ME-474 Numerical Flow Simulation

Exercise: 1D steady convection-diffusion

Fall 2021

So far, we have considered pure **diffusion**. This week we add one essential element of fluid flows: **convection**. At this stage, the velocity field is given, so we are not yet solving the Navier–Stokes equations (where the velocity field is part of the unknown), but we move one step in that direction.

Implement a FVM code in Matlab to solve the 1D steady **convection-diffusion equation**:

$$\frac{d(\rho u \phi)}{dx} = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right).$$

You can of course start from the 1D steady diffusion code. Use the following:

- Domain: $x \in [0, L]$, $L = 1$ m;
- Physical parameters: constant density: $\rho = 1$ kg/m^3; constant velocity $u > 0$ (try different values); constant diffusion coefficient: $\Gamma = 0.1$ kg/(m.s);
- Dirichlet boundary conditions: $\phi(0) = \phi_0 = 1$, $\phi(L) = \phi_L = 0$;
- Uniform mesh of n nodes (try different values of n), with the first and last nodes on the boundaries.

There are 3 main things to be done, as detailed below: (1) implement the convective term, (2) implement the boundary conditions, (3) compute and compare with the theoretical solution.

1. Use CD to discretize the diffusion term as usual. For the convection term, try different schemes: UD, CD, QUICK, and one TVD scheme of your choice. For QUICK and TVD, implement the deferred correction approach (dc). Below is a summary, for the different schemes, of the coefficients and source terms of the algebraic equation $a_P\phi_P = a_W\phi_W + a_E\phi_E + S$:

	a_P	a_W	a_E	S
CD	$a_W + a_E$	$D + F/2$	$D - F/2$	0
UD	$a_W + a_E$	$D + F$	D	0
QUICK + dc	$a_W + a_E$	$D + F$	D	$F(-\phi^*_{WW} - \phi^*_W + 5\phi^*_P - 3\phi^*_E)/8$
TVD + dc	$a_W + a_E$	$D + F$	D	$F[-\psi(r^*_e)(\phi^*_E - \phi^*_P) + \psi(r^*_w)(\phi^*_P - \phi^*_W)]/2$

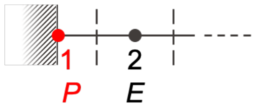
   Here, $F = \rho u > 0$ and $D = \Gamma/\Delta x$ are constant because ρ, u, Γ and Δx are constant. For TVD, the expression of the flux limiter $\psi(r)$ depends on the scheme you choose (see lecture slides), and

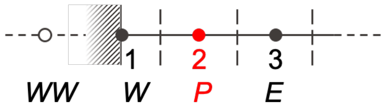
$$r^*_e = \frac{\phi^*_P - \phi^*_W}{\phi^*_E - \phi^*_P}, \quad r^*_w = \frac{\phi^*_W - \phi^*_{WW}}{\phi^*_P - \phi^*_W}.$$

2. Implementation of the boundary conditions:

   (i) In the first control volume (first node in $x = 0$), you can use the same method as for the diffusion equation, i.e "overwrite" the governing equation with the boundary condition $\phi(0) = \phi_0$ written as $a_W\phi_P = a_E\phi_E + b$ with $a_P = 1$, $a_E = 0$, $b = \phi_0$. See the first sketch below;

   (ii) The second control volume (second node in $x = \Delta x$) can be treated like other interior control volumes for CD and UD, but requires a special treatment for QUICK and TVD because these schemes





involve 2 upstream nodes. There are several options, one of them consisting in defining a "ghost" node WW outside the domain, using linear interpolation to express $\phi_W = (\phi_{WW} + \phi_P)/2$, and finally substituting $\phi_{WW} = 2\phi_W - \phi_P$ where needed. See the second sketch;

Don't forget to implement BCs in $x = L$ too.

3. Compute and compare with the theoretical solution:

$$\phi(x) = \phi_0 + (\phi_L - \phi_0)\,\frac{e^{Pe\,x/L} - 1}{e^{Pe} - 1}, \quad \text{where} \quad Pe = \frac{\rho u L}{\Gamma}.$$

Try different values of u and n, and see how they affect stability and accuracy.