Most flows are unstable...

Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

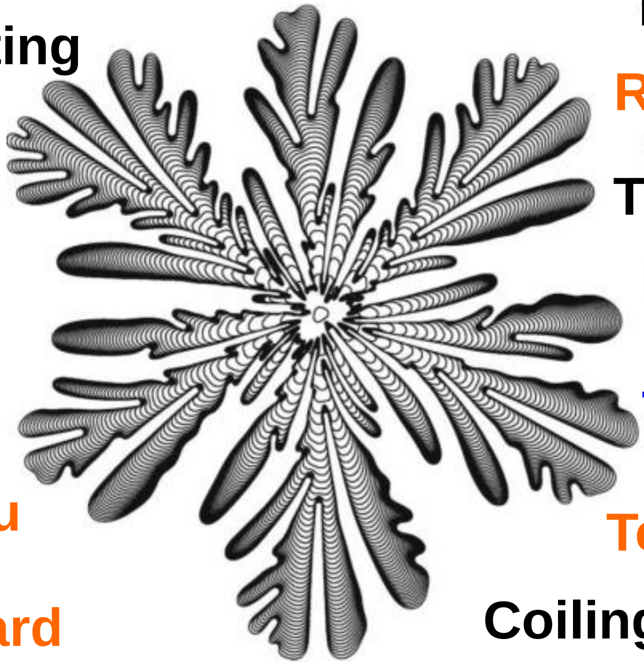
Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette



Rayleigh-Plateau

Tearing instability

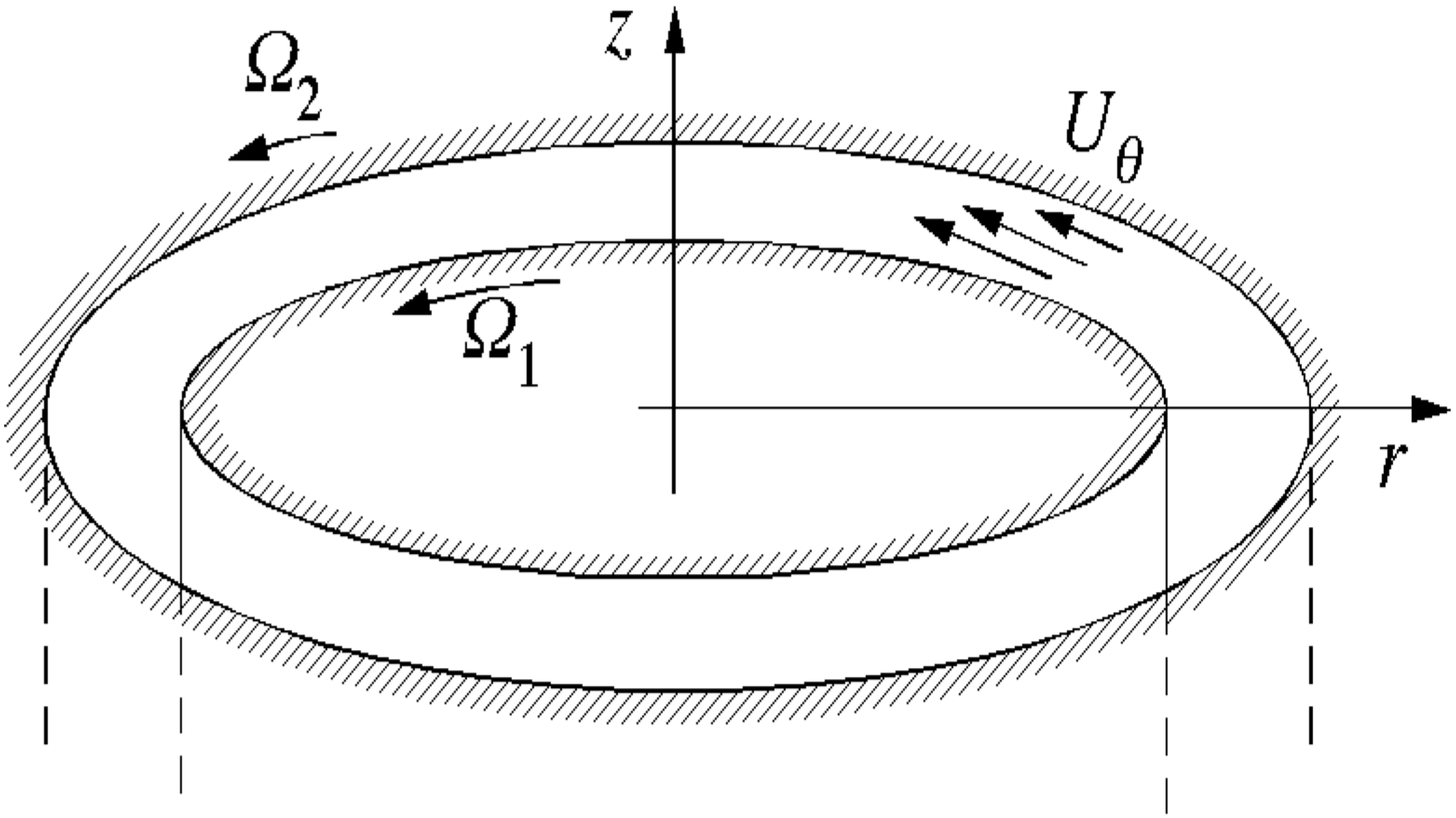
Rayleigh-Benard

Coiling instability

Benard-Marangoni

Kelvin-Helmholtz

Taylor Couette

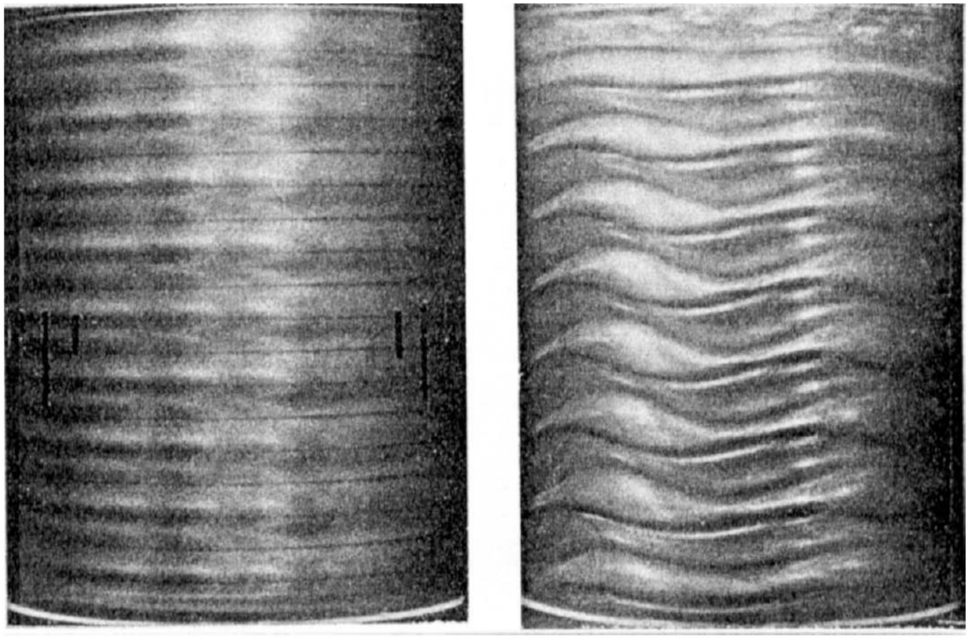


Taylor Couette

Movie by Garcia, Chomaz, Huerre, LadHyX, France

Taylor Couette

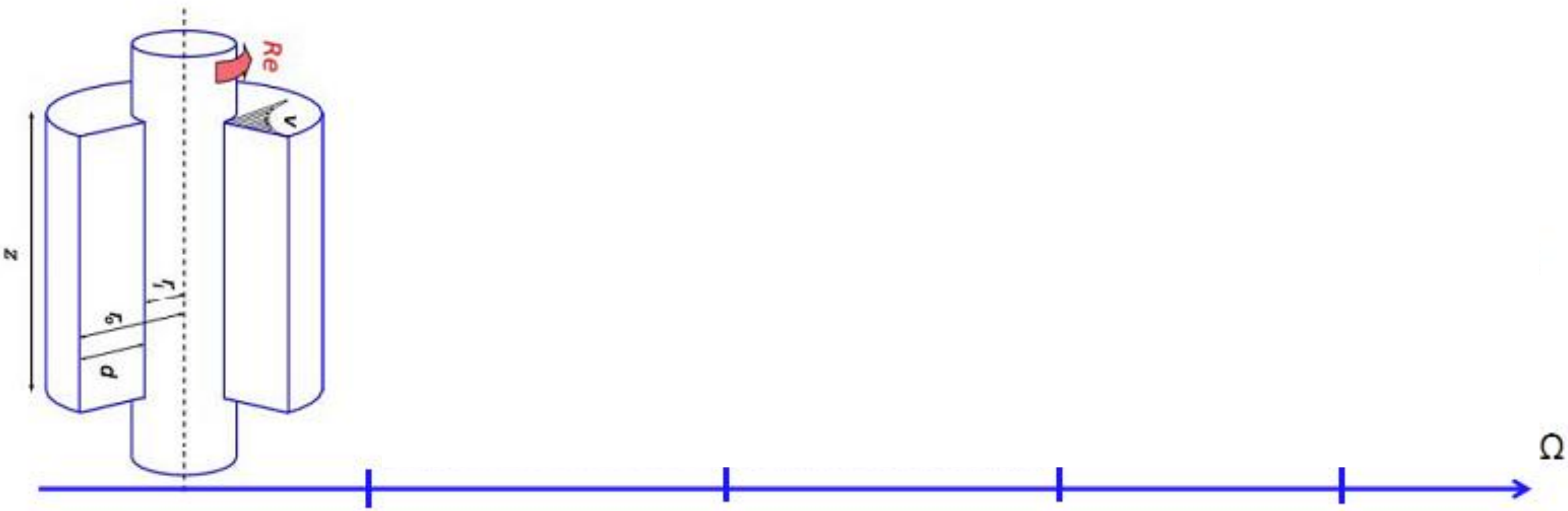
Movie by Garcia, Chomaz, Huerre, LadHyX, France



– *Rouleaux annulaires de Taylor. (a)* $Ta/Ta_c = 1.1$; *(b)* $Ta/Ta_c = 6.0$, *rouleaux ondulants apparus suite à une instablité secondaire* $(\lambda = 2\pi R/4)$. *(Fenstermacher, Swinney & Gollub 1979).*

Taylor Couette

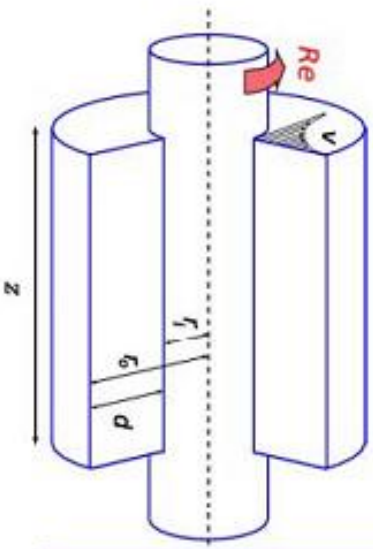
Cascade of instabilities



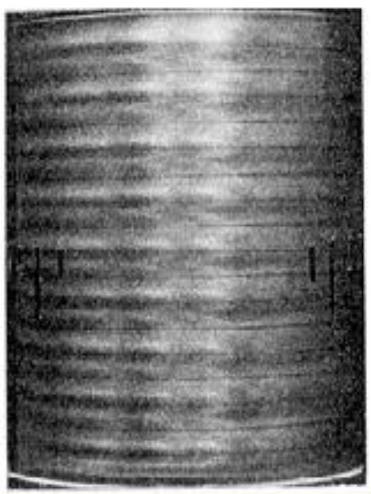
d/dt=0; d/dθ=0; d/dz=0;

Taylor Couette

Cascade of instabilities



Taylor vortices



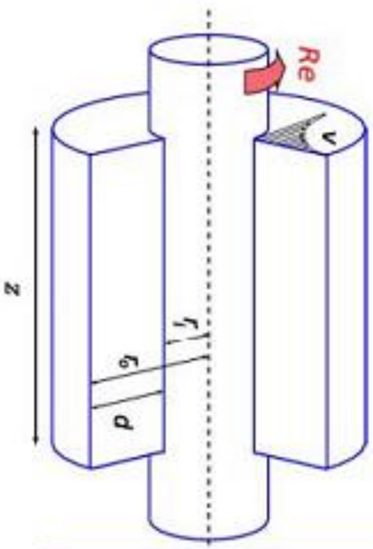
Ω

$d/dt=0$; $d/d\theta=0$; $d/dz=0$;

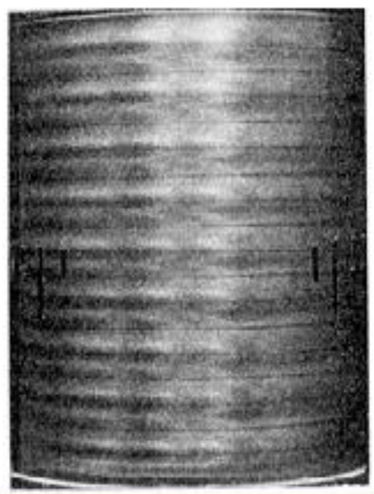
$d/dt=0$; $d/d\theta=0$; $d/d\cancel{z}=0$;

Taylor Couette

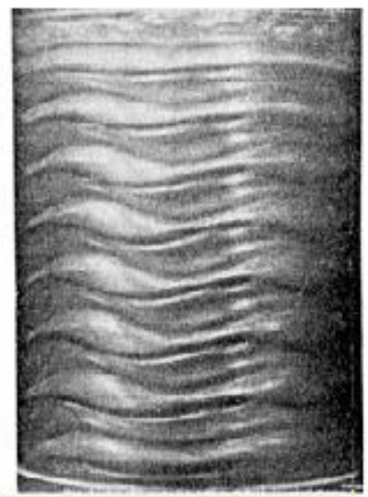
Cascade of instabilities



Taylor vortices



Ondulated vortices



Ω

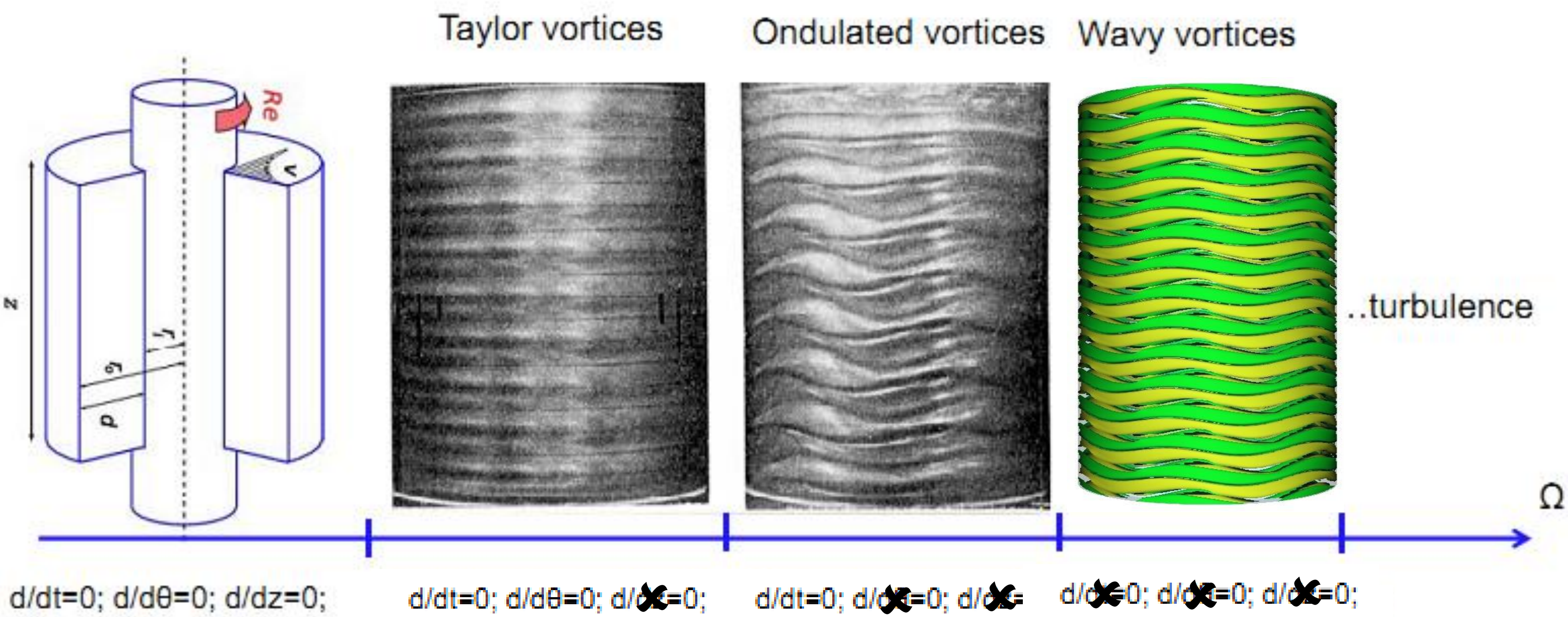
d/dt=0; d/dθ=0; d/dz=0;

d/dt=0; d/dθ=0; d/$\cancel{dz}$=0;

d/dt=0; d/$\cancel{d\theta}$=0; d/$\cancel{dz}$=0;

Taylor Couette

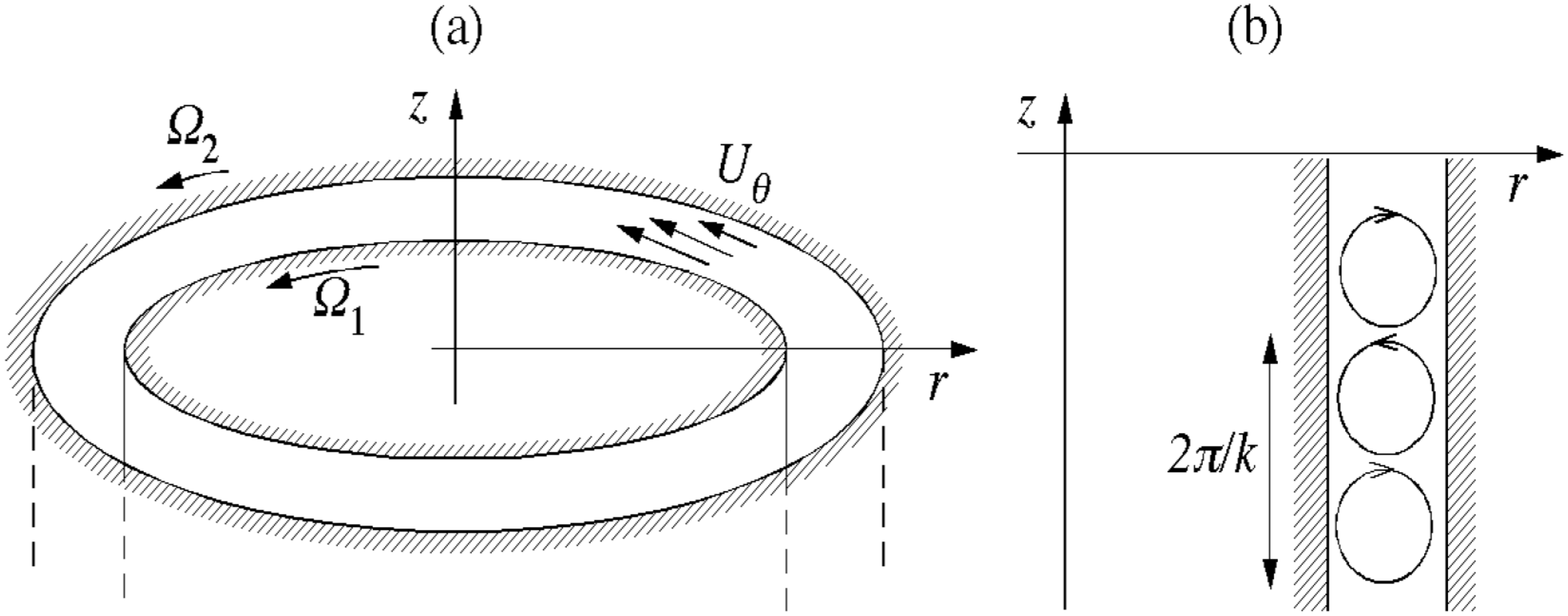
Cascade of instabilities



Instability analysis:

1. **Physical mechanism**
2. **Equations and boundary conditions**
3. **Base state**
4. **Linearized equations**
5. **Normal mode expansion**
6. **Dispersion relation**
7. **Analysis of the dispersion relation**

Taylor Couette



Navier-Stokes

$$\boxed{\begin{aligned}
\rho\left(\frac{Du_r}{Dt}-\frac{u_\theta^2}{r}\right) &= -\frac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{u_r}{r^2}\right)\\
\rho\left(\frac{Du_\theta}{Dt}-\frac{u_r u_\theta}{r}\right) &= -\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2}\right)\\
\rho\frac{Du_z}{Dt} &= -\frac{\partial p}{\partial z}+\mu\nabla^2 u_z
\end{aligned}}$$

$$\frac{D}{Dt}=\left(\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial \theta}+u_z\frac{\partial}{\partial z}\right)$$ convective derivative

$$\nabla^2=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)$$ Laplacian

Navier-Stokes

inertia | pressure stress | viscous stress

$$\rho\left(\frac{Du_r}{Dt}-\frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r^2}\right)$$

$$\rho\left(\frac{Du_\theta}{Dt}-\frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2}\right)$$

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z}+\mu\nabla^2 u_z$$

$$\frac{D}{Dt}=\left(\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)$$ convective derivative

$$\nabla^2=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)$$ Laplacian

Rayleigh criterion

Rayleigh criterion: A necessary and sufficient condition for stability of the $\{O, U_\theta(r), 0\}$ basic flow ($r \in]R_1, R_2[$) to inviscid axisymmetric perturbations, is that $\partial \left(rU_\theta\right)^2 / \partial r > 0$ everywhere in $[R_1, R_2]$.

Inviscid flow

steady flow

axisymmetric flow

parallel flow

unidirectional flow

Rayleigh criterion

Rayleigh criterion: A necessary and sufficient condition for stability of the $\{O, U_\theta(r), 0\}$ basic flow ($r \in]R_1, R_2[$) to inviscid axisymmetric perturbations, is that $\partial \left(rU_\theta\right)^2 / \partial r > 0$ everywhere in $[R_1, R_2]$.

Inviscid flow	$\mu = 0$
steady flow	$\partial / \partial t = 0$
axisymmetric flow	$\partial / \partial \theta = 0$
parallel flow	$\partial / \partial z = 0$
unidirectional flow	$\{0, U_\theta(r), 0\}$

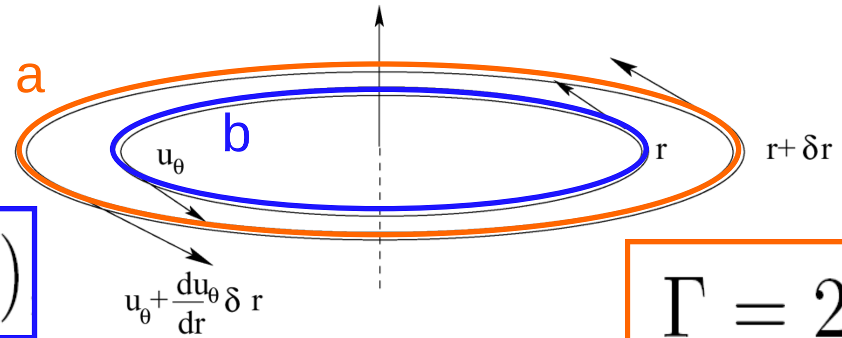
$$\boxed{\rho U_\theta^2 / r = \partial P / \partial r}$$

Aspirateurs à force centrifuge, Dyson

Mechanism for Rayleigh Criterion
Let us move an annulus of flow at r_b to another location $r_a > r_b$

Velocity profile before moving annulus

$$\boxed{\Gamma = 2\pi r_b U_\theta(r_b)}$$



$$\boxed{\Gamma = 2\pi r_a u_\theta(r_a)}$$

Velocity after moving annulus

Kelvin's theorem:

$$\boxed{u_\theta(r_a) = (r_b/r_a)U_\theta(r_b)}$$

Mechanism for Rayleigh Criterion
Let us move an annulus of flow at r_b to another location r_a

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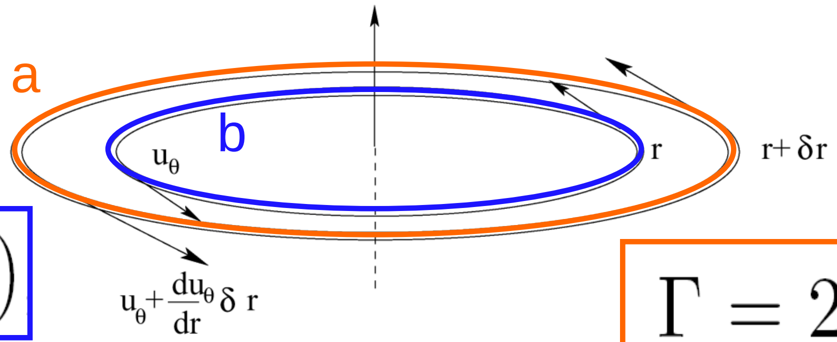


$$\boxed{\Gamma = 2\pi r_a u_\theta(r_a)}$$

Velocity after moving annulus

Kelvin's theorem:

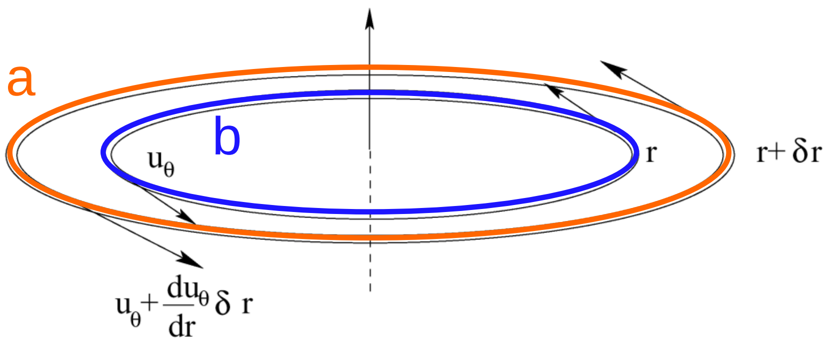
$$\boxed{u_\theta(r_a) = (r_b/r_a)U_\theta(r_b)}$$

Centrifugal force/ Pressure balance

$$\boxed{\rho U_\theta^2/r = \partial P/\partial r}$$

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



$$\text{if} \quad \rho \overbrace{u_\theta^2(r_a)}^{\text{new velocity}}/r_a > (dP/dr)(r_a) = \rho U_\theta^2(r_a)/r_a$$

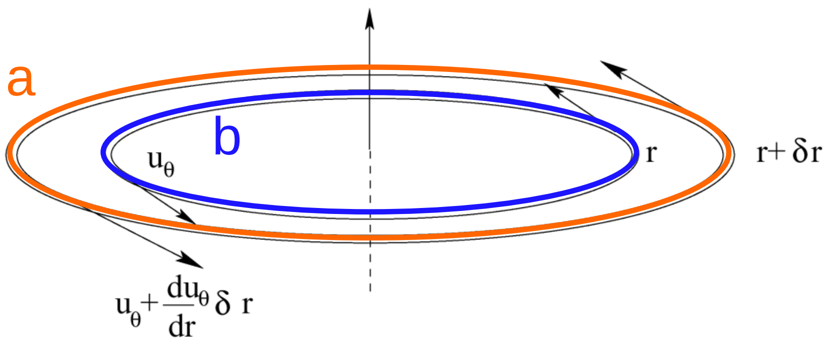
centrifugal force > pressure gradient

⇨ The annulus further escapes towards high r

⇨ UNSTABLE

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



$$\text{if} \quad \underbrace{\rho u_\theta^2}_{\text{new velocity}}(r_a)/r_a < (dP/dr)(r_a) = \rho U_\theta^2(r_a)/r_a$$

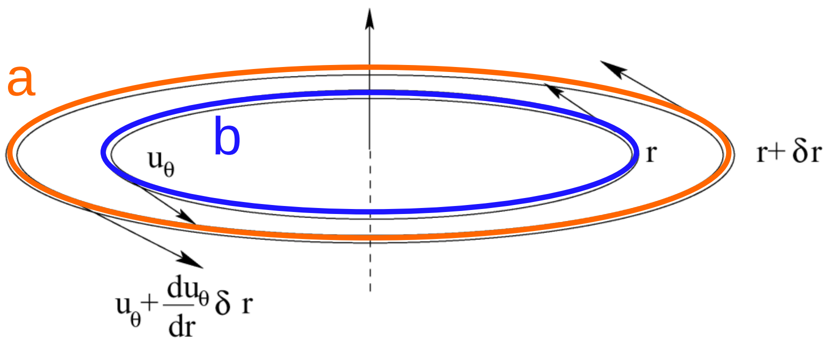
centrifugal force < pressure gradient

⇨ The annulus is brought back to its initial position

⇨ STABLE

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



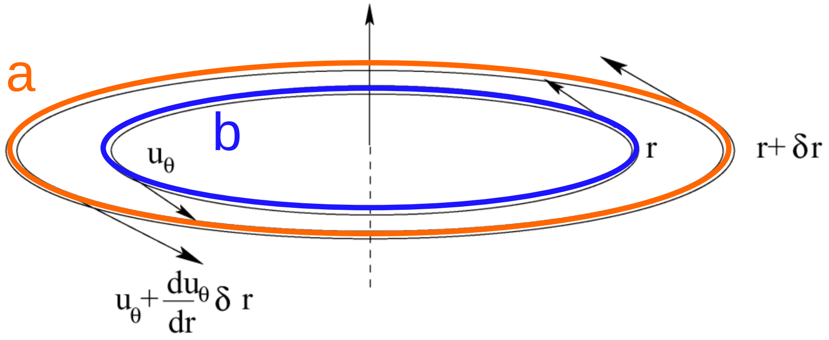
Condition for stability

$$\boxed{U_\theta^2(r_b)r_b^2 < U_\theta^2(r_a)r_a^2}$$

$$\boxed{d\left(rU_\theta\right)^2/dr > 0}$$

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



Condition for stability

$$\boxed{U_\theta^2(r_b)r_b^2 < U_\theta^2(r_a)r_a^2}$$

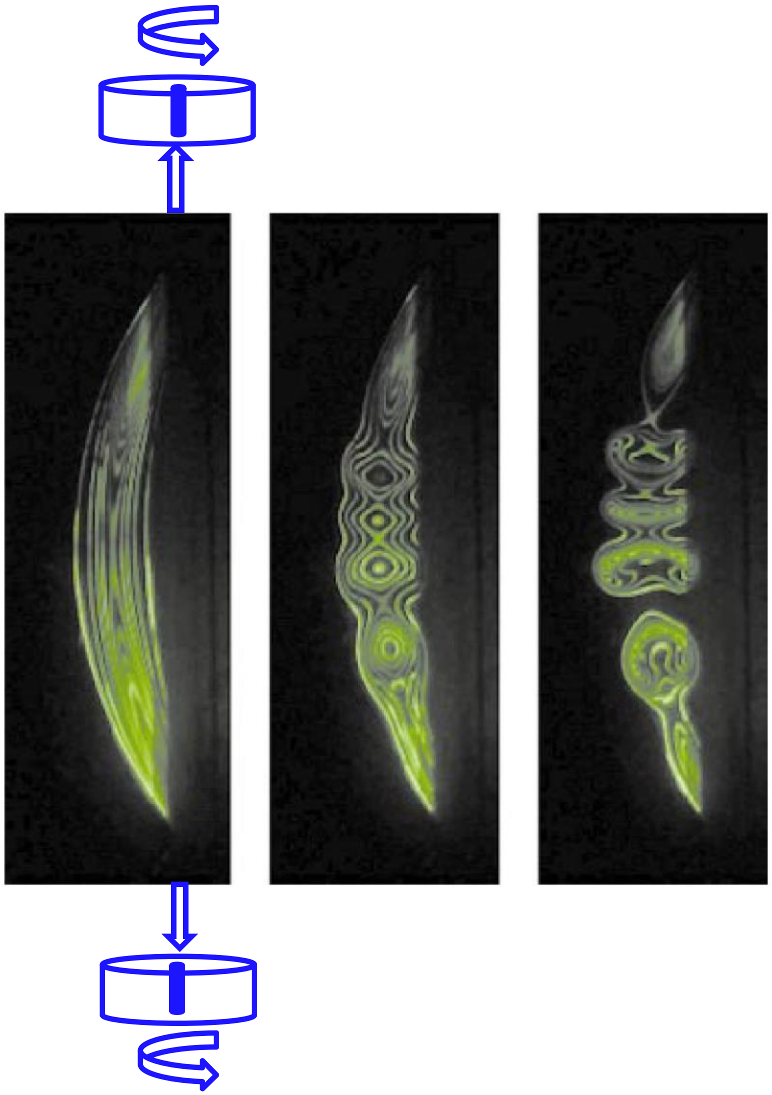
$$\boxed{d\left(rU_\theta\right)^2/dr > 0}$$

$\boxed{U_\theta \varsigma > 0}$

axial vorticity

$\varsigma = 1/r\, d(ru_\theta)/dr$

Centrifugal instability of a vortex



Bottausci & Petitjean 2002

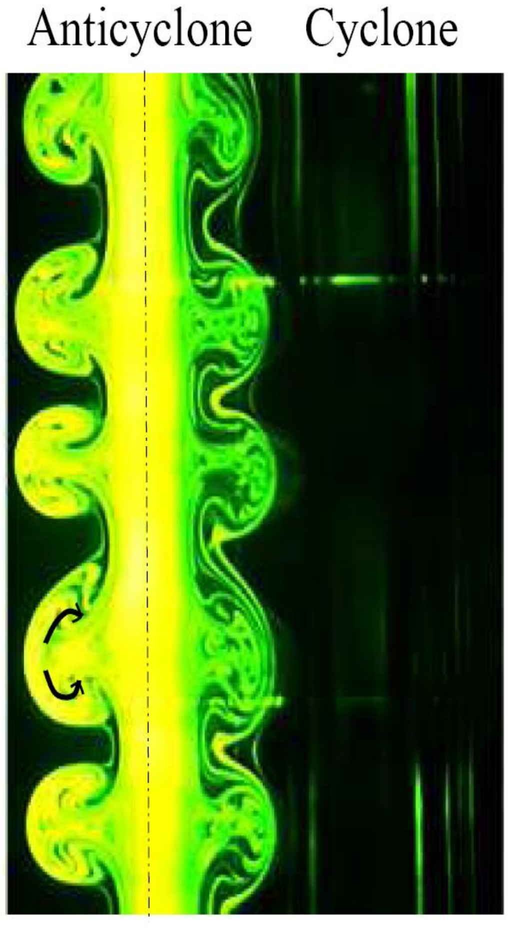
Anticyclone/Cyclone

Condition for instability in presence of background rotation Ω

$$\boxed{(\Omega + u_\theta/r)\ (2\Omega + \zeta)} > 0$$

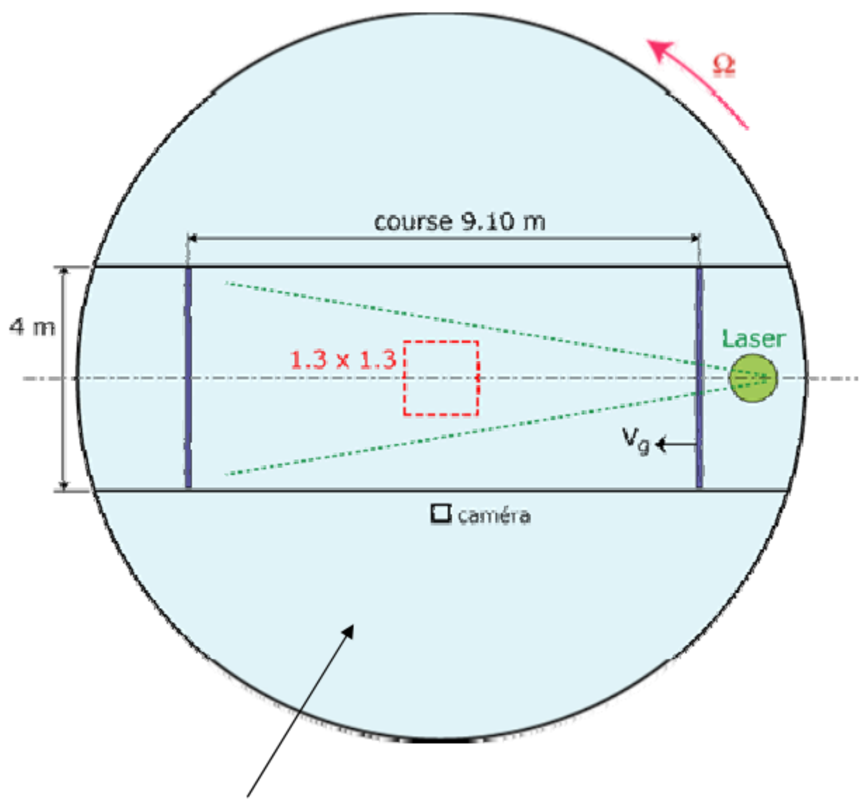
axial vorticity
$\varsigma = 1/r\ d(ru_\theta)/dr$

Anticyclone/Cyclone



Fontane et al. (2002)

Experimental setup: 'Coriolis' Rotating Plateform (LEGI, Grenoble)



150 tons of water

9 m x 4 m x 1 m channel

Grid (of square mesh M = 15 cm),
translated at $V_g = 0.3$ m s^{-1}

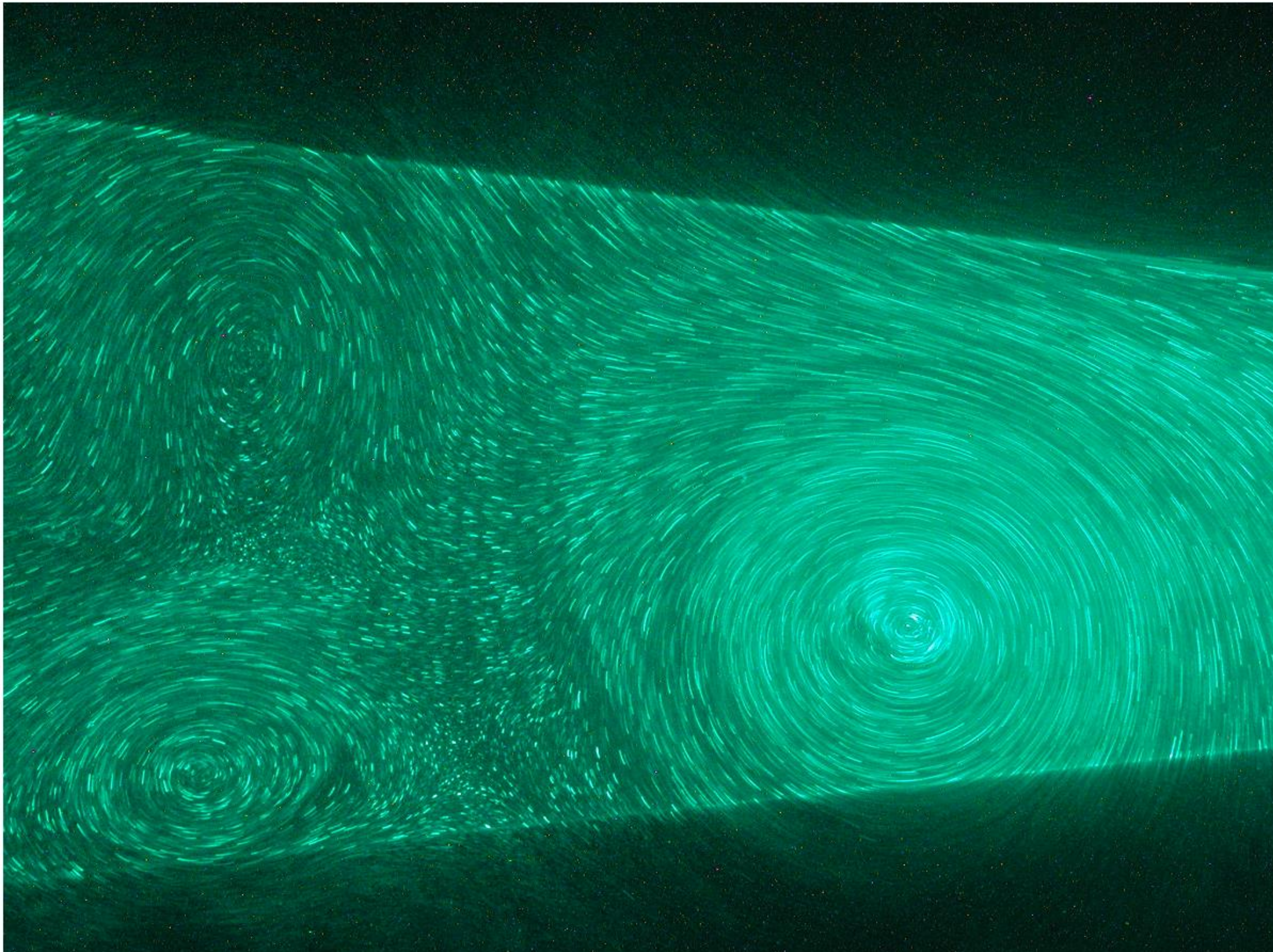
mounted on the **13 m** diameter
'Coriolis' rotating plateform

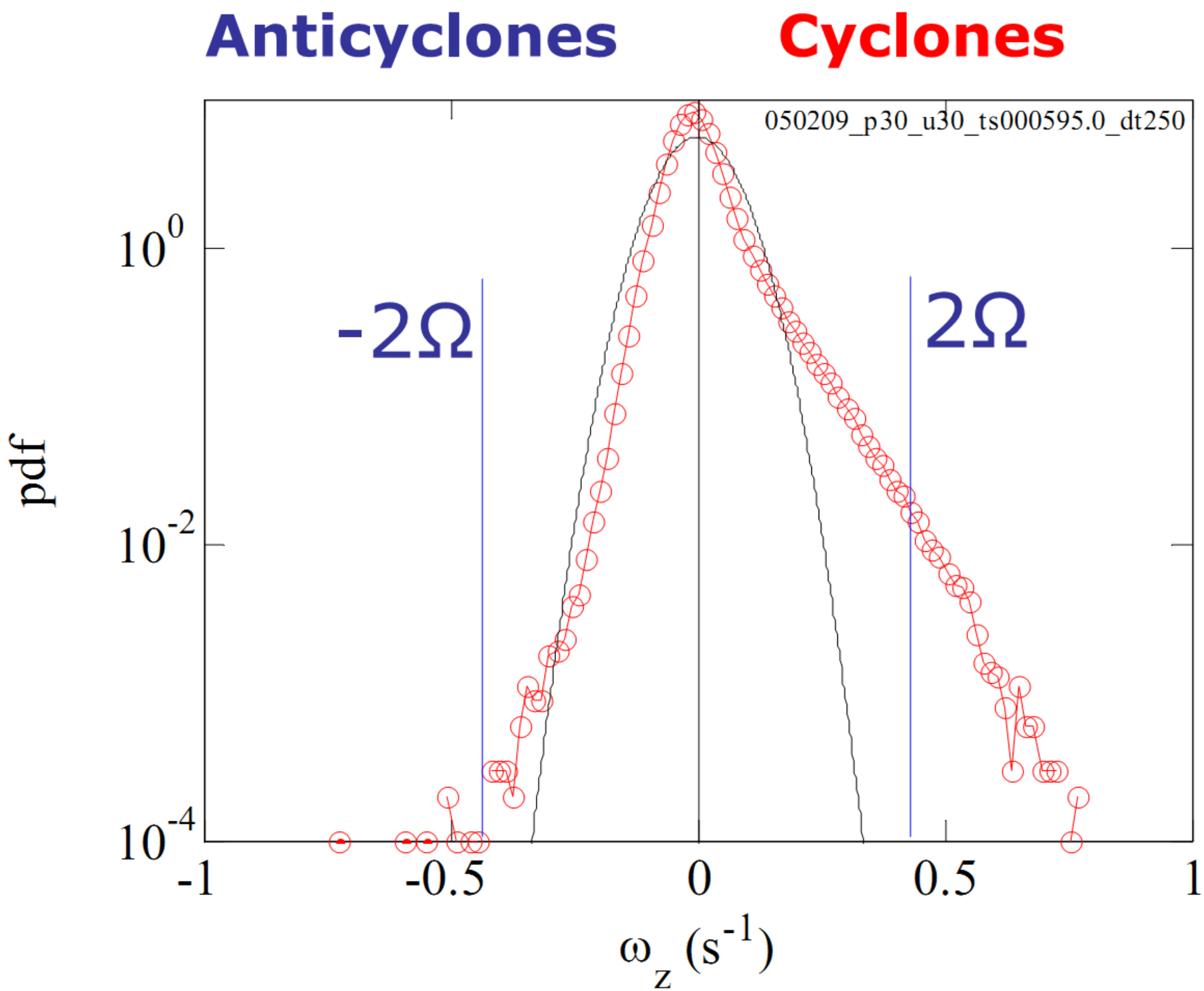
Rotation periods: T = 30, 60, 120 s
1 decay ~ 1 hour ~ 10^4 M/V_g

PIV measurements in horizontal
and vertical planes
2000x2000 HR camera

Experimental setup: ‘Coriolis’ Rotating Plateform (LEGI, Grenoble)

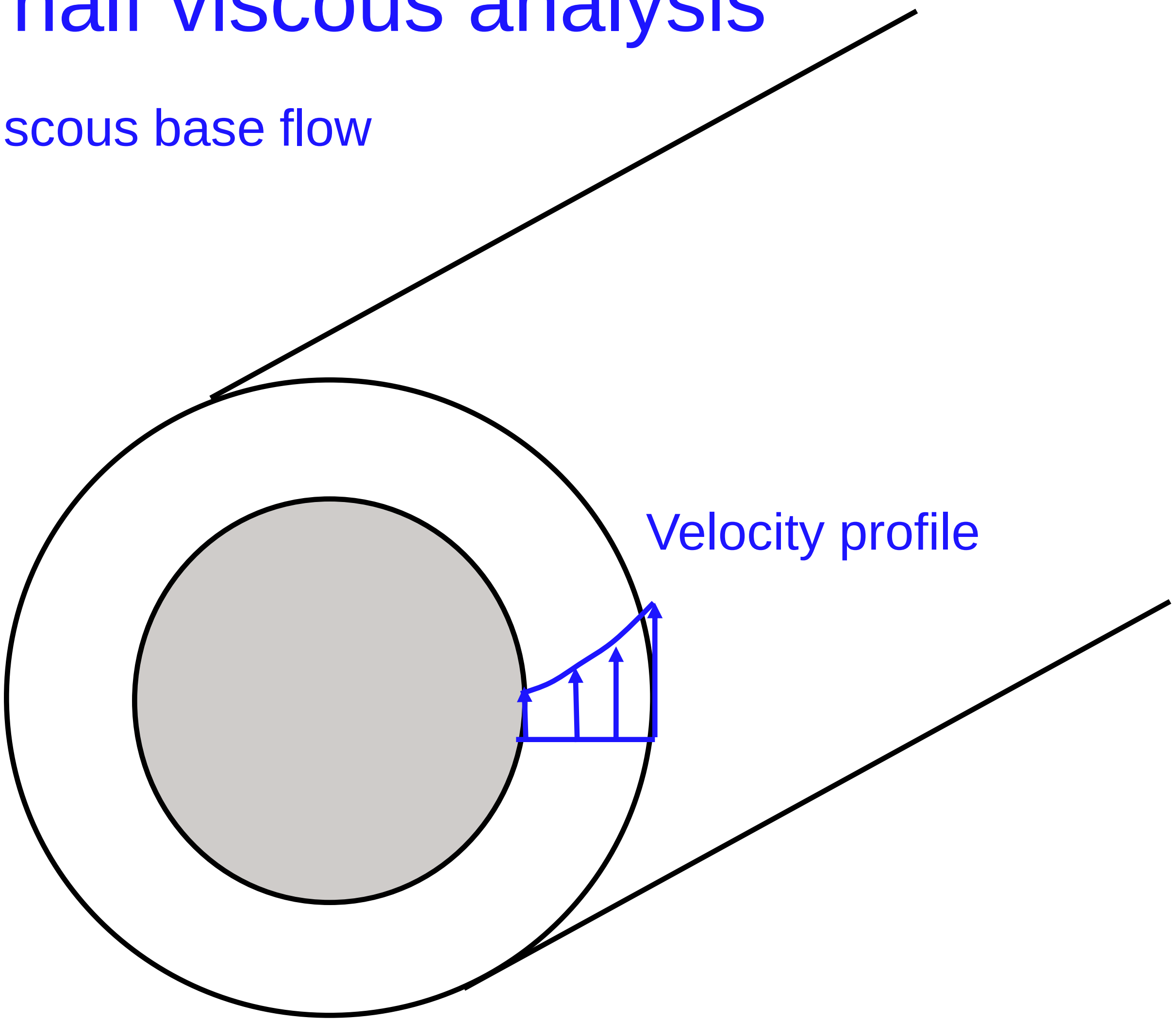
H. Didelle, S. Viboud




Anticyclones
Cyclones
050209_p30_u30_ts000595.0_dt250
-2Ω
2Ω
pdf
10^0
10^{-2}
10^{-4}
-1
-0.5
0
0.5
1
ω_z (s^{-1})

The half viscous analysis

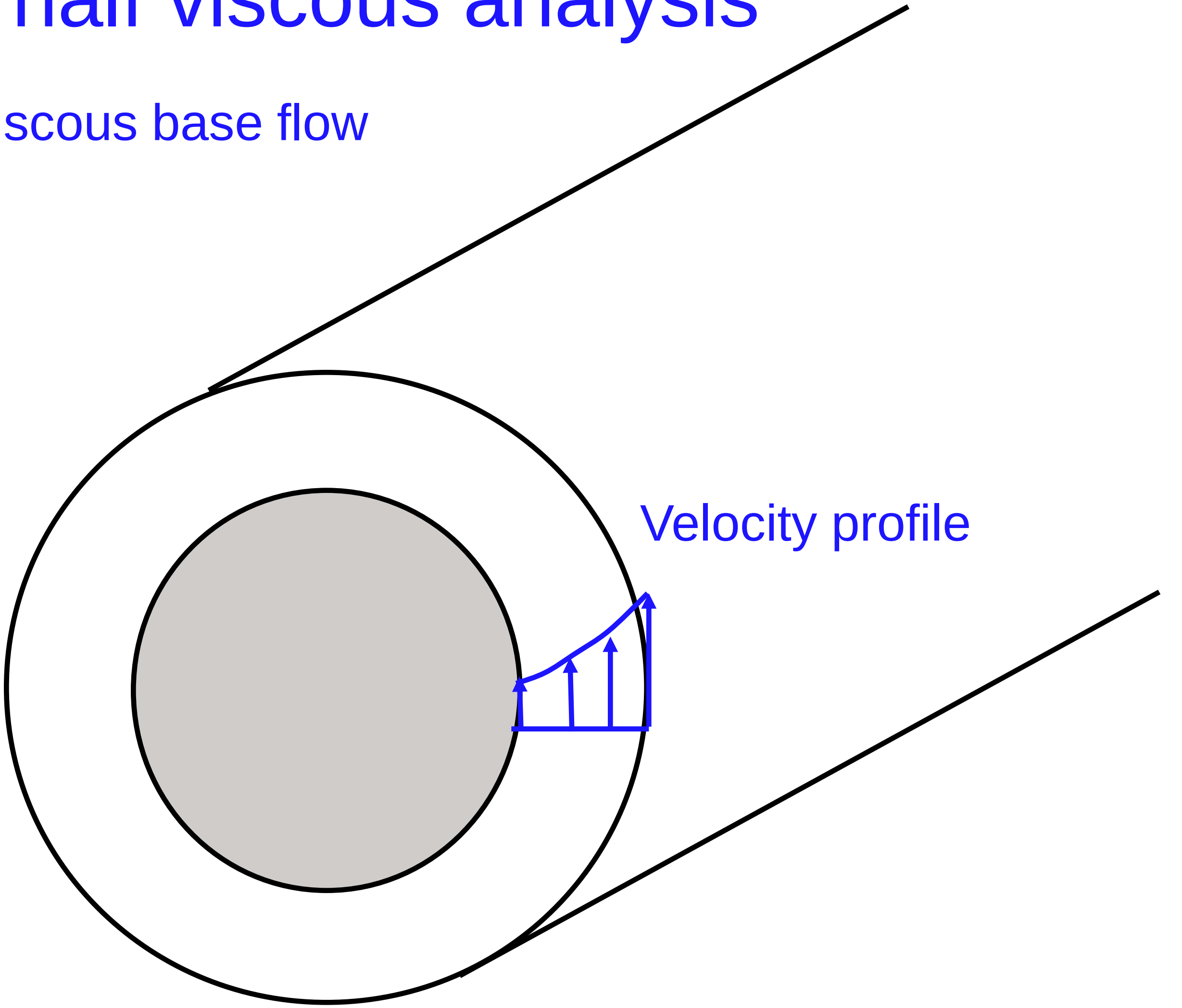
Inviscid analysis of a viscous base flow



The base flow velocity profile is only selected by viscosity!

The half viscous analysis

Inviscid analysis of a viscous base flow



if stable, viscosity is expected to be further stabilizing

if unstable, we expect a critical reynolds number

Base flow

$$\boxed{\begin{aligned} \rho\frac{U_\theta^2}{r} &= \frac{\partial P}{\partial r} \\ \frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} &= 0 \end{aligned}}$$

Boundary conditions

$$\boxed{U_\theta(R_1) = R_1\Omega_1}$$

$$\boxed{U_\theta(R_2) = R_2\Omega_2}$$

The half viscous analysis

Inviscid analysis of a viscous base flow?

viscous time?

convective time?

The half viscous analysis

Inviscid analysis of a viscous base flow

$$\boxed{\tau_\nu = (R_2 - R_1)^2/\nu}$$

viscous time

$$\boxed{\tau_{U_2} = (R_2 - R_1)/(\Omega_2 R_2)}$$

convective time

$$\boxed{\tau_\nu/\tau_{U_2} = (R_2 - R_1)\Omega_2 R_2/\nu = \mathrm{Re}_2}$$

Reynolds number

General solution

$$\boxed{U_\theta(r) = Ar + B/r}$$

Boundary conditions

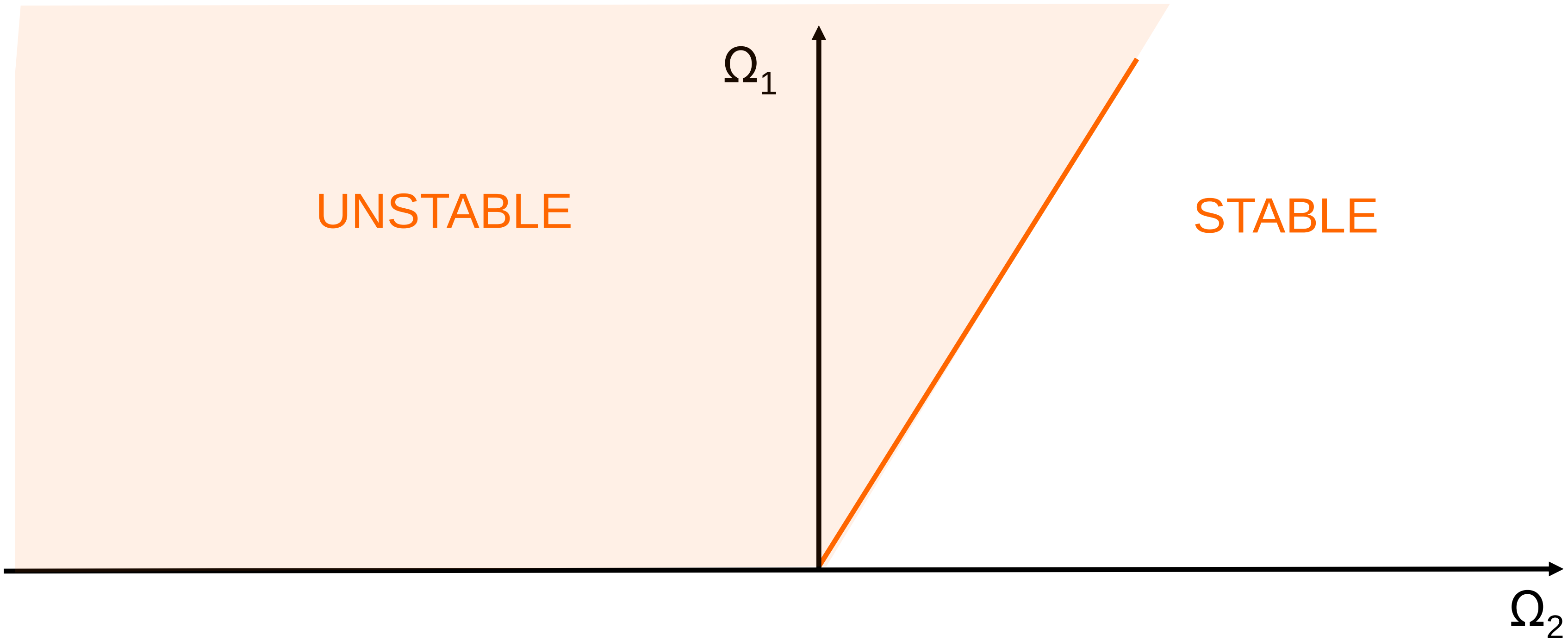
$$\boxed{A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad ; \quad B = \frac{(\Omega_1 - \Omega_2)\, R_1^2 R_2^2}{R_2^2 - R_1^2}}$$

Rayleigh criterion

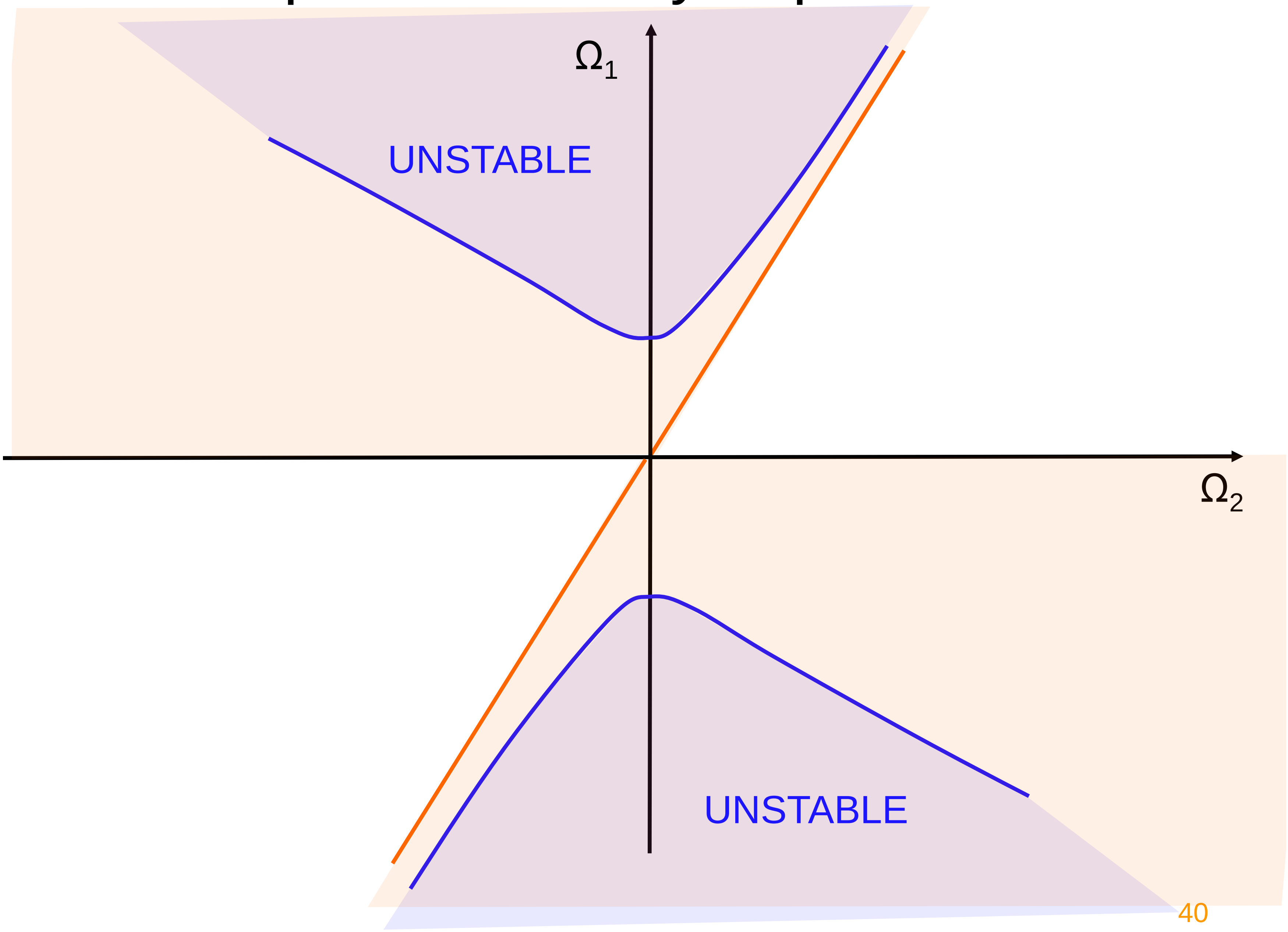
$$d(rU_\theta)^2/dr = 4Ar(Ar^2 + B) < 0$$

Unstable if $\boxed{\Omega_2 r_2^2 < \Omega_1 r_1^2.}$

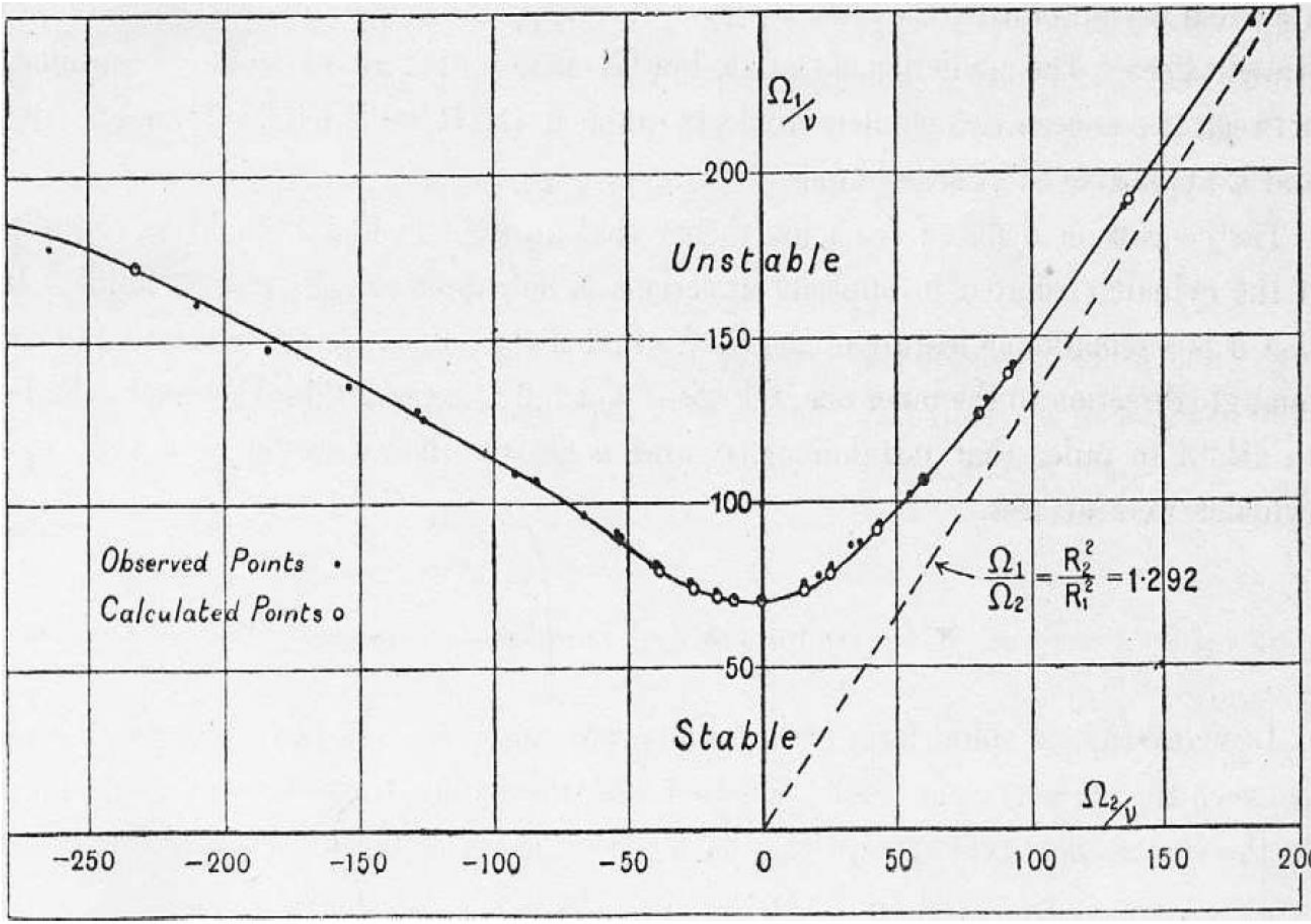
Comparison Theory/Experiment



Comparison Theory/Experiment

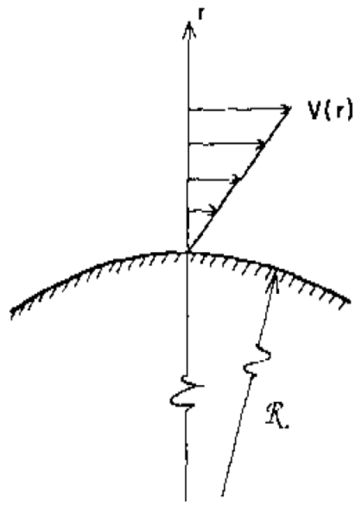


Comparison Theory/Experiment

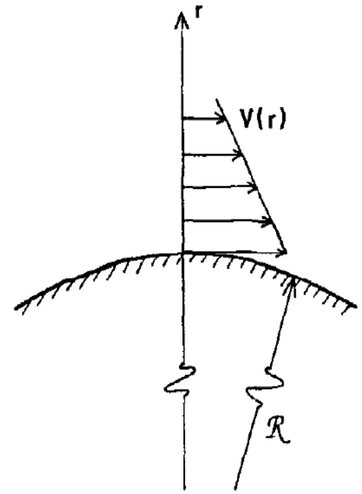


Comparison between observed and calculated speeds at which instability first appears in the case when $R_1 = 3.55$ cm, $R_2 = 4.035$ cm (from Taylor, 1923).

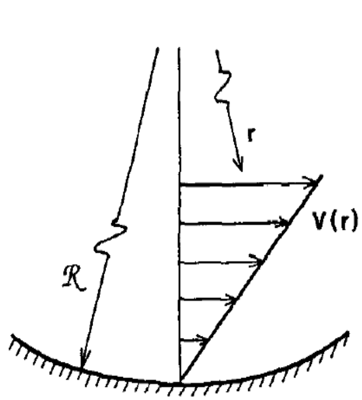
Wall bounded flows



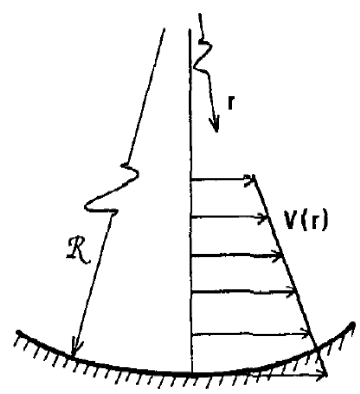
(a) stable



(b) unstable

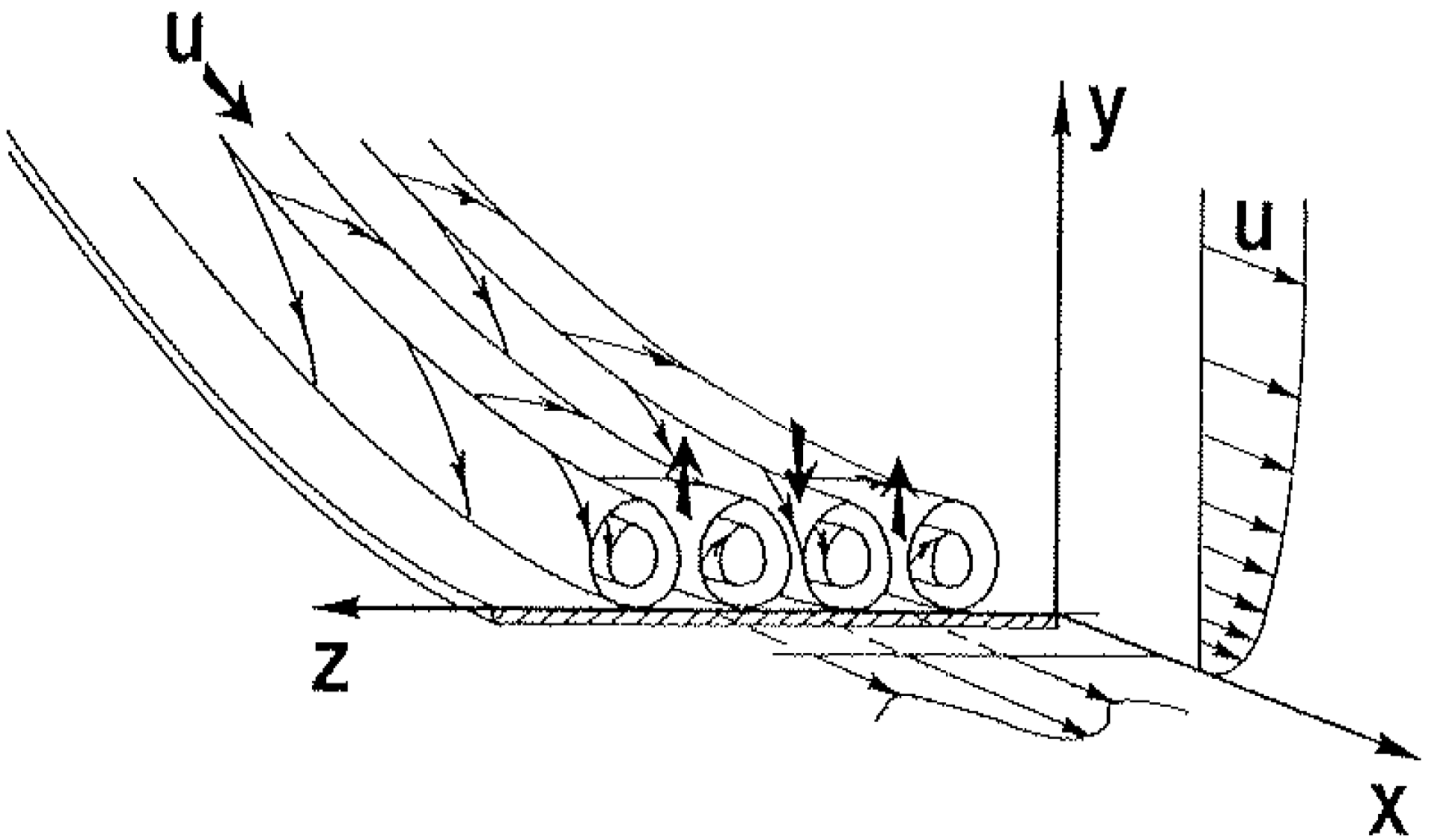


(c) unstable

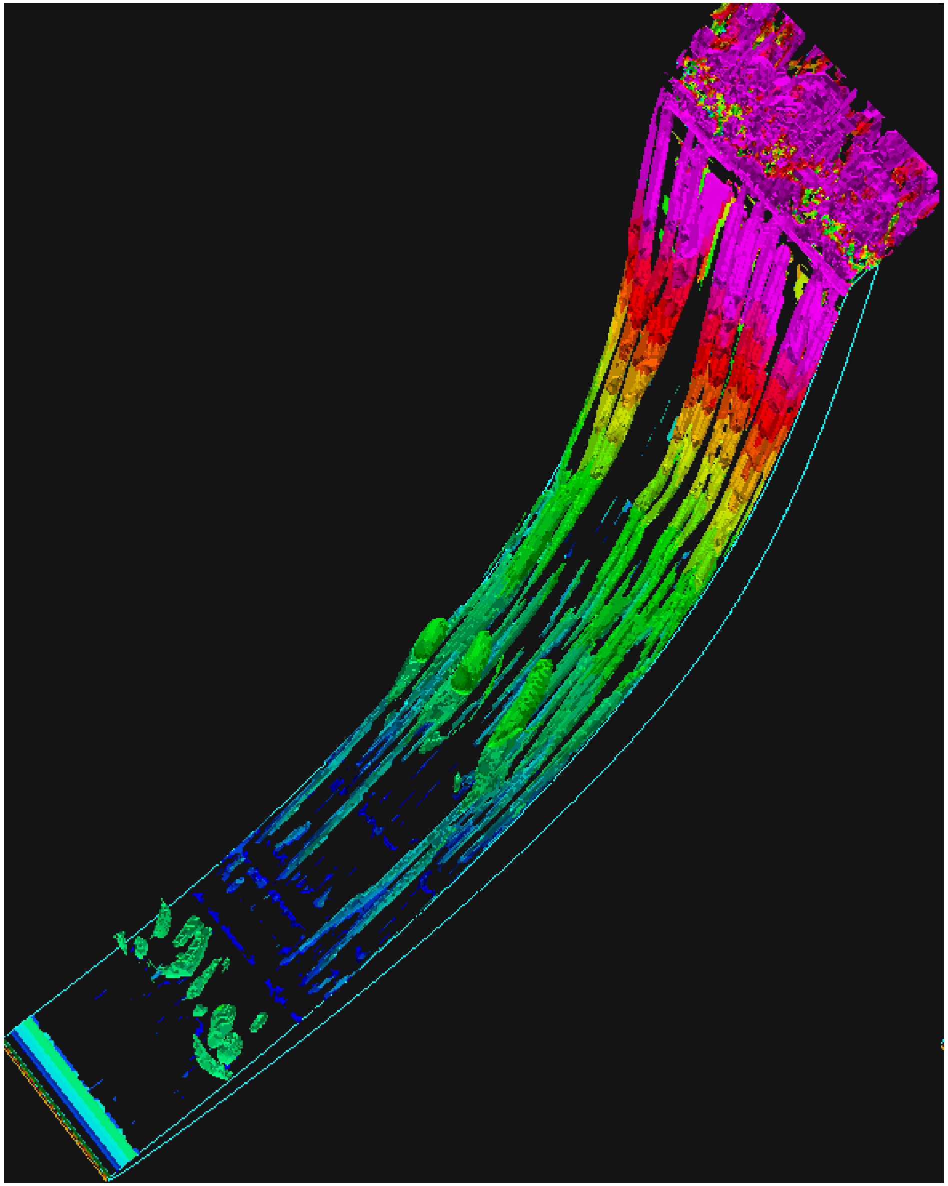


(d) stable

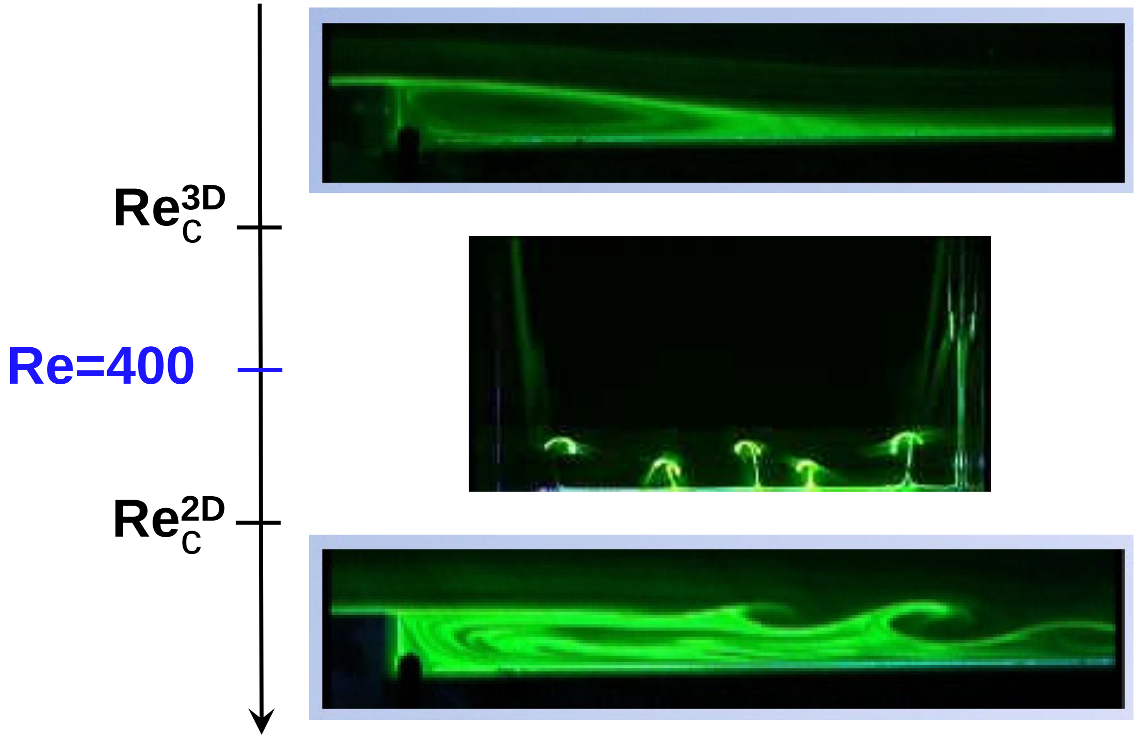
Gortler instability



Gortler instability



Backward facing step

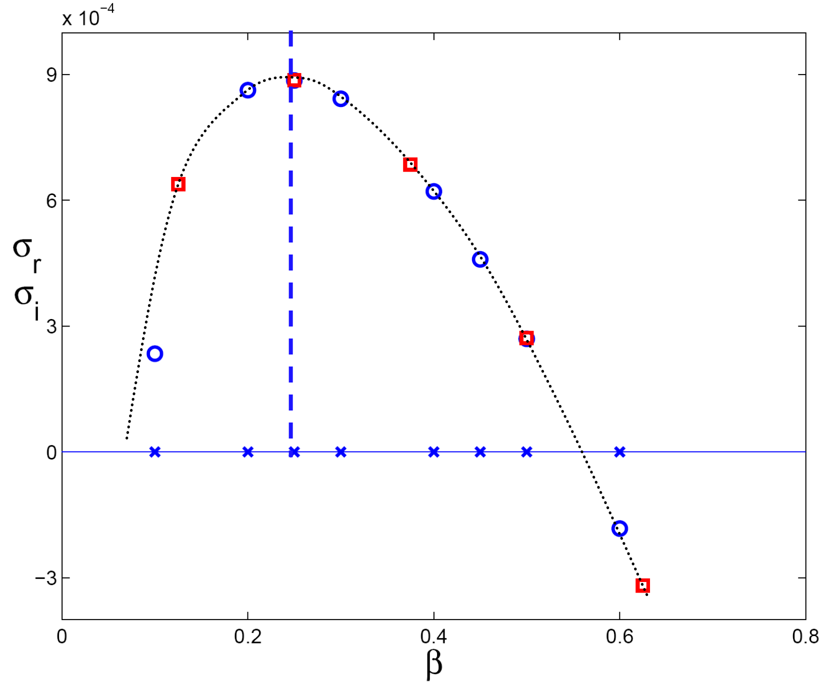


Beaudoin et al.
(Eur. J. Mech. 2004).

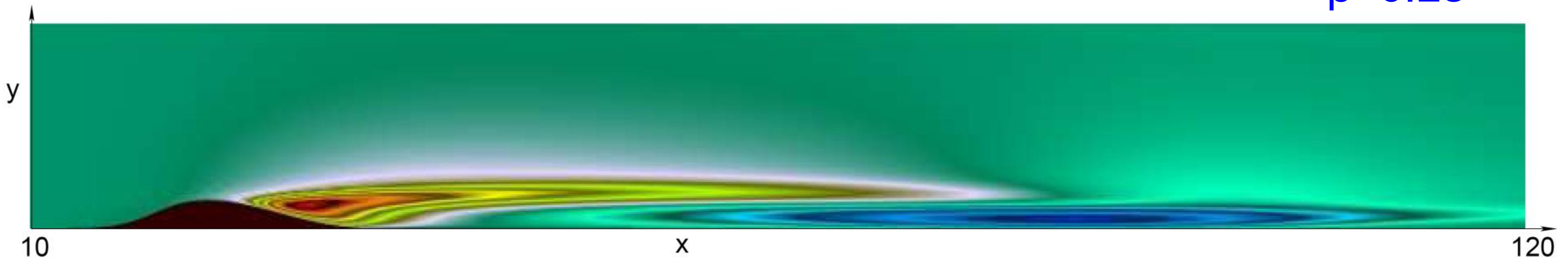
DNS by Kaiktsis et al. (JFM 93,96) and linear analysis by Barkley et al. (JFM 2002),

Global dispersion relation

A single stationnary unstable global mode



β=0.25

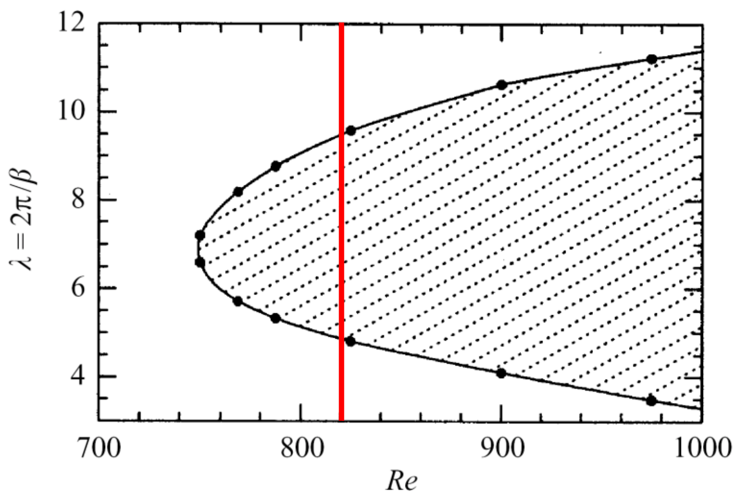


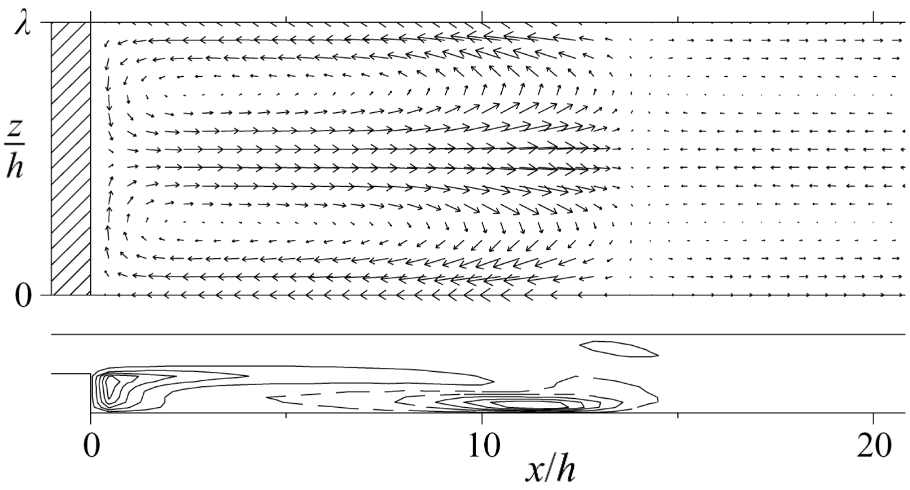
Confirms Barkley et al. (JFM 2002) on backward facing step

Global dispersion relation

A single stationnary unstable global mode

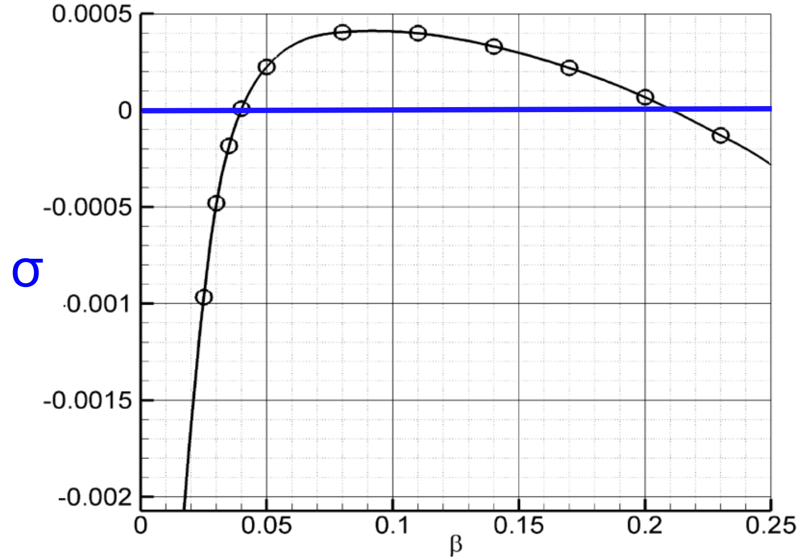
Confirms Barkley et al. (JFM 2002) on backward facing step

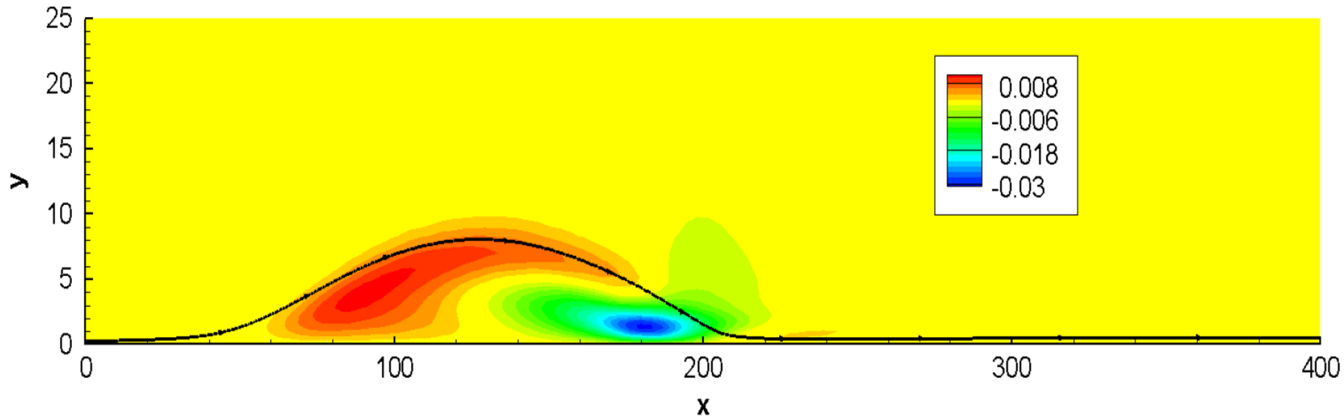




Global dispersion relation
A single stationnary unstable global mode

pressure induced separation (Alizard & Robinet 2007)

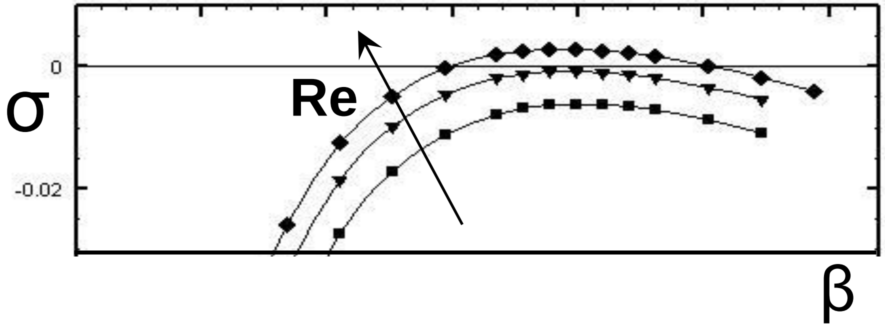


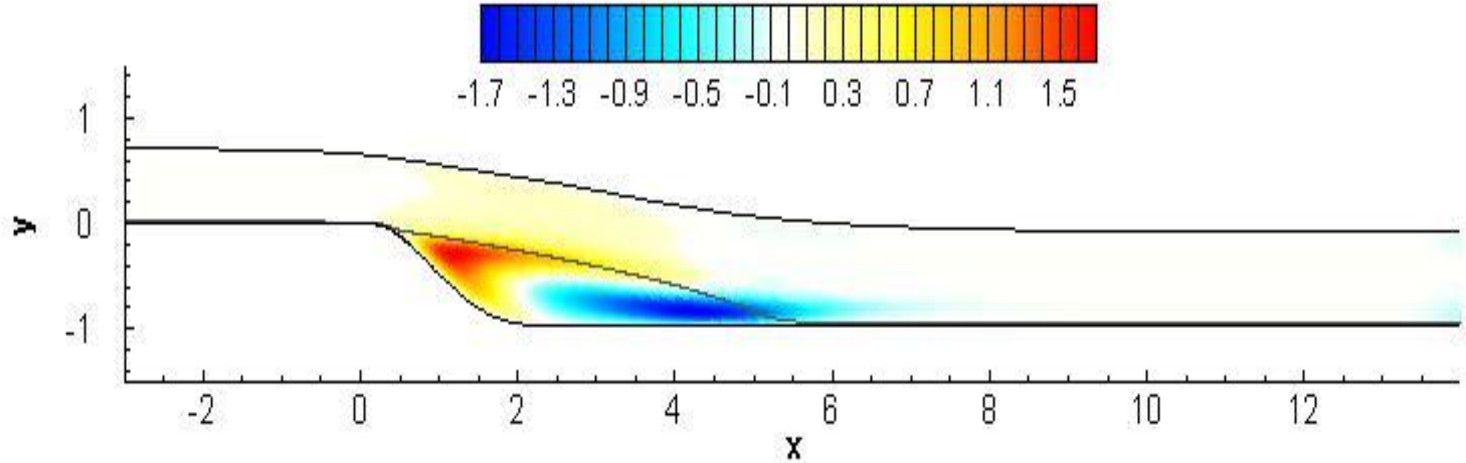


β=0.1

Global dispersion relation
A single stationnary unstable global mode

rounded facing step (Marquet, Chomaz, Sipp & Jacquin 2007)





β=2