ME470: Problem sheet 1 - Continuum mechanics and statistical mechanics

Solutions will be discussed in the exercise session on Thursday, October 3.

(1) Index notation

(a) write expressions for $\cos\theta$ and $\|\sin\theta\|$, where θ is the angle between vector $\mathbf{a}$ and $\mathbf{b}$ (Hint: use $\mathbf{a}\cdot\mathbf{b} = a_i b_i$, $\mathbf{a}\times\mathbf{b} = \epsilon_{ijk} a_i b_j \mathbf{e}_k$, and $\epsilon_{ijk}\epsilon_{pgk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$).
(b) show that if a tensor is antisymmetric, (i.e. $T_{ij} = -T_{ji}$), then $T_{ij}a_i a_j = 0$.
(c) show that $\epsilon_{ijk}a_j a_k = 0$, where ϵ_{ijk} is Levi-Civita symbol.

(2) Homogeneous isotropic linear elastic material Here, we will derive the constitutive equation for homogeneous isotropic linear elastic material by generalizing uniaxial stretching.

(a) Consider a uniaxial stress applied along the x-axis with a stress component, σ_{xx} . The resulting strains in x, y and z directions can be expressed using Young's modulus, E, and Poisson's ratio, ν, as follow:

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx}, \quad \epsilon_{yy} = \epsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} \tag{1}$$

Generalize this relationship to a situation involving a linear superposition of three uni-axial stretching applied independently along the x, y, and z axes to obtain the constitutive equation in the index notation. (Due to the applied stress, we obtain expressions for the diagonal elements of the strain tensor only, for instance. ϵ_{xx} in terms of σ_{xx}, σ_{yy}, and σ_{zz}. Rewrite ϵ_{xx} in terms of σ_{xx} and $\sigma_{ll} = \sigma_{xx}+\sigma_{yy}+\sigma_{zz}$, and generalize this expression to any arbitrary combination of ij. Note that σ_{ll} term should vanish for off-diagonal terms.)

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{ll}\delta_{ij} \tag{2}$$

(b) Invert the constitutive equation in (a) to express σ_{ij} in terms of ϵ_{ij} and ϵ_{ll}

(c) Find the expressions for the shear modulus, μ, and the bulk modulus, K, in terms of Young's modulus, E and Poisson's ratio, ν.

$$\mu = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)} \tag{3}$$

(d) Express the constitutive equation of σ_{ij} in terms of the bulk modulus K, and the shear modulus μ

$$\sigma_{ij} = 2\mu\left[\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}\right] + K\epsilon_{ll}\delta_{ij} \tag{4}$$

(e) Invert the constitutive equation in (a) to express σ_{ij} in terms of ϵ_{ij} and ϵ_{ll}

$$\epsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{1}{3}\sigma_{ll}\delta_{ij}\right) + \frac{1}{9K}\sigma_{ll}\delta_{ij} \tag{5}$$

(f) Utilizing that fact that the strain energy density can be written as $u = \frac{1}{2}\epsilon_{ij}\sigma_{ij}$, show that the strain energy density of homogeneous isotropic linear elastic material can be expressed as below.

$$u = \mu\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}\right)^2 + \frac{K}{2}\epsilon_{ll}^2 \tag{6}$$

(3) 2D elasticity Similar to 3D elasticity, the strain energy density of homogeneous isotropic two-dimensional material, u_A can be expressed as below.

$$u_A = \mu_A \left(\epsilon_{ij} - \frac{1}{2}\delta_{ij}\epsilon_{ll} \right)^2 + \frac{K_A}{2}\epsilon_{ll}^2, \qquad i,j \in \{x,y\} \tag{7}$$

(a) Derive from this the relation between the in-plane 2D stress, τ_{ij} and strain ϵ_{ij} (result: $\tau_{ij} = (K_A - \mu_A)\delta_{ij}\epsilon_{ll} + 2\mu_A\epsilon_{ij}$).

(b) Invert this expression to obtain ϵ_{ij} as a function of 2D stress.

(c) The material is placed under uniaxial stress $\tau_{xx} = \tau \neq 0$, $\tau_{yy} = 0$. Show that the Poisson ration is $\nu_A = -\frac{\epsilon_{yy}}{\epsilon_{xx}} = \frac{K_A - \mu_A}{K_A + \mu_A}$.

(d) The two-dimensional Young's modulus E_A is defined as $E_A \equiv \frac{\tau_{xx}}{\epsilon_{xx}}$. Show that $E_A = \frac{4K_A\mu_A}{K_A + \mu_A}$

(4) Two dice Two dice are rolled, each generating a number between 1 to 6 with equal probability. Consider this as a statistical system and the macroscopic states are represented by sum of numbers from two dice

(a) How many microstates are there for this system? How many macrostates are there for the system?

(b) What is the density of states, Ω, for each macrostate?

(c) What is the probability of each macrostate if the dice are rolled many times?

(5) Two-state system Consider a mechanosensitive ion channel that can be in one of two states: open or closed. Define the energy of the channel using a configuration parameter, s, that can be equal to 0 (closed) or 1(open). The energy is

$$E = s\epsilon_{open} + (1-s)\epsilon_{close} - s\tau\Delta A \tag{8}$$

where ϵ_{open} and ϵ_{closed} are the energies of the channel in the open and closed configurations, τ is the tension applied to the channel, and ΔA is the change in area of the channel in going from closed to open.

(a) Write an expression for the partition function.

(b) Write the probability of the channel being in the open state and the closed state

(c) Compute the average energy of the two-state system.