Solving the Navier-Stokes equations

Numerical Flow Simulation

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Reminder: the NS equations

- General conservation equation:

$$\boxed{\frac{\partial(\rho\phi)}{\partial t}} + \boxed{div(\rho\phi\mathbf{u})} = \boxed{div(\Gamma\, grad(\phi))} + \boxed{S}$$

- So far, solved for ϕ assuming known density and velocity.
- In general, velocity field not known and must be computed too.
- The Navier-Stokes eq. themselves have the same general form (week 2):

$$\boxed{\frac{\partial\rho}{\partial t}} + \boxed{div(\rho\mathbf{u})} = 0$$

$$\boxed{\frac{\partial(\rho\mathbf{u})}{\partial t}} + \boxed{div(\rho\mathbf{u}\mathbf{u})} = \boxed{div\left[\left(-p - \frac{2}{3}\mu\, div(\mathbf{u})\right)\mathbf{I} + 2\mu\mathbf{d}\right]} + \boxed{\rho\mathbf{f}}$$

- All the methods seen so far should apply?…

Reminder: notoriously difficult equations

- Some elements that make the Navier-Stokes equations difficult to solve:

Differential operators ✓ (weeks 2 & 4)

Coupled equations

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p - \dfrac{1}{Re} \nabla^2 \mathbf{u} \end{cases}$$

Unsteadiness (week 6)

Nonlinearity ✓ (week 3)

Pressure

Turbulence (week 9)

Focus of this week

Issue 1: coupled equations

- The variable solved for in each eq. appears in the other equations.

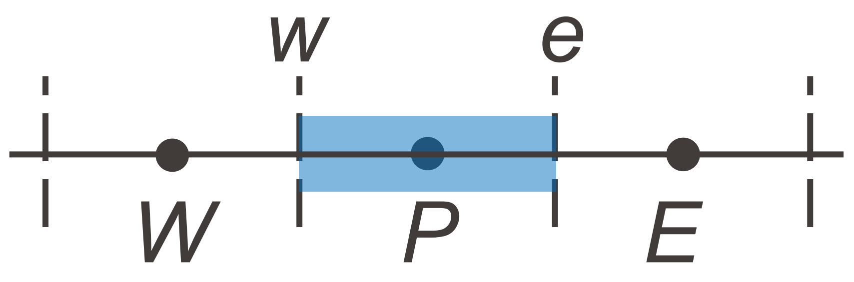
  For ex. steady incompressible 2D:

$$\begin{cases} \dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} = 0 \\ \dfrac{\partial(\rho uu)}{\partial x} + \dfrac{\partial(\rho uv)}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial x}\left(\mu\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial u}{\partial y}\right) \\ \dfrac{\partial(\rho uv)}{\partial x} + \dfrac{\partial(\rho vv)}{\partial y} = -\dfrac{\partial p}{\partial y} + \dfrac{\partial}{\partial x}\left(\mu\dfrac{\partial v}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial v}{\partial y}\right) \end{cases}$$

- **Coupled method:**
  - In principle, can solve all equations simultaneously. Define $\boldsymbol{\phi} = (u, v, w, \rho)$ or (u, v, w, p) and solve a single system.
  - Works best for linear, strongly coupled equations. Better convergence, but more memory requirement. Sometimes not possible with large meshes.

- **Segregated method:**
  - Solve each equation one by one, iteratively.
  - Works best for nonlinear equations. Slower convergence, but less memory requirement.

Issue 2: pressure checkerboard



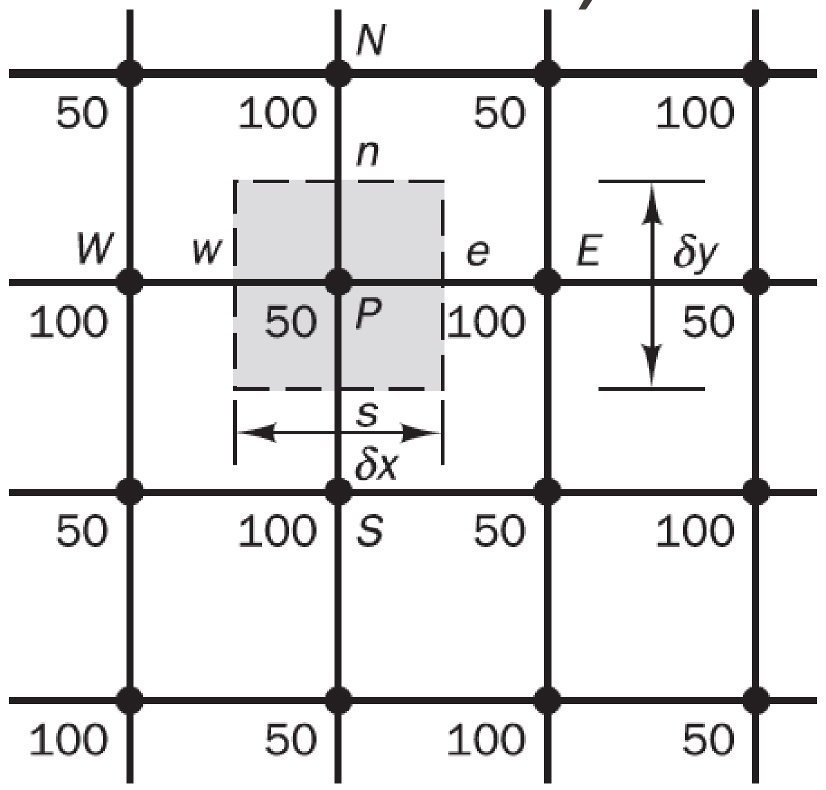
- Example: steady 1D momentum equation $$\frac{\partial(\rho u u)}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)$$
- Discretization of the pressure gradient with central differencing:

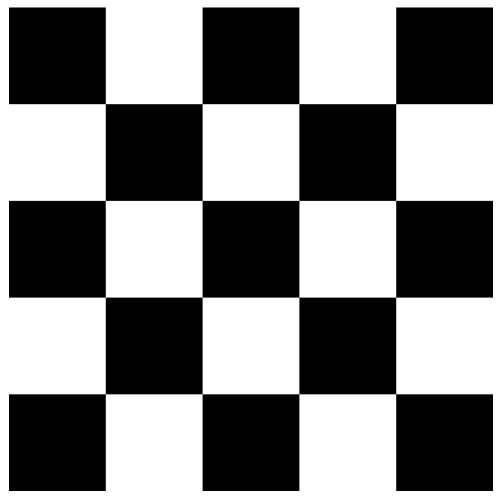
$$-\int_w^e \frac{\partial p}{\partial x} dx = p_w - p_e = \frac{p_P + p_W}{2} - \frac{p_E + p_P}{2} = \boxed{\frac{p_W - p_E}{2}}$$

- Involves nodes *W* and *E* that are 2 CVs apart. The central node *P* does not appear. Risk of "**checkerboard pressure mode**": not physical, but numerically possible because does not contribute to the momentum eq. (zero pressure gradient, just like a uniform pressure field).

1D

2D



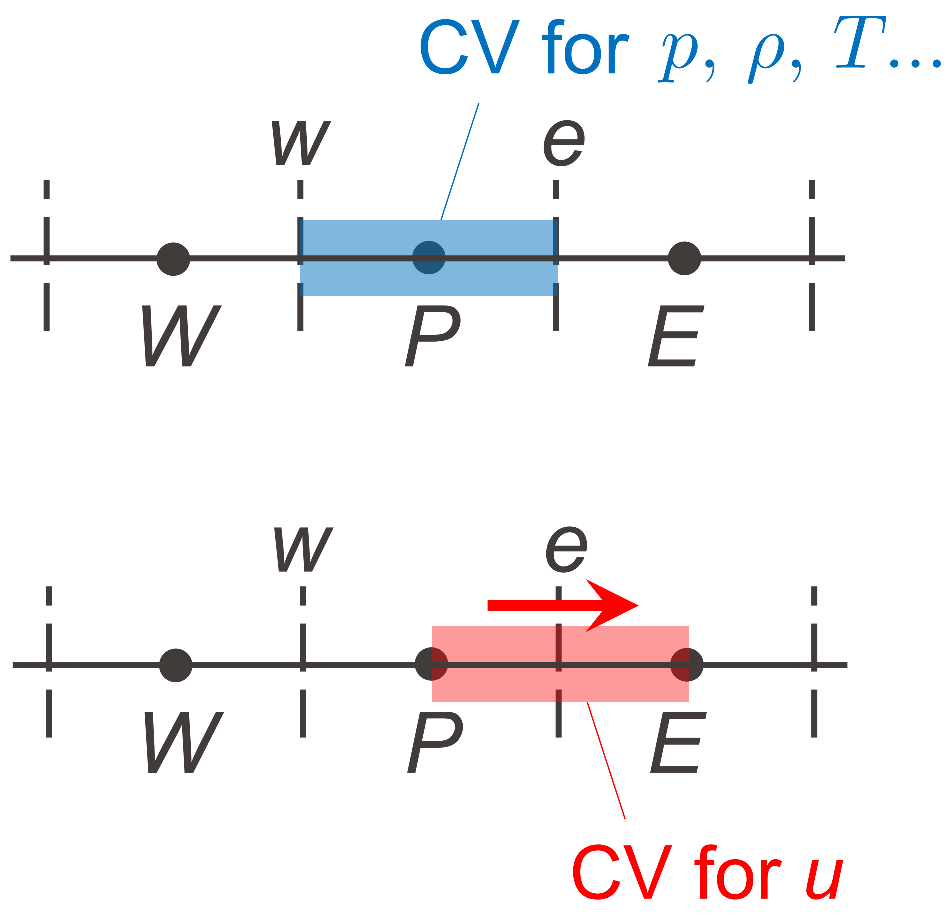


Staggered grid

- Remedy to checkerboard pressure: staggered grid
  - Evaluate pressure (and other scalars ϕ) at the nodes of the original mesh
  - Evaluate velocities at the nodes of a staggered mesh
  - CV centers of the staggered mesh correspond to face centers of the original mesh

1D

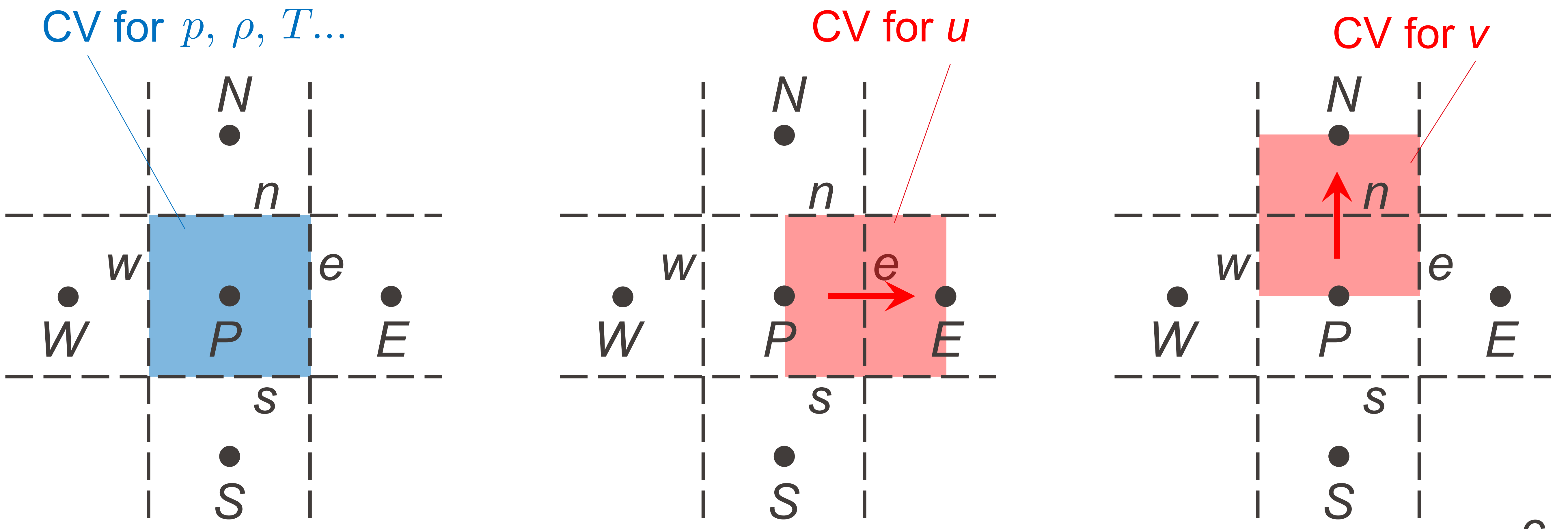
Original mesh



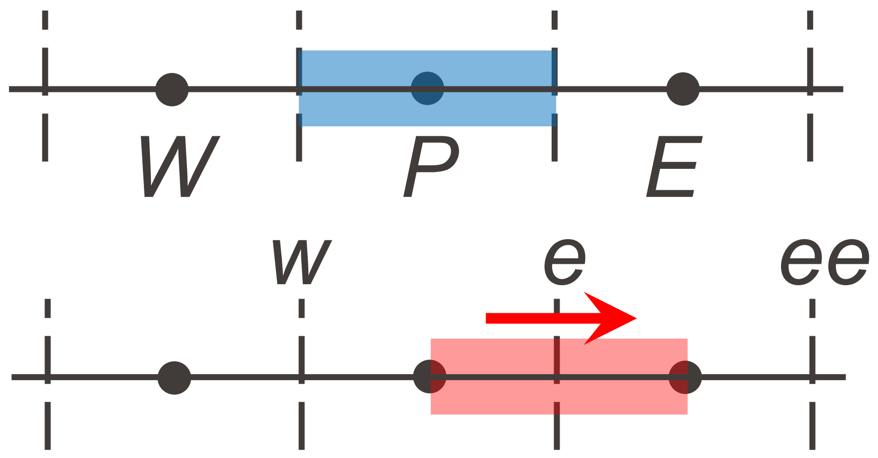
Staggered mesh

2D

Original mesh | Mesh staggered in *x* | Mesh staggered in *y*



Staggered grid



- 1D discretization for *u* on the staggered mesh:

$$\frac{\partial(\rho u u)}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)$$

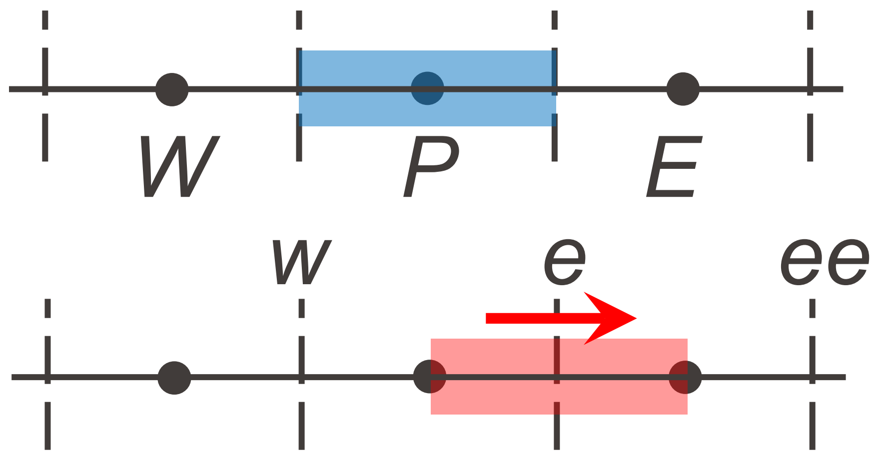
$$\int_P^E \frac{\partial(\rho u u)}{\partial x} dx = -\int_P^E \frac{\partial p}{\partial x} dx + \int_P^E \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)$$

(Assume constant cross-section area *A*)

$$(\rho u u)_E - (\rho u u)_P = \boxed{p_P - p_E} + \left(\mu \frac{\partial u}{\partial x}\right)_E - \left(\mu \frac{\partial u}{\partial x}\right)_P$$

- Now, no risk of checkerboard pressure, because would give a non-zero contribution.
- Physically consistent: pressure difference between CVs drives the flow across the face.

Staggered grid



- 1D discretization for *u* on the staggered mesh:

$$(\rho uu)_E - (\rho uu)_P = p_P - p_E + \left(\mu \frac{\partial u}{\partial x}\right)_E - \left(\mu \frac{\partial u}{\partial x}\right)_P$$

For instance with CD for the diffusion term:

$$F_E u_E - F_P u_P = p_P - p_E + D_E(u_{ee} - u_e) - D_P(u_e - u_w)$$

Notations (week 4): $F = \rho u^*$ $\quad D = \frac{\mu}{\Delta x}$

Algebraic equation:

$$a_e u_e = a_w u_w + a_{ee} u_{ee} + (p_P - p_E)$$

Coefficients of face values depend on *D* and *F* at the nodes (CV centers):

$$a_f = a_f(D_C, F_C)$$

Compare with discretization on original mesh (previous weeks)… and don't get lost!

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + b$$

Coefficients of nodal values depend on *D* and *F* at the faces:

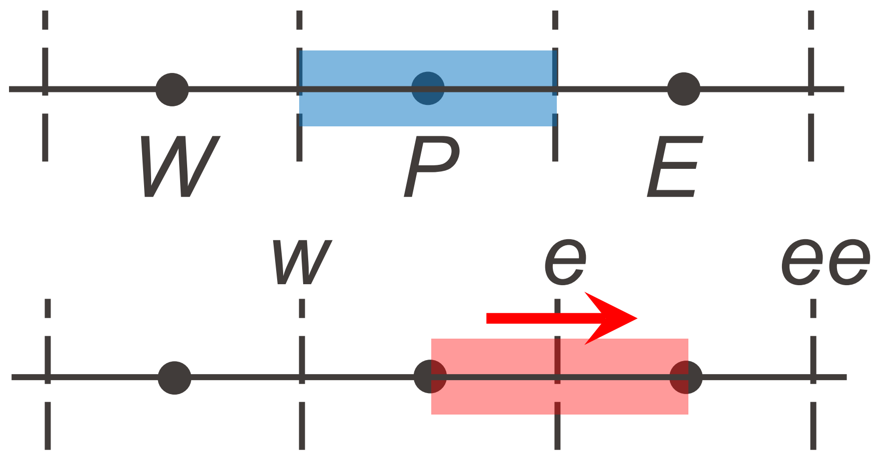
$$a_C = a_C(D_f, F_f)$$

Need for *C*→*f* and *f*→*C* interpolation:

$$F_C = (\rho u^*)_C = \frac{1}{2}\left((\rho u^*)_{f_1} + (\rho u^*)_{f_2}\right) \qquad \rho_f = \frac{1}{2}(\rho_{C_1} + \rho_{C_2})$$

Numerical Flow Simulation

Staggered grid



- 1D discretization of the continuity eq. on the original mesh:

$$\frac{d(\rho u)}{dx} = 0 \quad \rightarrow \quad (\rho u)_e - (\rho u)_w = 0$$
$$F_e - F_w = 0$$

Velocities at faces available (no interpolation required).

Staggered grid

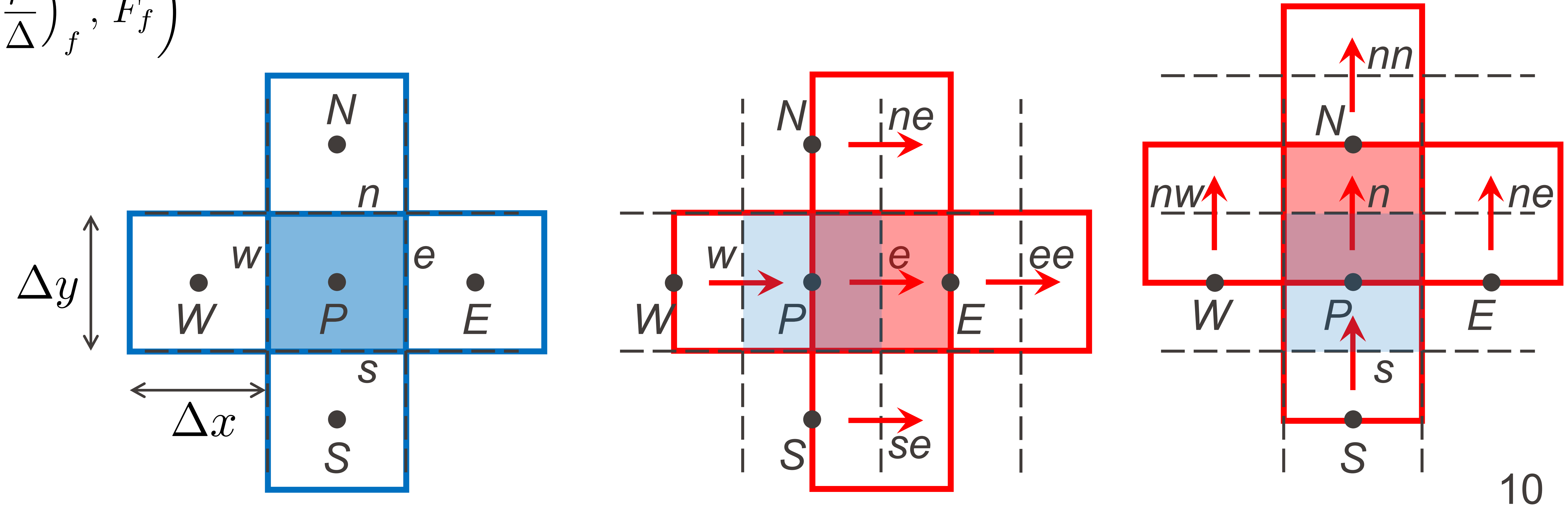
- Momentum equations in 2D (staggered mesh):

$$\begin{cases} a_e u_e = \sum\limits_{f=w,\,ee,\,ne,\,se} a_f u_f + (p_P - p_E) A_e \\ a_n v_n = \sum\limits_{f=nw,\,ne,\,nn,\,s} a_f v_f + (p_P - p_N) A_n \end{cases}$$

$$a_f = a_f \left(D_C = \left(\frac{\mu}{\Delta}\right)_C,\ F_C = \frac{F_{f1} + F_{f2}}{2} \right)$$

- Continuity eq. in 2D (original mesh): $F_e - F_w + F_n - F_s = 0$
- Other scalar unknowns (original mesh): $a_P \phi_P = \sum\limits_C a_C \phi_C + b$

$$a_C = a_C \left(D_f = \left(\frac{\mu}{\Delta}\right)_f,\ F_f \right)$$



Numerical Flow Simulation

Issue 3: no equation for pressure?

- **Compressible** flows: continuity eq. = eq. for density. Can deduce pressure with an equation of state such as $p = p(\rho)$. (Pressure = “thermodynamic” variable.)

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0 \qquad \frac{\partial (\rho \mathbf{u})}{\partial t} + div(\rho \mathbf{u}\mathbf{u}) = div\left[\left(-p - \frac{2}{3}\mu\, div(\mathbf{u})\right)\mathbf{I} + 2\mu \mathbf{d}\right]$$

- **Incompressible** flows: constant density, no eq. for pressure. The role of pressure is to enforce a constraint: continuity. (Pressure = “mathematical” variable.)

$$div(\mathbf{u}) = 0 \qquad \frac{\partial (\rho \mathbf{u})}{\partial t} + div(\rho \mathbf{u}\mathbf{u}) = -grad(p) + div(\mu\, grad(\mathbf{u}))$$

No problem if use a coupled method. However, if use a segregated method, need to find a way to compute pressure.

Poisson eq. for pressure

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

- Key idea: obtain a scalar eq. for pressure (Poisson eq.) by taking the divergence of the momentum eq.

$$\nabla \cdot \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \right]$$

$$\frac{\partial (\nabla \cdot \mathbf{u})}{\partial t} + \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] = -\frac{1}{\rho}\nabla^2 p + \nu \nabla^2 (\nabla \cdot \mathbf{u})$$

- If the velocity field satisfies continuity (divergence free):

$$\nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] = -\frac{1}{\rho}\nabla^2 p \longrightarrow$$

Can solve for *p* if **u** is known.

- Need a boundary condition: project the momentum eq. on the boundary normal

$$\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \right] \cdot \mathbf{n} \quad \rightarrow \quad \nabla p \cdot \mathbf{n} = \rho \left[-\frac{\partial \mathbf{u}}{\partial t} - (\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} \right] \cdot \mathbf{n}$$

Unsteady incompressible NS

$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

- Assume an **explicit** temporal scheme (week 6) for velocity:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n = -\frac{1}{\rho}\nabla p^{n+1} + \nu\nabla^2\mathbf{u}^n$$

Take the divergence:
$$\frac{\cancel{\nabla\cdot\mathbf{u}^{n+1}} - \cancel{\nabla\cdot\mathbf{u}^n}}{\Delta t} + \nabla\cdot[(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n] = -\frac{1}{\rho}\nabla^2 p^{n+1} + \nu\nabla^2\cancel{(\nabla\cdot\mathbf{u}^n)}$$

Require divergence-free velocity at new time step

$$\nabla\cdot[(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n] = -\frac{1}{\rho}\nabla^2 p^{n+1}$$

Divergence-free velocity from previous time step

1. Solve Poisson eq. for new p^{n+1} (linear system).
2. Compute new velocity: $$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\left[-\frac{1}{\rho}\nabla p^{n+1} + \nu\nabla^2\mathbf{u}^n - (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n\right]$$

Not applicable to **implicit** temporal schemes (because $\mathbf{u}^{n+1}$ would appear in eq. for p^{n+1}).

HH decomposition / Pressure projection

- Helmholtz-Hodge decomposition: any vector field is the sum of a solenoidal (divergence-free) part and an irrotational part:

$$\mathbf{w} = \mathbf{v} + \nabla\psi$$

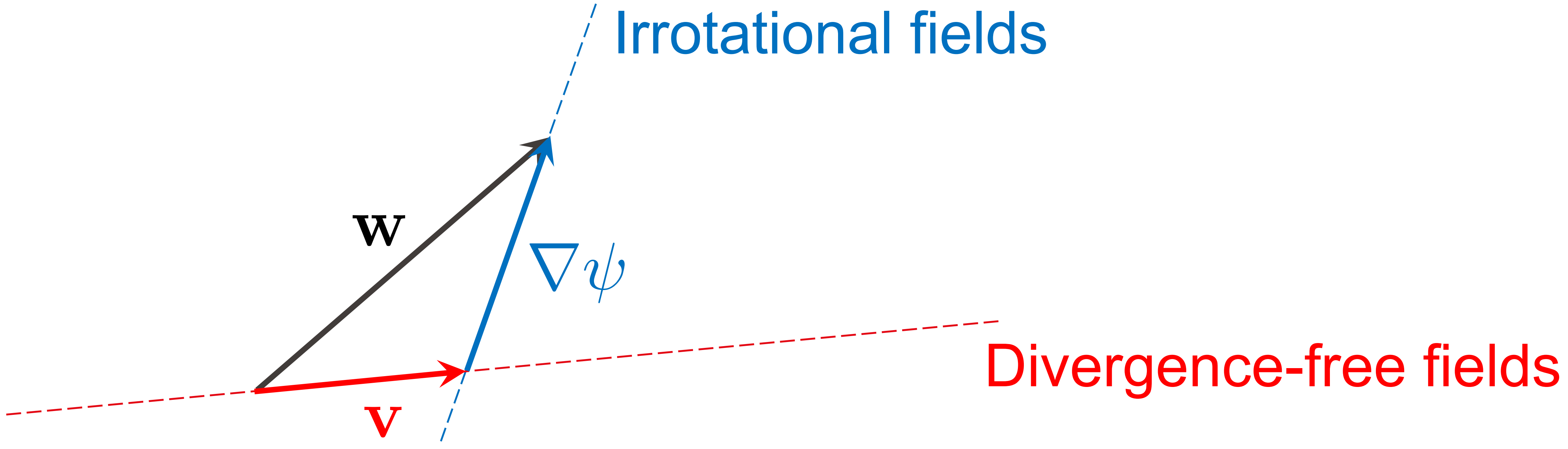
solenoidal $\nabla \cdot \mathbf{v} = 0$

irrotational $\nabla \times (\nabla\psi) = 0$

ψ is solution of a Poisson eq.: $\nabla \cdot \mathbf{w} = \cancel{\nabla \cdot \mathbf{v}} + \nabla^2\psi = \nabla^2\psi$

→ can extract the divergence-free part of any field: $\mathbf{v} = \mathbf{w} - \nabla\psi$

- Geometric interpretation: projection onto the space of divergence-free fields.



- The incompressible NS eq. can be seen as such a projection onto divergence-free velocity fields. This projection is done via the pressure *p*.

Unsteady incompressible NS: fractional step method

- Idea:
  1. Compute tentative velocity from momentum eq., ignoring pressure,
  2. Compute the divergence-free component part of **u***.

- Assuming again an **explicit** temporal scheme:

Split $\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}+(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n=-\dfrac{1}{\rho}\nabla p^{n+1}+\nu\nabla^2\mathbf{u}^n$ into $\begin{cases}\dfrac{\mathbf{u}^*-\mathbf{u}^n}{\Delta t}+(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n=-\nu\nabla^2\mathbf{u}^n\\[2ex] \dfrac{\mathbf{u}^{n+1}-\mathbf{u}^*}{\Delta t}=-\dfrac{1}{\rho}\nabla p^{n+1}\end{cases}$

1. Compute tentative velocity **u*** with 1st eq.

Take divergence of 2nd eq.: $\dfrac{\nabla\cdot\mathbf{u}^{n+1}-\nabla\cdot\mathbf{u}^*}{\Delta t}=-\dfrac{1}{\rho}\nabla^2 p^{n+1}$ $\rightarrow$ (require div.-free $\mathbf{u}^{n+1}$) $\dfrac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^*=\nabla^2 p^{n+1}$

2.a. Solve Poisson eq. for new p^{n+1} (linear system)

2.b. Compute new velocity: $\mathbf{u}^{n+1}=\mathbf{u}^*-\dfrac{\Delta t}{\rho}\nabla p^{n+1}$ It's indeed a HH decomposition: $\mathbf{v}=\mathbf{w}-\nabla\psi$

Also applicable to **implicit** temporal schemes ($\mathbf{u}^{n+1}$ doesn't appear in eq. for p^{n+1}).

Unsteady incompressible NS: fractional step method

- Can also keep pressure term, solve with p^n as a guess, and solve for a pressure correction (instead of pressure). Example with **implicit** scheme:

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} = -\frac{1}{\rho}\nabla p^{n+1} + \nu\nabla^2\mathbf{u}^{n+1}$$

$$\begin{cases} \dfrac{\mathbf{u}^*-\mathbf{u}^n}{\Delta t} + (\mathbf{u}^*\cdot\nabla)\mathbf{u}^* = -\dfrac{1}{\rho}\nabla p^n + \nu\nabla^2\mathbf{u}^* \\ \dfrac{\mathbf{u}^{n+1}-\mathbf{u}^*}{\Delta t} = -\dfrac{1}{\rho}\nabla(p^{n+1}-p^n) = -\dfrac{1}{\rho}\nabla p' \end{cases}$$

1. Compute tentative velocity **u*** with 1st eq.

Take divergence of 2nd eq. (require divergence-free $\mathbf{u}^{n+1}$): $\dfrac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^* = \nabla^2 p'$

2.a. Solve Poisson eq. for correction *p*' (linear system)

2.b. Compute new velocity: $\mathbf{u}^{n+1} = \mathbf{u}^* - \dfrac{\Delta t}{\rho}\nabla p'$

Actually, the sum of the 2 eq. is not exactly the intended scheme: $\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + (\mathbf{u}^*\cdot\nabla)\mathbf{u}^* = -\dfrac{1}{\rho}\nabla p^{n+1} + \nu\nabla^2\mathbf{u}^*$

But the error is of 2nd order in time, consistent with other errors of 2nd-order temporal schemes.

Steady incompressible NS

$$\nabla \cdot \mathbf{u} = 0$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

- For steady equations, can’t split the time derivative into 2 parts. Use instead other types of “predictor-corrector” approaches.
- Idea:
  1. Solve the momentum eq., using a guess velocity to linearize the nonlinear convective term (Picard) + a guess pressure,
  2. Use the continuity eq. to deduce an eq. for pressure or pressure correction,
  3. Update velocity and pressure; iterate.

Schematically:

Momentum eq.

$$\boxed{(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p^* + \nu \nabla^2 \mathbf{u}}$$

$\mathbf{u}(p^*)$ New velocity as fct of guess pressure

$\mathbf{u}'(p')$ Express velocity correction as fct of pressure correction

$$\boxed{\nabla \cdot \mathbf{u}'(p') = 0}$$

Divergence of continuity eq.
= eq. for pressure correction p'

Pressure correction p'

Compute vel. correction $\mathbf{u}'(p')$

Update velocity and pressure

Steady incompressible NS: the SIMPLE algorithm

"Semi-Implicit Method for Pressure-Linked Equations" (originally developed for unsteady flows).

1. Solve momentum eq. with guess velocity (for nonlinear term) and guess pressure p^* → new (tentative) velocity $\mathbf{u}^*$. For ex. (2D, staggered grid):

$$a_e u_e^* = \sum_f a_f u_f^* + (p_P^* - p_E^*) A_e \qquad \rightarrow u_e^*$$
$$a_n v_n^* = \sum_f a_f v_f^* + (p_P^* - p_N^*) A_n \qquad \rightarrow v_n^*$$

At this stage, $\mathbf{u}^*$ has no reason to satisfy continuity → introduce velocity correction $\mathbf{u}'$ and pressure correction p', aimed at enforcing continuity:

$$\mathbf{u} = \mathbf{u}^* + \mathbf{u}', \quad p = p^* + p'$$

Steady incompressible NS: the SIMPLE algorithm

2. Derive an equation for the pressure correction as follows:
   - Subtract momentum eq. for **u*** from those for **u**

$$a_e(u_e - u_e^*) = \sum a_f(u_f - u_f^*) + \left[(p_P - p_E) - (p_P^* - p_E^*)\right] A_e \quad \rightarrow \quad a_e u_e' = \cancel{\sum a_f u_f'} + (p_P' - p_E')A_e$$

$$a_n(v_n - v_n^*) = \sum a_f(v_f - v_f^*) + \left[(p_P - p_N) - (p_P^* - p_N^*)\right] A_n \quad \rightarrow \quad a_n v_n' = \cancel{\sum a_f v_f'} + (p_P' - p_N')A_n$$

   - Main approximation of the SIMPLE algorithm: neglect neighbor contributions in the eq. for velocity corrections

$$u_e' = (p_P' - p_E')d_e \quad \rightarrow \quad u_e = u_e^* + (p_P' - p_E')d_e$$

$$v_n' = (p_P' - p_N')d_n \quad \rightarrow \quad v_n = v_n^* + (p_P' - p_N')d_n$$

where $d_e = \dfrac{A_e}{a_e}$ $\quad d_n = \dfrac{A_n}{a_n}$

   - Substitute **u** in continuity equation: $(\rho u A)_e - (\rho u A)_w + (\rho v A)_n - (\rho v A)_s = 0$

$$\rho_e \left[u_e^* + (p_P' - p_E')d_e\right] A_e - \rho_w \left[u_w^* + (p_W' - p_P')d_w\right] A_w \ldots$$
$$+\rho_n \left[v_n^* + (p_P' - p_N')d_n\right] A_n - \rho_s \left[v_s^* + (p_S' - p_P')d_s\right] A_s = 0$$

Equation for *p*': $\quad a_P p_P' = a_E p_E' + a_W p_W' + a_N p_N' + a_S p_S' + b$

Steady incompressible NS: the SIMPLE algorithm

2. Pressure correction equation: $a_P p'_P = a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S + b$

- Coefficients: $a_E = \rho_e d_e A_e, \quad a_W = \rho_w d_w A_w, \quad a_N = \rho_n d_n A_n, \quad a_S = \rho_s d_s A_s$

$$a_P = a_E + a_W + a_N + a_S$$

- Source term: $b = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n$
  - Balance of convective fluxes, based on guess velocities (discrete version of *div*(**u***)).
  - Before convergence, continuity not satisfied, therefore *b*≠0 ⇒ *p'*≠0 ⇒ **u'**≠0.
  - Convergence when continuity satisfied: *b*=0 ⇒ *p'*=0, **u'**=0 ⇒ *p*=*p**, **u**=**u***.
  - *b* is an indicator of convergence (in addition to the residuals)
  - The omission of neighbor contributions doesn't affect the final converged solution.

Steady incompressible NS: the SIMPLE algorithm

3. Update and iterate:
   - New value = old guess + correction: $\mathbf{u} = \mathbf{u}^* + \mathbf{u}', \quad p = p^* + p'$
   - Generally need under-relaxation to stabilize the procedure:

$$\mathbf{u}_{new} = \alpha_{\mathbf{u}} \mathbf{u} + (1 - \alpha_{\mathbf{u}}) \mathbf{u}_{old}$$

$$p_{new} = \alpha_p p + (1 - \alpha_p) p^* = p^* + \alpha_p p'$$

   - With this type of velocity under-relaxation, the discretized momentum eq. become

$$\left(\frac{a_e}{\alpha_u}\right) u_e^{(k+1)} = \sum a_f u_f^{(k+1)} + (p_P - p_E) A_e + \left(\frac{1-\alpha_u}{\alpha_u}\right) a_e u_e^{(k)}$$

$$\left(\frac{a_n}{\alpha_v}\right) v_n^{(k+1)} = \sum a_f v_f^{(k+1)} + (p_P - p_N) A_n + \left(\frac{1-\alpha_v}{\alpha_v}\right) a_n v_n^{(k)}$$

Analogous to pseudo-transient simulation, i.e. unsteady simulation with space-dependent time step (week 6).

The SIMPLE algorithm, and improved variants

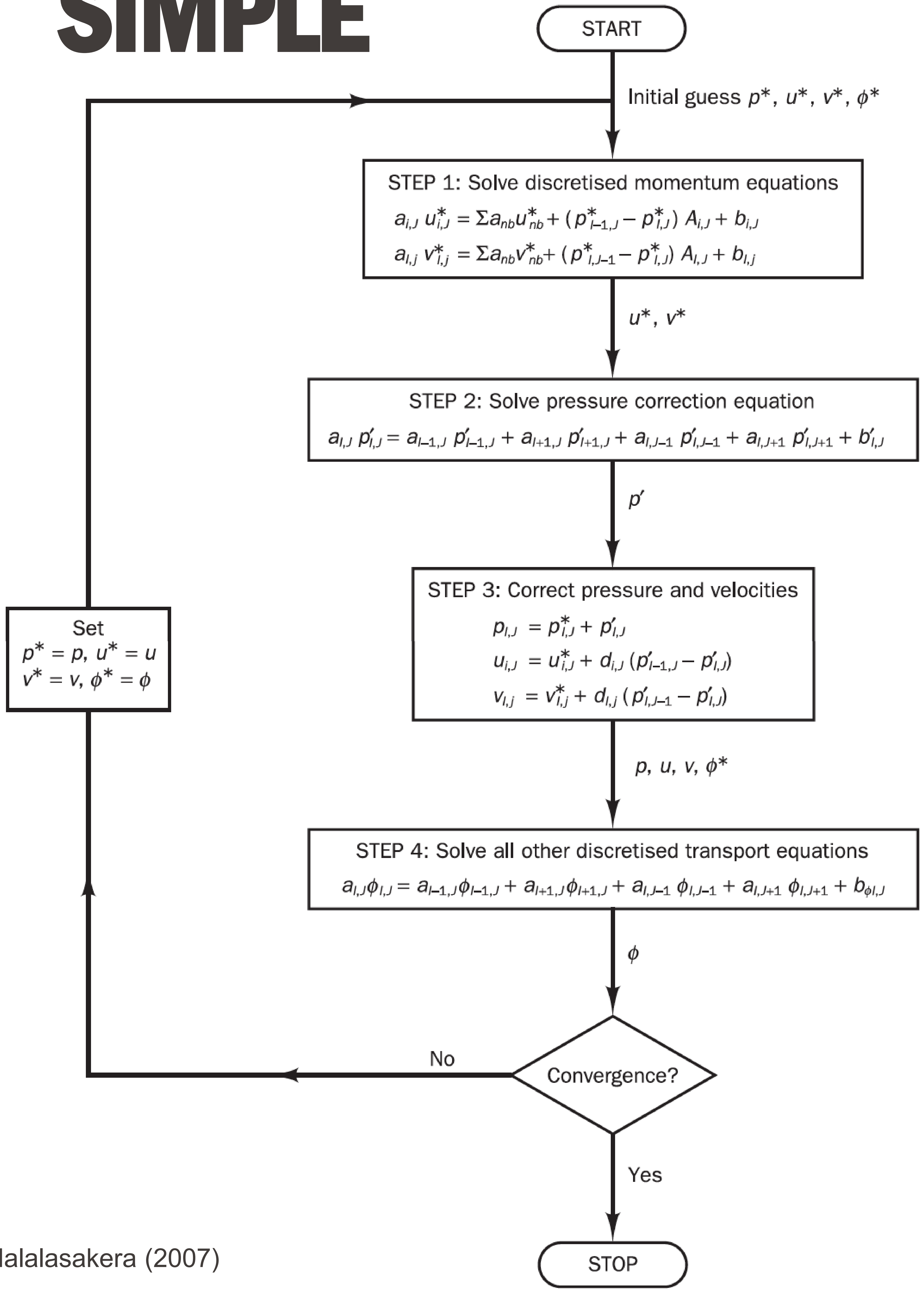
- SIMPLE
  - Routinely implemented in CFD codes.
  - The pressure correction is satisfactory for correcting velocities, but not so good for correcting pressure itself. A correct velocity field does not yield a correct pressure.
  - Rather slow convergence due to crude approximation.

- SIMPLER
  - Velocity: same method as SIMPLE (velocity correction from pressure correction).
  - Pressure: not corrected, directly calculated from discretized continuity equation.

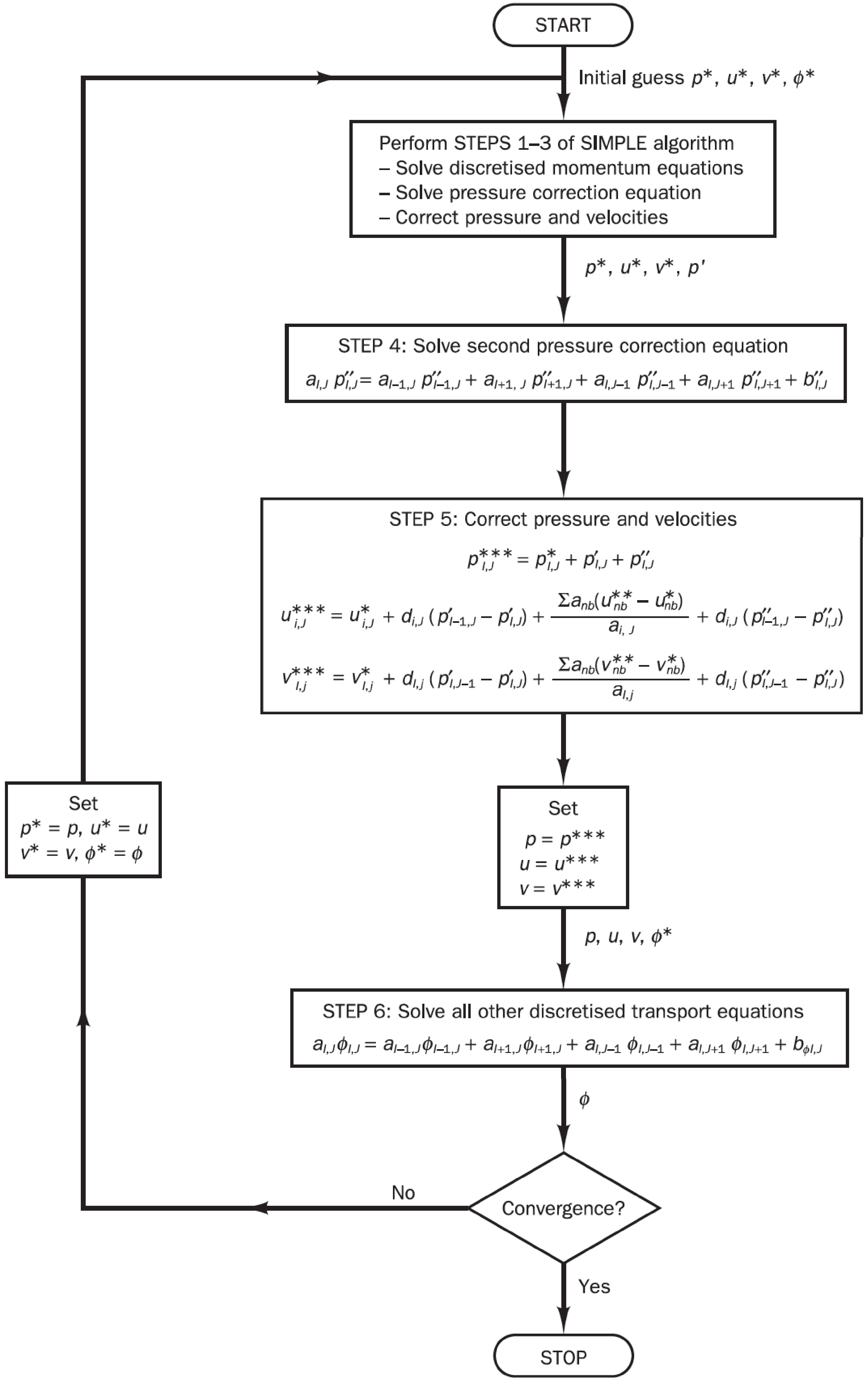
- SIMPLEC
  - Less crude approximation in the velocity correction eq. (omit less terms).

- PISO (“Pressure Implicit with Splitting of Operators”)
  - One additional correction step.
  - More expensive per iteration, but converges faster.

SIMPLE



PISO

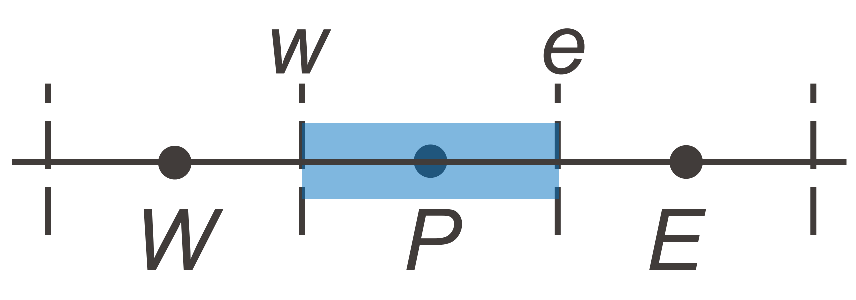


Numerical Flow Simulation

Versteeg & Malalasakera (2007)

Staggered grid vs. collocated grid

- Staggered grid: popular until a few decades ago. But several drawbacks: must store more geometric information; difficulties at boundaries.
- Alternative: stick to collocated grid (same grid for velocity and pressure); add higher-order pressure term to avoid checkerboard mode.



Discretize momentum eq. and continuity eq. on same grid:

$$a_P u_P = \sum a_C u_C + \frac{p_W - p_E}{2} A_P \qquad a_E u_E = \sum a_C u_C + \frac{p_P - p_{EE}}{2} A_E \qquad \sum (\rho \mathbf{u} \cdot \mathbf{n} A)_f = 0$$

If interpolate face velocities as usual, $u_e = \boxed{(u_P + u_E)/2}$ doesn't contain the physically expected pressure difference p_E-p_P across face *e* → pressure correction can be oscillatory (checkerboard).

- Example of alternative discretization: Rhie-Chow interpolation (expression on 1D uniform grid)

Usual term Higher-order pressure term, with desired pressure difference (if all *d* values equal)

$$u_e = \boxed{\frac{u_P + u_E}{2}} + \boxed{\frac{1}{2}(d_P + d_E)\boxed{(p_P - p_E)} - \frac{1}{4} d_P (p_W - p_E) - \frac{1}{4} d_E (p_P - p_{EE})} \approx \frac{u_P + u_E}{2} + \frac{d}{4} \left.\frac{\partial^3 p}{\partial x^3}\right|_e (\Delta x)^3$$

Additional term is small → doesn't compromise solution accuracy.

Damping of spurious pressure oscillations → also called "pressure dissipation".

Summary and guidelines

- Momentum and continuity eq. are coupled. Usually solved with segregated (sequential) method. Coupled (simultaneous) method can be used too; better convergence rate but requires more memory.

- Risk of checkerboard pressure mode on collocated grid. Remedies: staggered grids for velocities, or higher-order pressure term (Rhie-Chow).

- Incompressible flows: no eq. for pressure. Most methods derive a Poisson eq. for pressure (taking the divergence of the momentum eq., and enforcing continuity).
  - Fractional step method: non-iterative, 2-step method (unsteady)
  - SIMPLE algorithm: iterative, 2-step method (steady or unsteady)

- Fluent: 1 coupled solver + 4 segregated solvers: SIMPLE, SIMPLEC, PISO, Fractional Step Method. Collocated grid.

Appendix: pressure as a Lagrange multiplier

- Rigorous demonstration for the Stokes equations (*Re*=0): $\nabla \cdot \mathbf{u} = 0, \quad -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} = \mathbf{0}$

- The solution of the Stokes equations satisfies a minimum dissipation problem MP:

$$\min_{\mathbf{u}} \int_V \left(\frac{1}{2} \mu \left(\nabla \mathbf{u} \right)^2 - \mathbf{f} \cdot \mathbf{u} \right) dV \qquad \text{such that} \qquad \nabla \cdot \mathbf{u} = 0$$

- Transform this constrained optimization problem into an unconstrained one by introducing the Lagrangian

$$\mathcal{L}(\mathbf{u}, \lambda) = \int_V \left(\frac{1}{2} \mu \left(\nabla \mathbf{u} \right)^2 - \mathbf{f} \cdot \mathbf{u} - \lambda (\nabla \cdot \mathbf{u}) \right) dV$$

where *λ* is a (yet unknown) Lagrange multiplier enforcing the incompressibility constraint.

- A solution of the optimization problem corresponds to a stationary point of the Lagrangian, i.e. $\delta \mathcal{L} = 0$, or:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

Appendix: pressure as a Lagrange multiplier

- Variation with respect to λ: $$\int_V -\delta\lambda(\nabla \cdot \mathbf{u})\, dV = 0 \quad \forall\, \delta\lambda \qquad \Rightarrow \boxed{\nabla \cdot \mathbf{u} = 0}$$

  By construction, obtain the constraint.

- Variation with respect to **u**: $$\int_V \left(\mu\nabla\mathbf{u} : \nabla\boldsymbol{\delta}\mathbf{u} - \mathbf{f}\cdot\boldsymbol{\delta}\mathbf{u} - \lambda(\nabla\cdot\boldsymbol{\delta}\mathbf{u})\right)\, dV = 0 \quad \forall\, \boldsymbol{\delta}\mathbf{u}$$

$$\Rightarrow \int_V \left(-\mu\nabla^2\mathbf{u}\cdot\boldsymbol{\delta}\mathbf{u} - \mathbf{f}\cdot\boldsymbol{\delta}\mathbf{u} + \nabla\lambda\cdot\delta\mathbf{u}\right)\, dV = 0 \quad \forall\, \boldsymbol{\delta}\mathbf{u}$$

$$\Rightarrow \int_V \left(-\mu\nabla^2\mathbf{u} - \mathbf{f} + \nabla\lambda\right)\cdot\delta\mathbf{u}\, dV = 0 \quad \forall\, \boldsymbol{\delta}\mathbf{u}$$

$$\Rightarrow \boxed{\mu\nabla^2\mathbf{u} + \mathbf{f} - \nabla\lambda = \mathbf{0}}$$

Therefore, the velocity field and the Lagrange multiplier that are solution of the minimization problem MP are also solution of the Stokes equations.

Additionally, one recognizes that the pressure *p* is the Lagrange multiplier λ enforcing the incompressibility constraint.