Nonlinearity

Numerical Flow Simulation

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Linearity

- General conservation equation

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\Gamma\, grad(\phi)) + S$$

- Linear in ϕ if:
  - coefficients independent of ϕ,
  - source term independent of/linear in ϕ.

- Example: steady diffusion $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + S = 0$ with

$$\left|\begin{array}{l} k = cst \text{ or } k(x) \\ S = cst \text{ or } S = S(x) \\ \qquad\qquad \text{or } S = S_c + S_l T \end{array}\right.$$

- Discretization yields a **linear** algebraic system:

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{b} \quad \rightarrow \quad \boldsymbol{\phi} = \mathbf{A}^{-1}\mathbf{b}$$

Nonlinearity

- General conservation equation

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\Gamma\, grad(\phi)) + S$$

- Nonlinear in ϕ if:
  - Coefficients depend on ϕ
  - Other terms nonlinear in ϕ

- Example: Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \boxed{div(\rho\mathbf{u})} = 0$$

Nonlinear if compressible
(both ρ and $\mathbf{u}$ unknown)

Always nonlinear

$$\boxed{\frac{\partial(\rho\mathbf{u})}{\partial t}} + \boxed{div(\rho\mathbf{u}\mathbf{u})} = div\left[\left(-p - \frac{2}{3}\mu\, div(\mathbf{u})\right)\mathbf{I} + 2\mu\mathbf{d}\right]$$

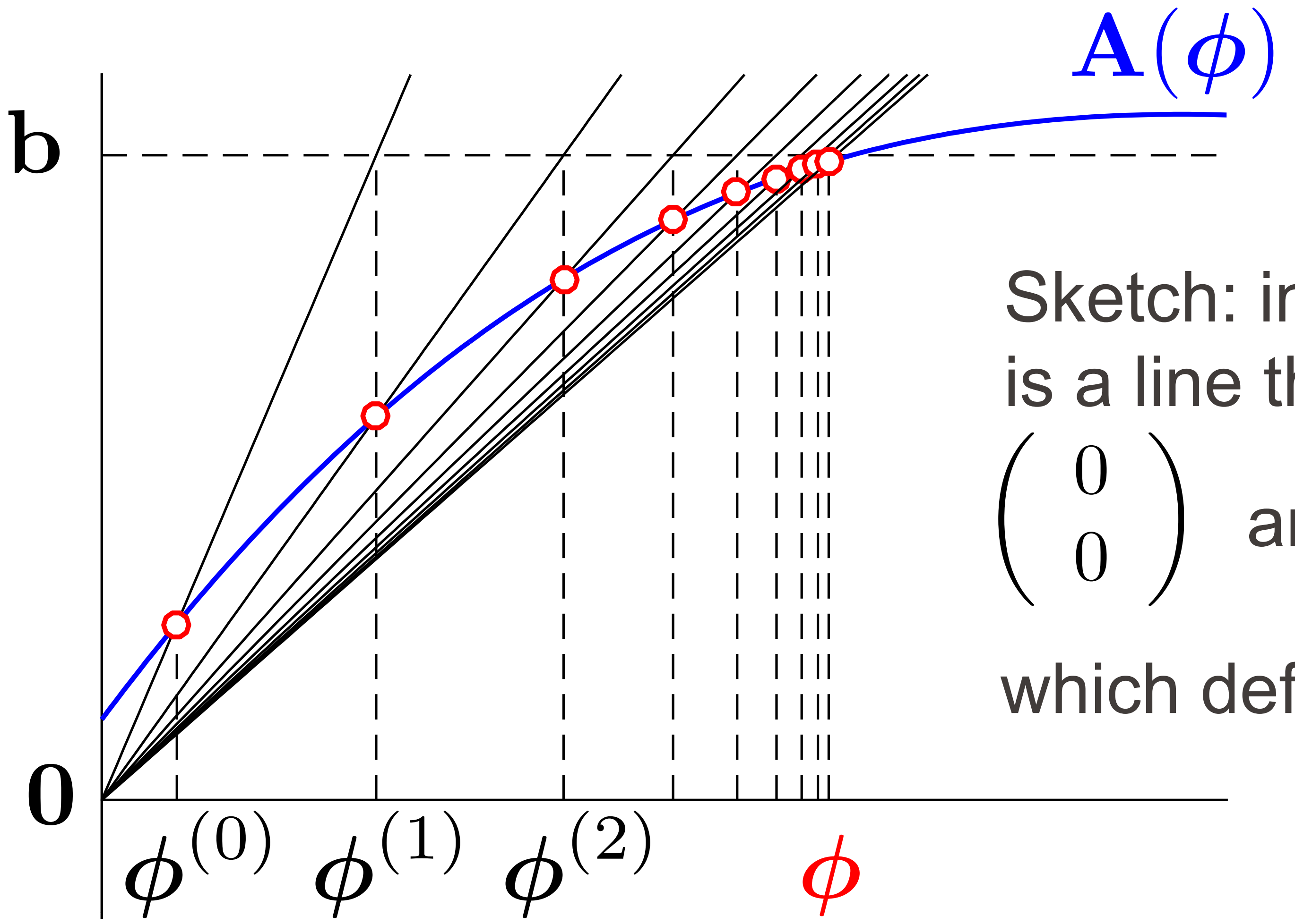
- Discretization yield a **nonlinear** system: $\mathbf{A}(\phi) = \mathbf{b} \quad \rightarrow \quad \phi = ?$
- We only know how to solve linear systems → must linearize.

Numerical Flow Simulation

Linearization: Picard's method

$$\mathbf{A}(\phi) = \mathbf{b} \quad \rightarrow \quad \phi = ?$$

- If the system can be written in the "quasi-linear" form $\widetilde{\mathbf{A}}(\phi)\phi = \mathbf{b}$
  - take a guess solution ϕ^* to evaluate the matrix $\widetilde{\mathbf{A}}(\phi^*)$
  - solve $\widetilde{\mathbf{A}}(\phi^*)\phi = \mathbf{b}$
  - take ϕ as new guess,
  - iterate until convergence.



Sketch: in the scalar case, $\widetilde{A}(\phi^*)\phi$ is a line that goes through the 2 points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \phi^* \\ \widetilde{A}(\phi^*)\phi^* \end{pmatrix} = \begin{pmatrix} \phi^* \\ A(\phi^*) \end{pmatrix}$, which defines a 3rd point $\begin{pmatrix} \phi^*_{new} \\ b \end{pmatrix}$.

Guess ϕ^* → Linearize + discretize + solve linear system →(ϕ) $||\phi - \phi^*|| < \mathrm{tol}?$ → yes → converged solution ϕ

no → update the guess $\phi^* = \phi$

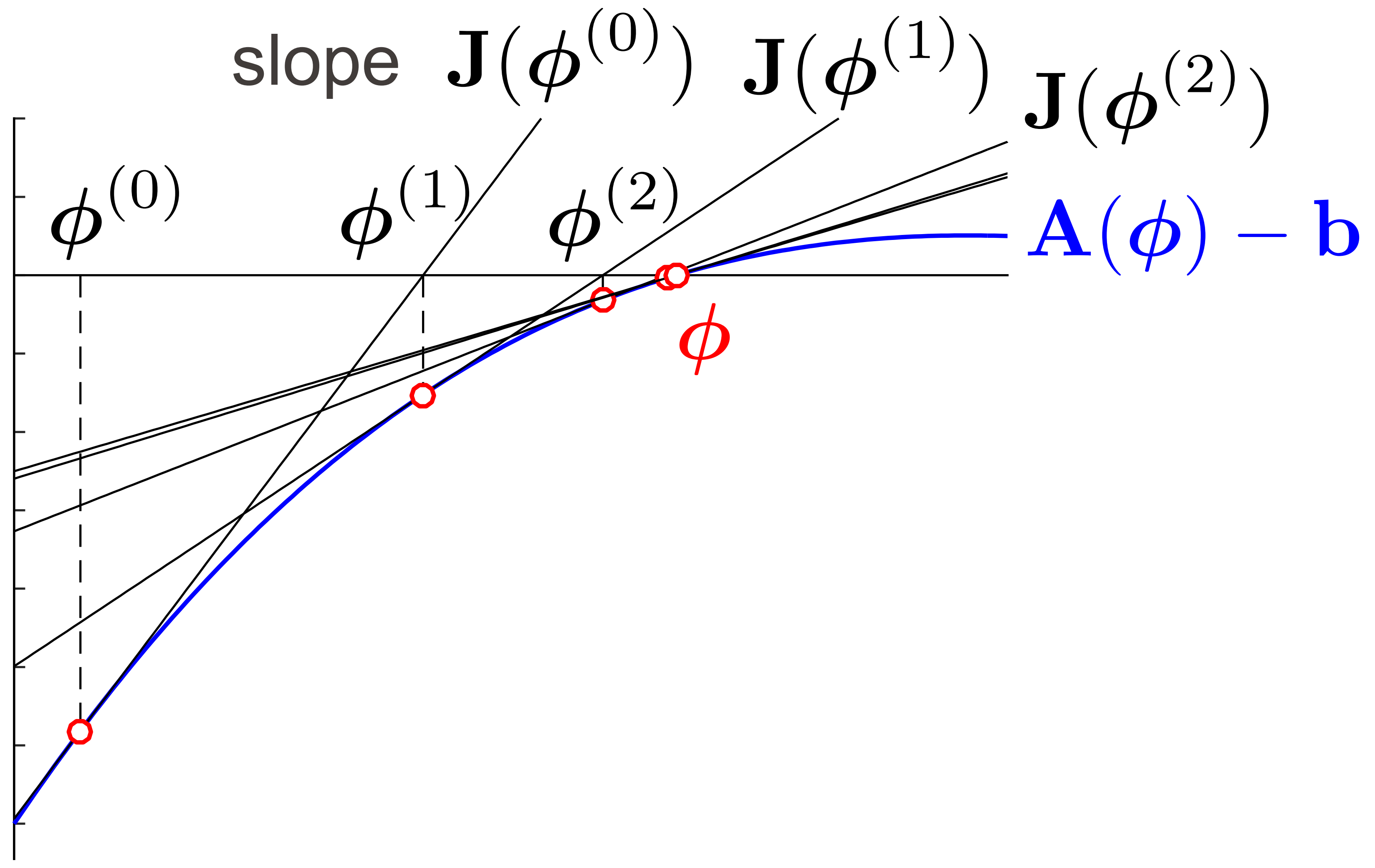
Linearization: Newton's method

$$\mathbf{A}(\phi) = \mathbf{b} \quad \rightarrow \quad \phi = ?$$

- Linearize with 1st-order Taylor expansion:

$$\mathbf{A}(\phi^* + \delta\phi) \approx \mathbf{A}(\phi^*) + \left.\frac{\partial \mathbf{A}}{\partial \phi}\right|_{\phi^*} \delta\phi \approx \mathbf{b}$$

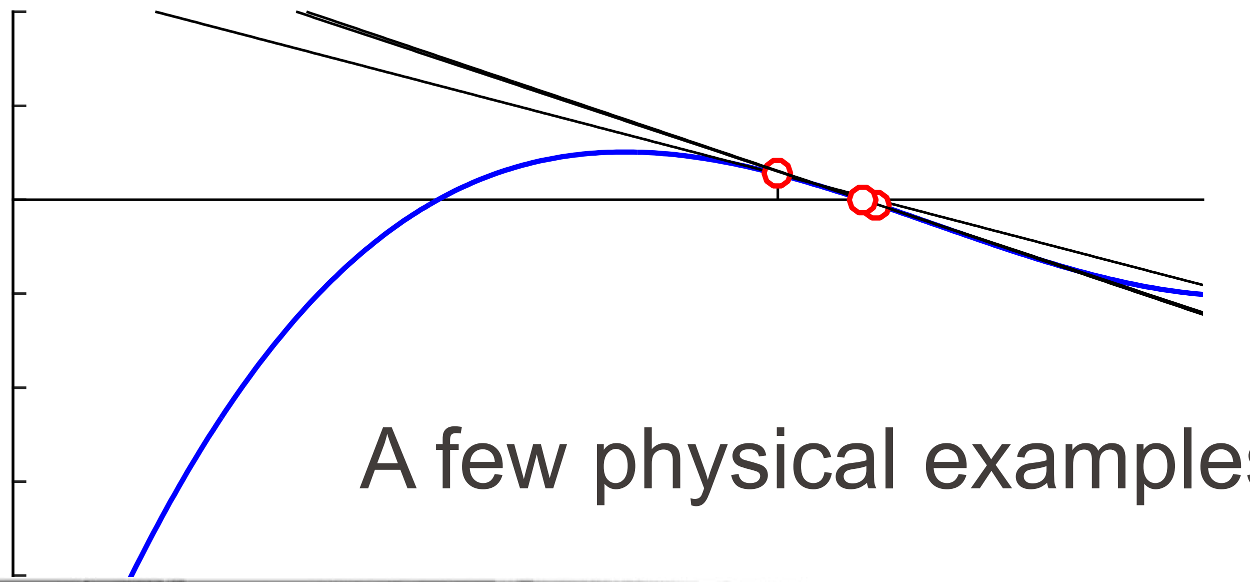
  - take a guess solution to evaluate the Jacobian matrix $\mathbf{J}(\phi^*) = \left.\frac{\partial \mathbf{A}}{\partial \phi}\right|_{\phi^*}$
  - solve for the increment $\delta\phi$
  - take $\phi = \phi^* + \delta\phi$ as new guess,
  - iterate until convergence.



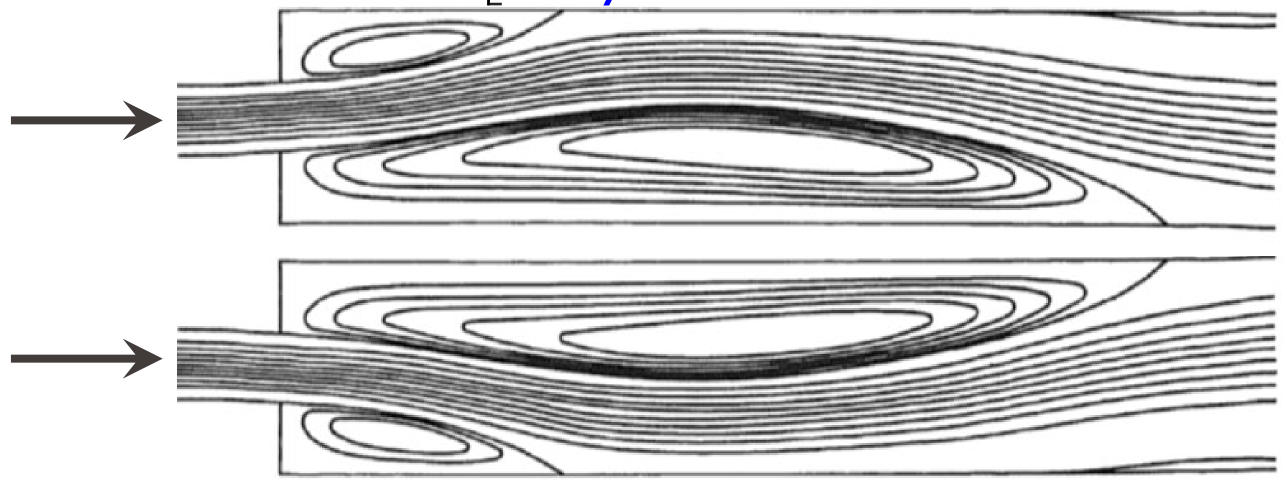
- Alternatively, can write $\mathbf{A}(\phi^*) + \mathbf{J}(\phi^*)(\phi - \phi^*) \approx \mathbf{b}$ and solve directly for ϕ.
- Faster convergence than Picard's method, but more complicated to implement and costly to solve.

Possible problems

- May fail:
  - Can converge to “another” solution (when multiple solutions)
  - Can oscillate / diverge
- Remedies:
  - Choose a good initial guess
  - Use under-relaxation

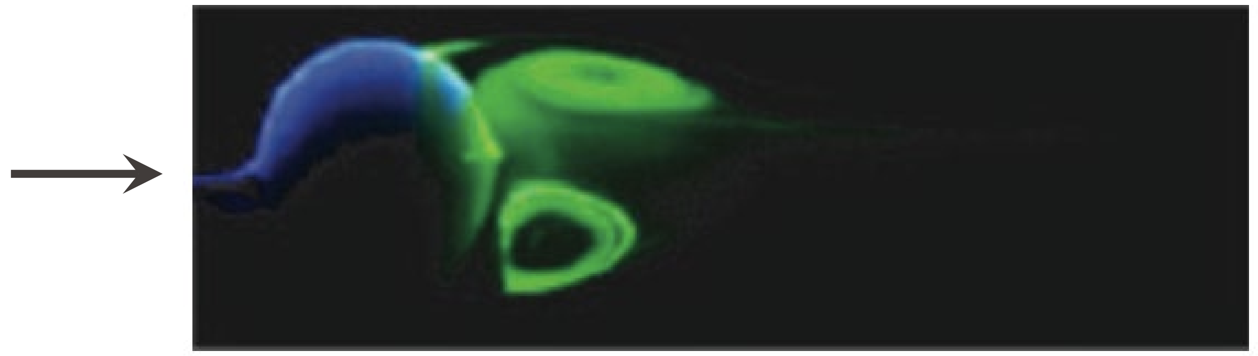


A few physical examples:



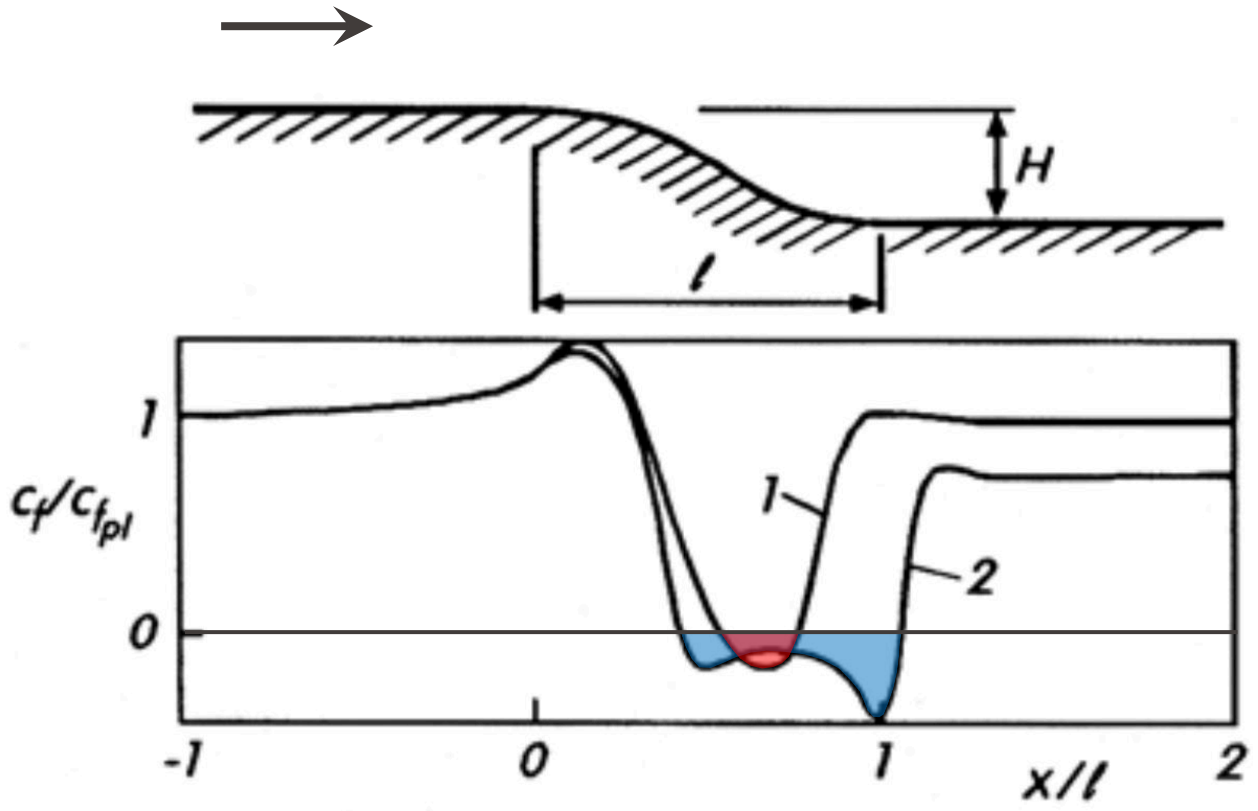
Sudden expansion
(2 steady solutions)

Fearn *et al.* 1990



Sphere wake,
212<*Re*<273 (infinitely many steady solutions)

Przadka *et al.* 2008



Smooth step
(2 solutions)

Schlichting & Gersten 2005

Under-relaxation

- Instead of taking ϕ or $\phi^* + \delta\phi$ as new guess,

  take $\omega\phi + (1-\omega)\phi^*$ or $\phi^* + \omega\,\delta\phi$, with $\omega < 1$

- Can help to stabilize the iterative process.
- Generally, slows down convergence.
- Suitable value of the relaxation factor is case-dependent.

Examples

- Nonlinear source term in the steady diffusion eq., for example:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \underbrace{(4 - 5T^3)}_{S(T)} = 0$$

Picard

$$S(T) \approx 4 - (5{T^*}^2)T$$

Newton

$$S(T^* + \delta T) \approx 4 - 5{T^*}^3 - (15{T^*}^2)\delta T$$

or $$4 + 10{T^*}^3 - (15{T^*}^2)T$$

→ Obtain an equation linear in *T*, to be solved for *T* knowing the current guess *T**.

Examples

- Convective term $\mathbf{C}(\phi) = \rho \mathbf{u}\mathbf{u}$ in the incompressible Navier-Stokes equations (constant density ρ):

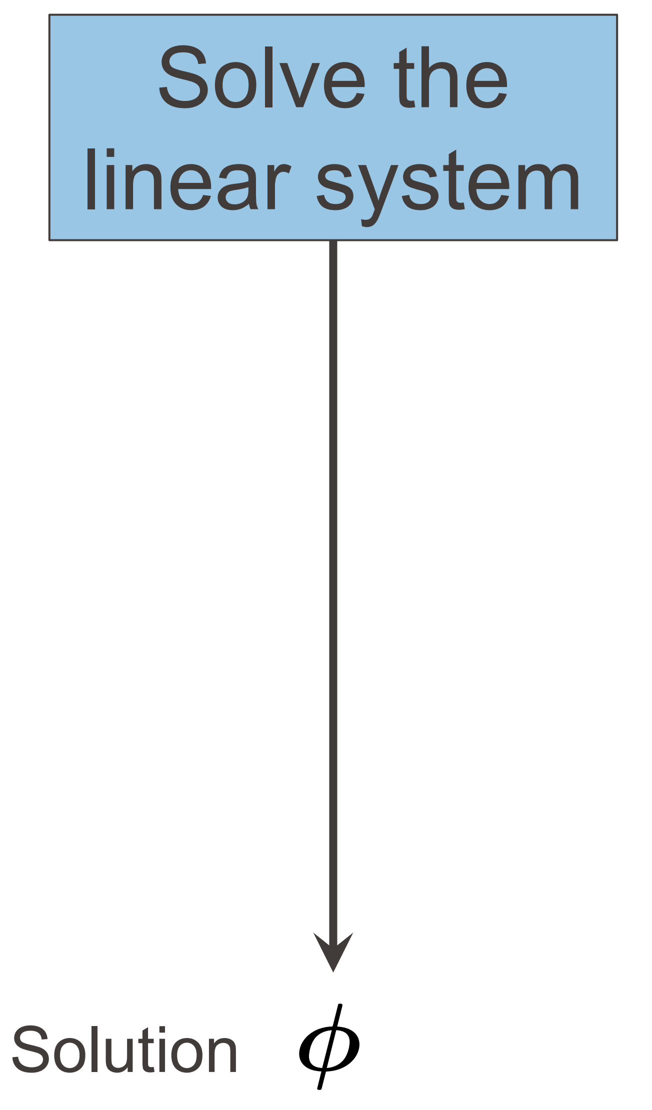
Picard	Newton
$\mathbf{C}(\phi) \approx (\rho^* \mathbf{u}^*)\mathbf{u}$	$\mathbf{C}(\phi^* + \delta\phi) \approx \rho^* \mathbf{u}^* \mathbf{u}^* + (2\rho^* \mathbf{u}^*)\, \delta\mathbf{u}$
	or $-2\rho^* \mathbf{u}^* \mathbf{u}^* + (2\rho^* \mathbf{u}^*)\, \mathbf{u}$

→ Obtain an equation linear in **u**, to be solved for **u** knowing the current guess **u***.

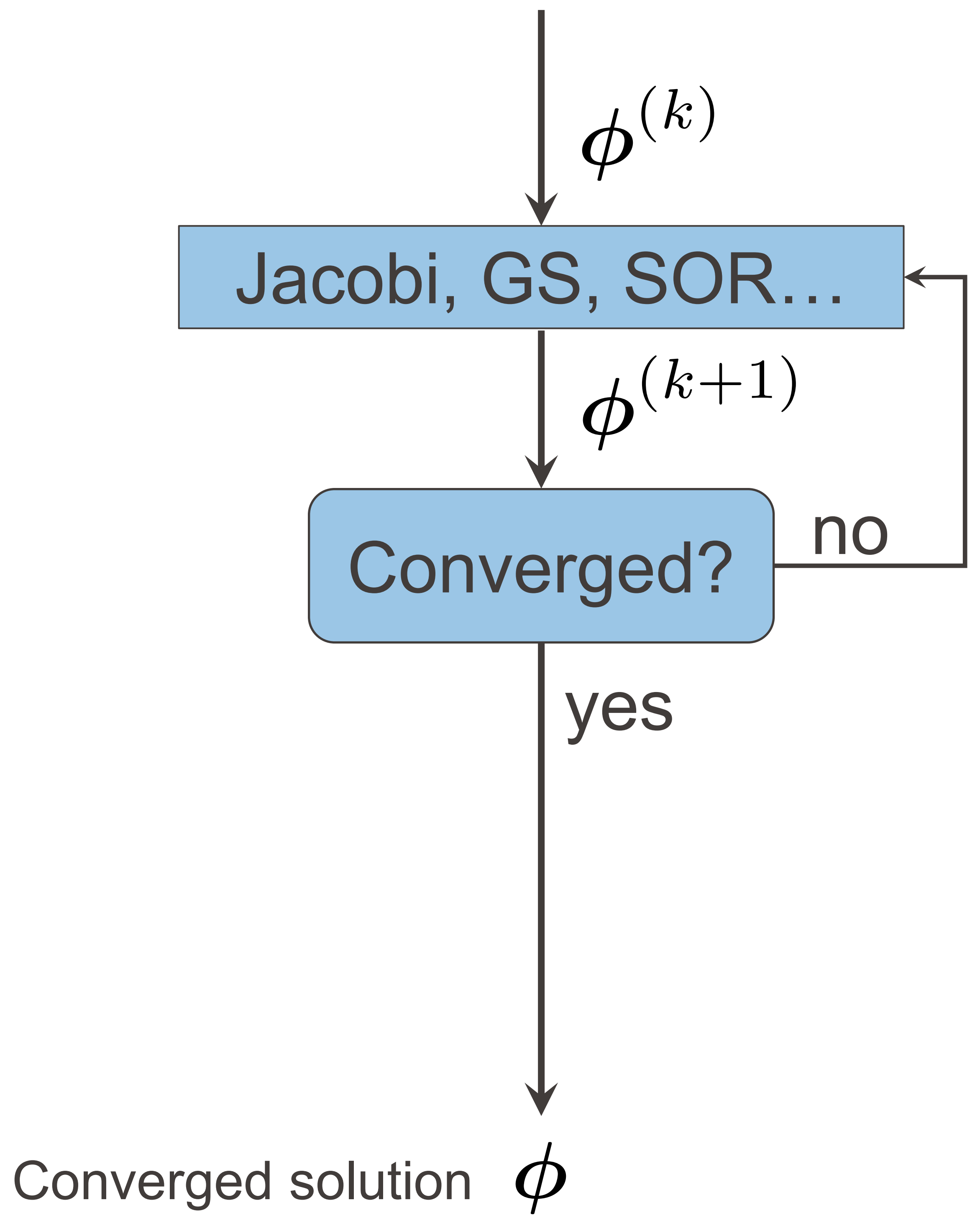
Exercise: what would you obtain for the nonlinear terms in the compressible NS equations?
Hints: (i) work with the full state vector $\boldsymbol{\phi}$=(ρ,$\mathbf{u}$); (ii) consider the continuity and momentum equations simultaneously.

Summary

Linear system, direct methods



Linear system, iterative methods



Nonlinear system, iterative methods