ME-474 Numerical Flow Simulation

Exercise: 1D steady diffusion

Fall 2021

Implement a FVM code in Matlab to solve a 1D steady-state heat conduction problem.
Equation:

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial T}{\partial x}\right) + S(x) = 0$$

Domain: $x \in [0, L]$, $L = 1$ m.
Assume the thermal conductivity is constant: $k = 400$ W/(K.m).

1. Consider Dirichlet boundary conditions: $T(0) = T_a = 300$ K, $T(L) = T_b = 320$ K. Assume the source term is constant: $S = S_c = 5000$ W/m^3.

   - Define a uniform grid of n nodes: $x_1 = 0$, $x_2 = \Delta x = L/(n-1)\ldots$, $x_n = L$. Start with $n = 21$.
   - Recall the discretized equation

     $$a_P T_P = a_W T_W + a_E T_E + b,$$

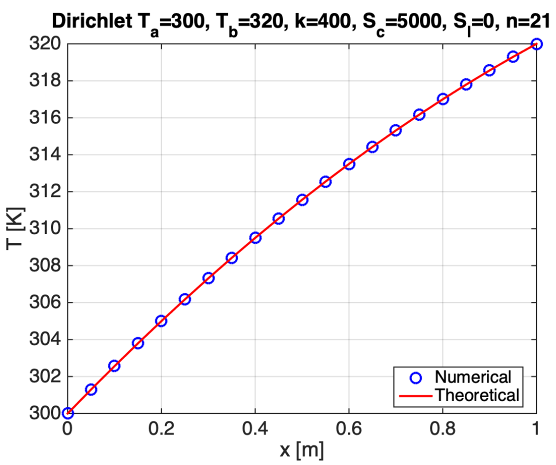
     or in vectorial form $\mathbf{AT} = \mathbf{b}$. Define the $n \times n$ matrix $\mathbf{A}$, and the $n \times 1$ right-hand side vector $\mathbf{b}$. Implement boundary conditions in equations 1 and n.
   - Solve for $\mathbf{T}$ and plot $T(x)$.
   - Compare with the theoretical solution ($T'' = -S/k = cst \to$ quadratic $T(x)$):

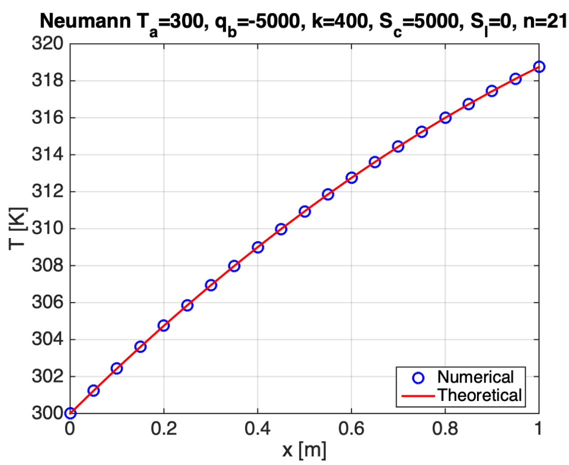
     $$T_{theo}(x) = \left(-\frac{S_c}{2k}\right)x^2 + \left(\frac{T_b - T_a}{L} + \frac{S_c L}{2k}\right)x + T_a.$$

   - Check that the mean error

     $$\frac{1}{n}\sum_i |T_i - T_{theo,i}|$$

     is exactly zero whatever the value of n. Why?





2. Consider now the same Dirichlet boundary condition on the left, $T(0) = T_a = 300$ K, but a Neumann boundary condition on the right, $q_b = -k(\partial T/\partial x)_{x=L} = -5000$ W/m^2.

   - Modify the implementation of the boundary conditions. (It may be a good idea to save two different versions of your code.)
   - Solve for $\mathbf{T}$ and plot $T(x)$.
   - Compare with the theoretical solution:

$$T_{theo}(x) = \left(-\frac{S_c}{2k}\right) x^2 + \left(\frac{S_c L - q_b}{k}\right) x + T_a.$$

   - Check that the mean error per control volume is exactly zero whatever the value of n. Why?

3. Finally, come back to Dirichlet boundary conditions on both ends like in Q1, but assume now that the source term varies linearly with temperature:

$$S = S_c + S_l T.$$

   Take for instance $S_c = 5000$ for the constant component, and $S_l = -100$ for the linear coefficient. Note that the integration of this source term over a control volume yields

$$\int_{x_w}^{x_e} S\,dx \approx \overline{S}\Delta x = (S_c + S_l T_P)\Delta x,$$

   so now the constant right-hand side is $b = S_c \Delta x$, while the solution-dependent term $S_l T_P \Delta x$ goes into the diagonal coefficient $a_P T_P$.

   - Modify the matrix $\mathbf{A}$ accordingly.
   - Solve for $\mathbf{T}$ and plot $T(x)$.
   - Compare with the theoretical solution, that can be obtained as the sum of (i) a particular solution of the full equation $kT'' + S_l T = -S_c$, i.e. $T = -S_c/S_l$, and (ii) the general solution of the homogeneous equation $kT'' + S_l T = 0$, which is $T = c_1 e^{\mu x} + c_2 e^{-\mu x}$, with $\mu = \sqrt{-S_l/k}$, and c_1 and c_2 such that boundary conditions are satisfied, which yields:

$$T_{theo}(x) = -\frac{S_c}{S_l} + c_1 e^{\mu x} + c_2 e^{-\mu x}, \quad c_1 = \frac{T_b - \left(\frac{S_c}{S_l} + T_a\right) e^{-\mu L} + \frac{S_c}{S_l}}{e^{\mu L} - e^{-\mu L}}, \quad c_2 = T_a + \frac{S_c}{S_l} - c_1.$$

   - Observe that the mean error per control volume is not zero. Why? How does it decrease with n? (Plot the mean error as a function of n in log-log scale.)