$. Mesh should have at least 10-20 cells in the boundary layer.



Near-wall treatment

- How to determine y^+ ?
 - Defined as distance to the wall normalized by flow quantities → in principle, need to know flow solution to determine y^+ values!

$$y^+ = \frac{yu_\tau}{\nu} \quad \text{where} \quad u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \frac{\partial u_x}{\partial y}}$$

- Estimation of y^+ for the first mesh cell:
 - trial and error (compute, remesh, recompute),
 - from experience,
 - based on simple analytic formula.
- Online calculators based on formula for flat-plate boundary layer (only useful if flat plate is a good analogy for problem being solved). For example:
 - www.pointwise.com/yplus
 - www.cfd-online.com/Tools/yplus.php

Turbulence models: guidelines

1. Choosing a **turbulence model**

- Depends on flow physics, accuracy required, computer resources available, etc.
- Understand capabilities and limitations of turbulence model before using it.
- For industrial applications: use RANS approach.
 - Aeronautical applications involving wall-bounded flows (boundary-layer flows): consider Spalart-Allmaras or $k-\omega$.
 - “General” flow situations: consider $k-\varepsilon$ or $k-\omega$.
 - If low- Re effects, compressibility or shear flow spreading are important: consider $k-\omega$.
 - More complex flows (e.g. streamline curvature, swirl, rotation, rapidly changing strain rate): consider RSM.
- For detailed unsteady turbulent flow studies: use LES approach.

Turbulence models: guidelines

2. Choosing a **model variation**

- For complex near-wall phenomena (e.g. separation from curved surface), use:
 - RNG or realizable (rather than standard) k - ε model,
 - SST (rather than standard) k - ω model.
- If solution stability is a problem, use:
 - standard (rather than RNG or realizable) k - ε model,
 - standard (rather than SST) k - ω model.

3. Choosing a **near-wall treatment**

- High- Re flows and/or limited computer resources: consider wall function.
- For complex near-wall phenomena (e.g. separation from curved surface), use:
 - near-wall model approach (if practical),
 - non-equilibrium wall functions.

Turbulence models: guidelines

4. Choosing **model parameters**

- Never change model constants, unless expert user.
- If solution stability is a problem, consider:
 - decreasing under-relaxation factors for turbulence equations
 - using first-order solution procedure for turbulence equations
- Include all model options that may be important in the flow (e.g. buoyancy, swirl, compressibility, thermal effects)

5. Choosing an appropriate **mesh**

- LES: use fine mesh to resolve all scales down to smallest resolved eddies
- RANS with near-wall model: set first mesh cell at $y^+ \sim 1$
- RANS with wall function: set first mesh cell at $y^+ \sim 30$

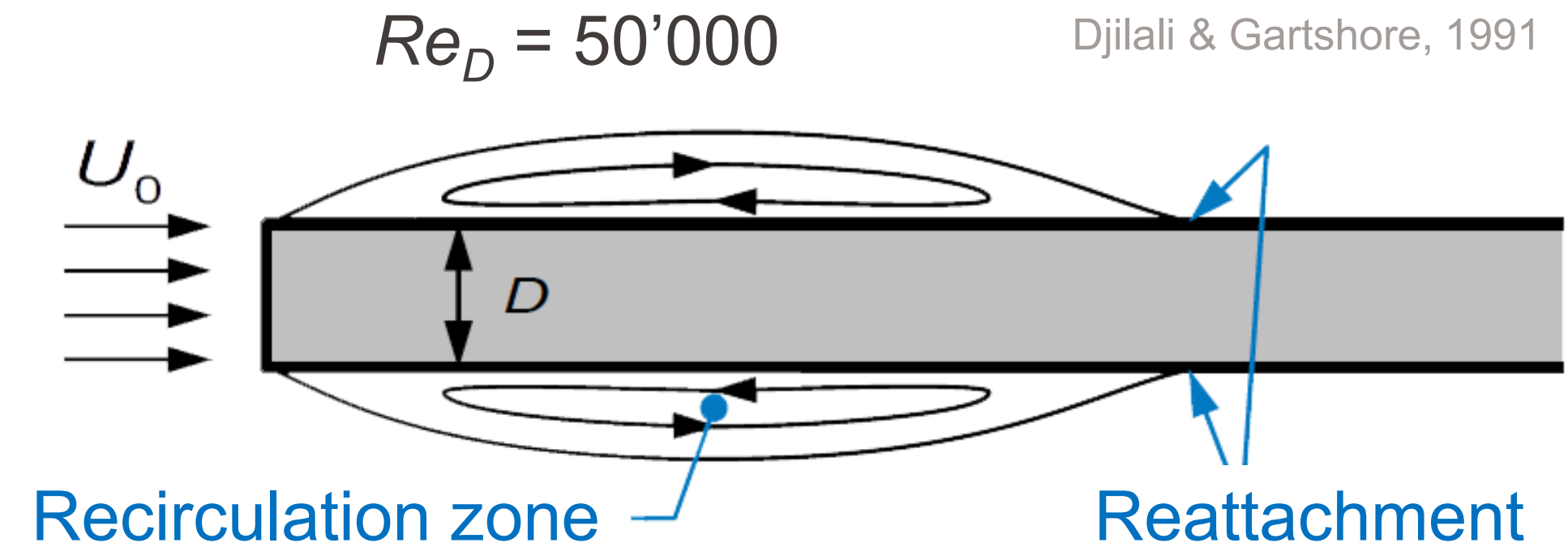
Turbulence models: guidelines

- **General comments**

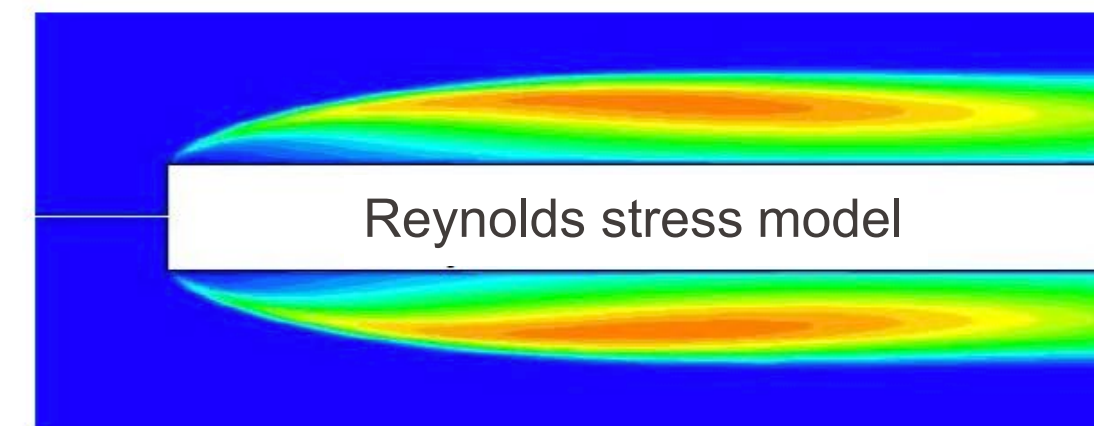
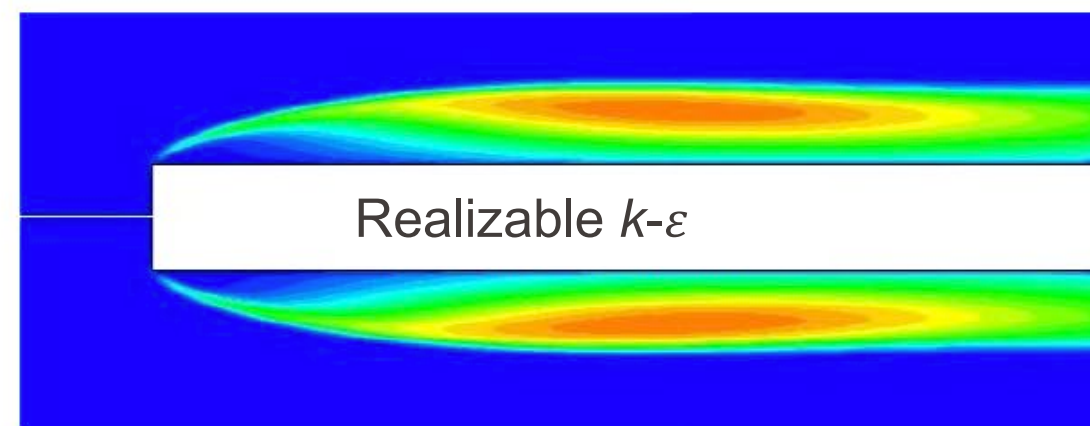
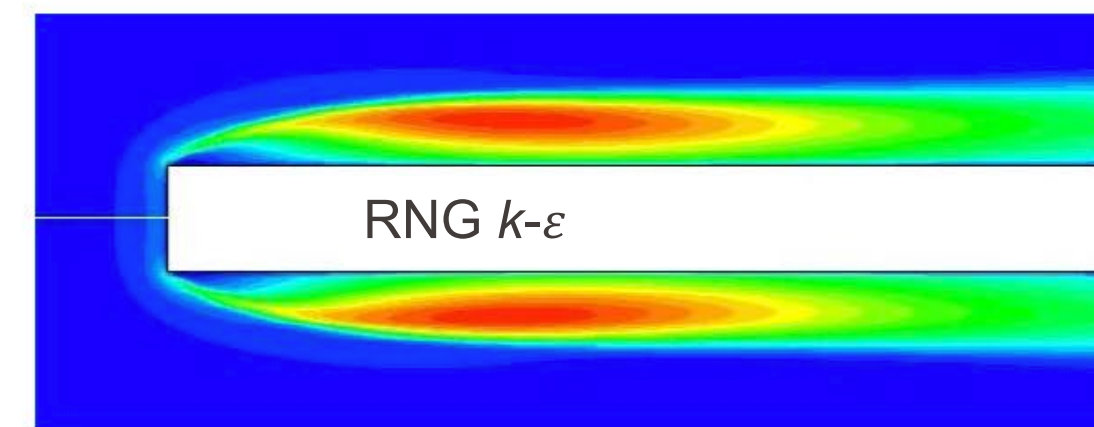
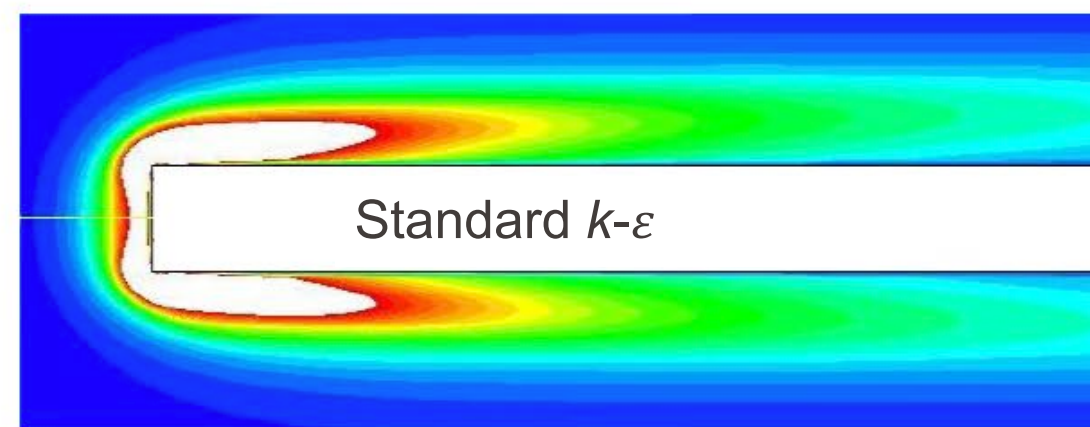
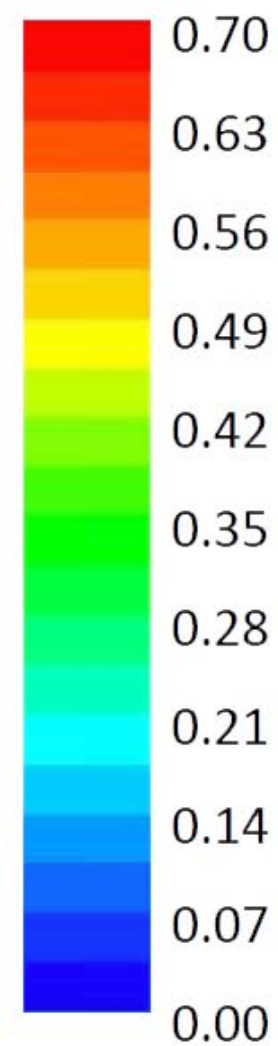
- Read Fluent Theory Guide and User's Guide before applying turbulence models.
- Study influence of turbulence model on flow solution.
- Validate flow solution whenever possible (compare with experimental data).

Turbulence models: example

- Turbulent flow past a blunt flat plate.
- Four turbulence models considered.

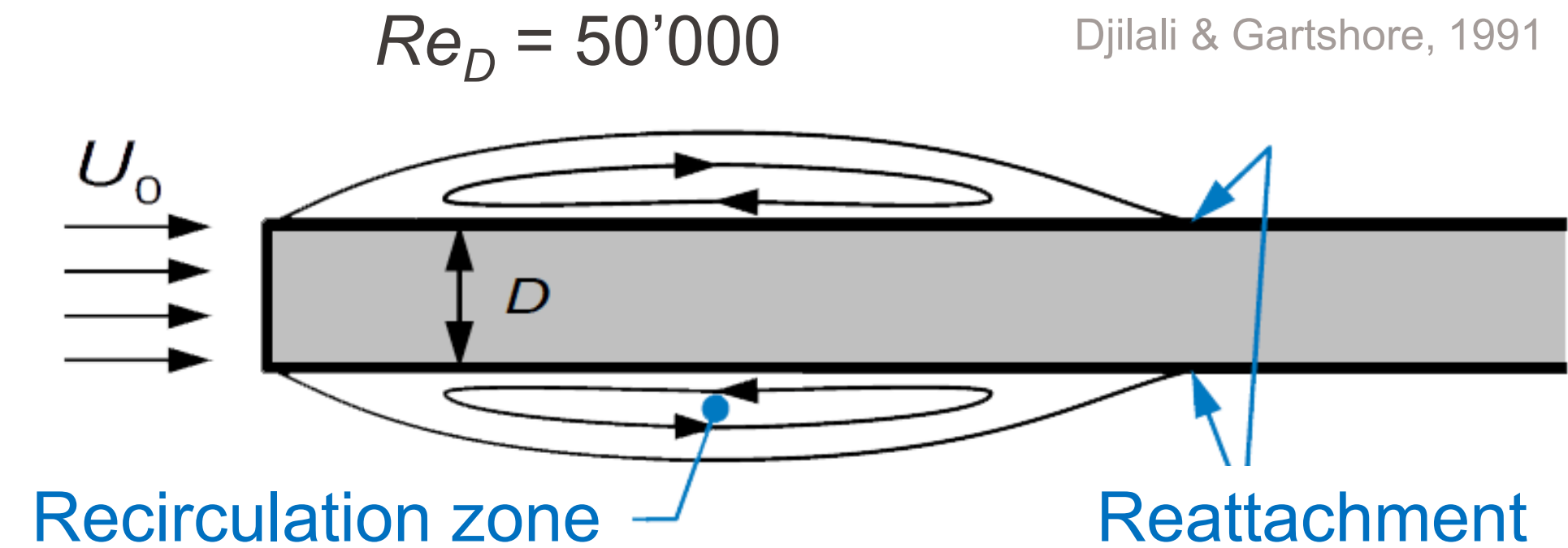


Turbulent kinetic energy:

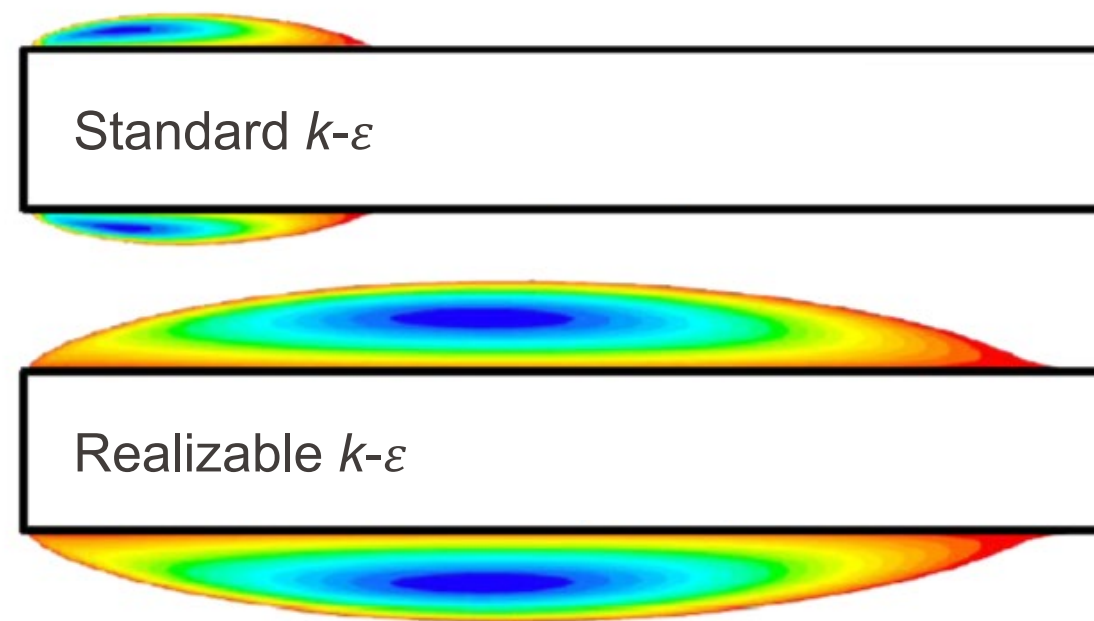


Turbulence models: example

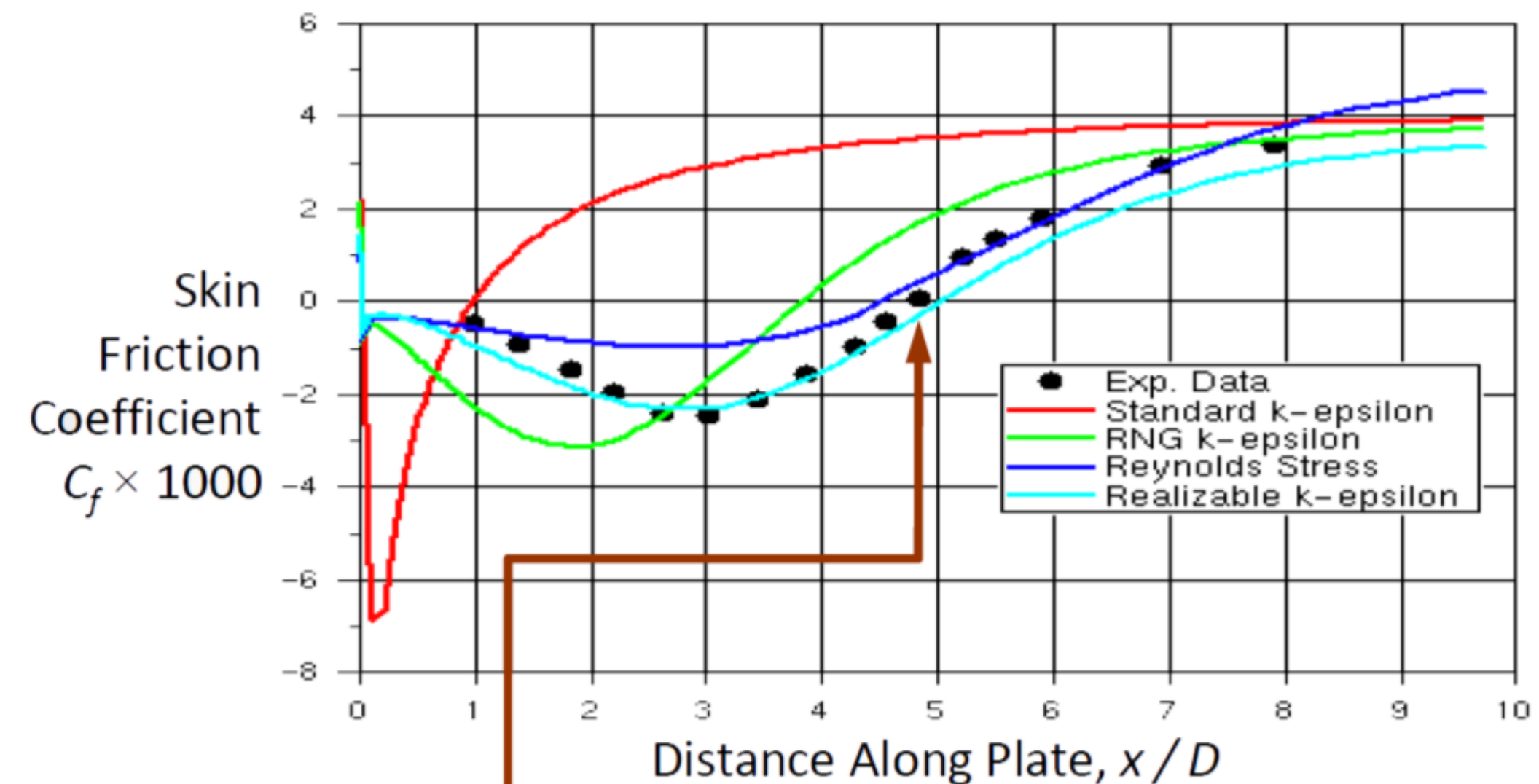
- Turbulent flow past a blunt flat plate.
- Four turbulence models considered.



Predicted recirculation zone:



Standard $k-\epsilon$ severely underpredicts the size of the recirculation zone.
Realizable $k-\epsilon$ is more accurate.



Experimentally observed reattachment point is at $x/D = 4.7$

Summary

- Turbulent flows are complex and difficult to simulate numerically.
- Hierarchy of different approaches: DNS, LES, RANS.
- Modeling turbulent flow is difficult. Many different RANS models of varying complexity.
- Choice of wall treatment is important for accuracy and mesh size (y^+).
- Choice of most suitable turbulence model is generally difficult. Should be based on physical guidelines (see manual) and computer resources.
- Validation is particularly important to gauge accuracy of solution.
- For more details on turbulence and its modeling:
 - EPFL SGM Master course “Turbulence” (ME-467, T. Schneider)
 - *Turbulent Flows*, S. Pope

Appendix: derivation of some equations

- Equation for the **fluctuations**:

- Subtract the RANS eq. (mean flow) from the NS eq. (total flow), and recall the Reynolds decomposition $\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t)$:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \\ - \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} &= -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}} - \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} &= -\frac{1}{\rho} \nabla p' + \nu \nabla^2 \mathbf{u}' + \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} \\ &= \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}' \cdot \nabla) \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{u}' \end{aligned}$$

$$\Leftrightarrow \frac{\partial \mathbf{u}'}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}' = -\frac{1}{\rho} \nabla p' + \nu \nabla^2 \mathbf{u}' + \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} - (\mathbf{u}' \cdot \nabla) \bar{\mathbf{u}} - (\mathbf{u}' \cdot \nabla) \mathbf{u}'$$

Appendix: derivation of some equations

- Equation for the **turbulent kinetic energy**:

- Start from the eq. for the fluctuations (previous slide):

$$\frac{\partial u'_i}{\partial t} + \bar{u}_j \partial_j u'_i = -\frac{1}{\rho} \partial_i p' + \nu \partial_{jj} u'_i + \overline{u'_j \partial_j u'_i} - u'_j \partial_j \bar{u}_i - u'_j \partial_j u'_i$$

- Multiply each eq. i by u'_i , sum the 3 eqs., and take the mean:

$$\overline{u'_i \frac{\partial u'_i}{\partial t}} + \overline{u'_i \bar{u}_j \partial_j u'_i} = -\frac{1}{\rho} \overline{u'_i \partial_i p'} + \nu \overline{u'_i \partial_{jj} u'_i} + \overline{u'_i u'_j \partial_j u'_i} - \overline{u'_i u'_j \partial_j \bar{u}_i} - \overline{u'_i u'_j \partial_j u'_i}$$

- Rearrange each term (use product rule and/or incompressibility where needed):

- 1st term: $\overline{u'_i \frac{\partial u'_i}{\partial t}} = \frac{1}{2} \frac{\partial \overline{u'^2_i}}{\partial t}$

- 3rd term: $-\frac{1}{\rho} \overline{u'_i \partial_i p'} = -\frac{1}{\rho} \partial_i (\overline{u'_i p'})$

- 2nd term: $\overline{\bar{u}_j u'_i \partial_j u'_i} = \frac{1}{2} \bar{u}_j \partial_j \overline{u'^2_i}$

- 4th term: $\overline{\nu u'_i \partial_{jj} u'_i} = \frac{1}{2} \nu \partial_{jj} \overline{u'^2_i} - \nu \overline{(\partial_j u'_i)^2}$

Appendix: derivation of some equations

- Equation for the **turbulent kinetic energy** (continued):

- 5th term: $\overline{u'_i u'_j \partial_j u'_i} = \overline{u'_i} \overline{u'_j \partial_j u'_i} = 0 \cdot \overline{u'_j \partial_j u'_i} = 0$

- 6th term: $-\overline{u'_i u'_j \partial_j \bar{u}_i}$

- 7th term: $-\overline{u'_i u'_j \partial_j u'_i} = -\frac{1}{2} \overline{u'_j \partial_j (u_i'^2)} = -\frac{1}{2} \partial_j \overline{u'_j u_i'^2}$

- Gather:
$$\frac{1}{2} \frac{\partial \overline{u_i'^2}}{\partial t} + \frac{1}{2} \bar{u}_j \partial_j \overline{u_i'^2} = -\frac{1}{\rho} \partial_i (\overline{u'_i p'}) + \frac{1}{2} \nu \partial_{jj} \overline{u_i'^2} - \nu \overline{(\partial_j u'_i)^2} - \overline{u'_i u'_j \partial_j \bar{u}_i} - \frac{1}{2} \partial_j \overline{u'_j u_i'^2}$$
$$\frac{\partial k}{\partial t} + \bar{u}_j \partial_j k = -\frac{1}{\rho} \partial_j (\overline{u'_j p'}) + \nu \partial_{jj} k - \nu \overline{(\partial_j u'_i)^2} - \overline{u'_i u'_j \partial_j \bar{u}_i} - \frac{1}{2} \partial_j \overline{u'_j u_i'^2}$$
$$\frac{\partial k}{\partial t} + \bar{u}_j \partial_j k = \partial_j \left(-\frac{1}{\rho} \overline{u'_j p'} + \nu \partial_j k - \frac{1}{2} \overline{u'_j u_i'^2} \right) - \nu \overline{(\partial_j u'_i)^2} - \overline{u'_i u'_j \partial_j \bar{u}_i}$$
$$\frac{\partial k}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) k = \nabla \cdot \left(-\frac{1}{\rho} \overline{\mathbf{u}' p'} + \nu \nabla k - \frac{1}{2} \overline{\mathbf{u}'^2 \mathbf{u}'} \right) - \nu \overline{\|\nabla \mathbf{u}'\|^2} + \boldsymbol{\tau} : \nabla \bar{\mathbf{u}}$$

Appendix: derivation of some equations

- Equation for the **Reynolds stresses**:

- Same idea as for the turbulent kinetic energy (2 previous slides).
- Start from the eq. for the fluctuations:

$$\frac{\partial u'_i}{\partial t} + \bar{u}_k \partial_k u'_i = \dots$$

- Multiply by fluctuations and combine to form the quantity

$$u'_j \left(\frac{\partial u'_i}{\partial t} + \bar{u}_k \partial_k u'_i \right) + u'_i \left(\frac{\partial u'_j}{\partial t} + \bar{u}_k \partial_k u'_j \right) = \dots$$

- Take the mean:

$$\overline{u'_j \left(\frac{\partial u'_i}{\partial t} + \bar{u}_k \partial_k u'_i \right)} + \overline{u'_i \left(\frac{\partial u'_j}{\partial t} + \bar{u}_k \partial_k u'_j \right)} = \dots$$

- Rearrange:

$$\frac{\partial \overline{u'_i u'_j}}{\partial t} + \bar{u}_k \partial_k \overline{u'_i u'_j} = \dots$$