Most flows are unstable...

Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

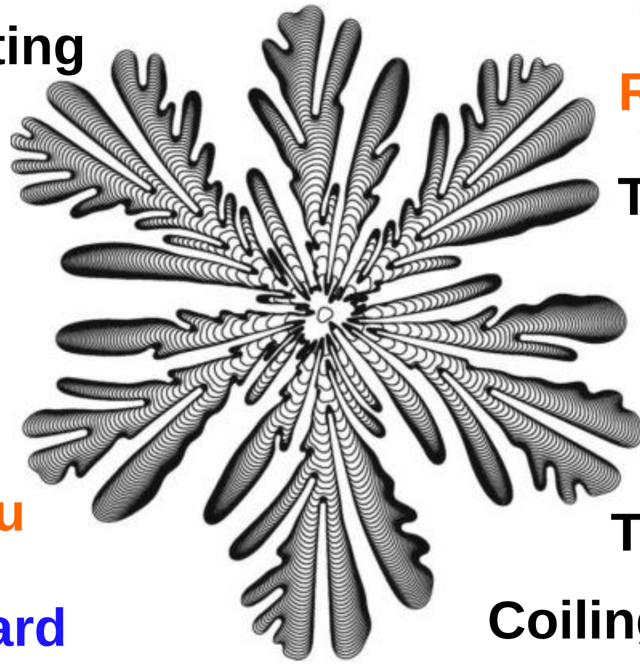
Tearing instability

Rayleigh-Benard

Coiling instability

Benard-Marangoni

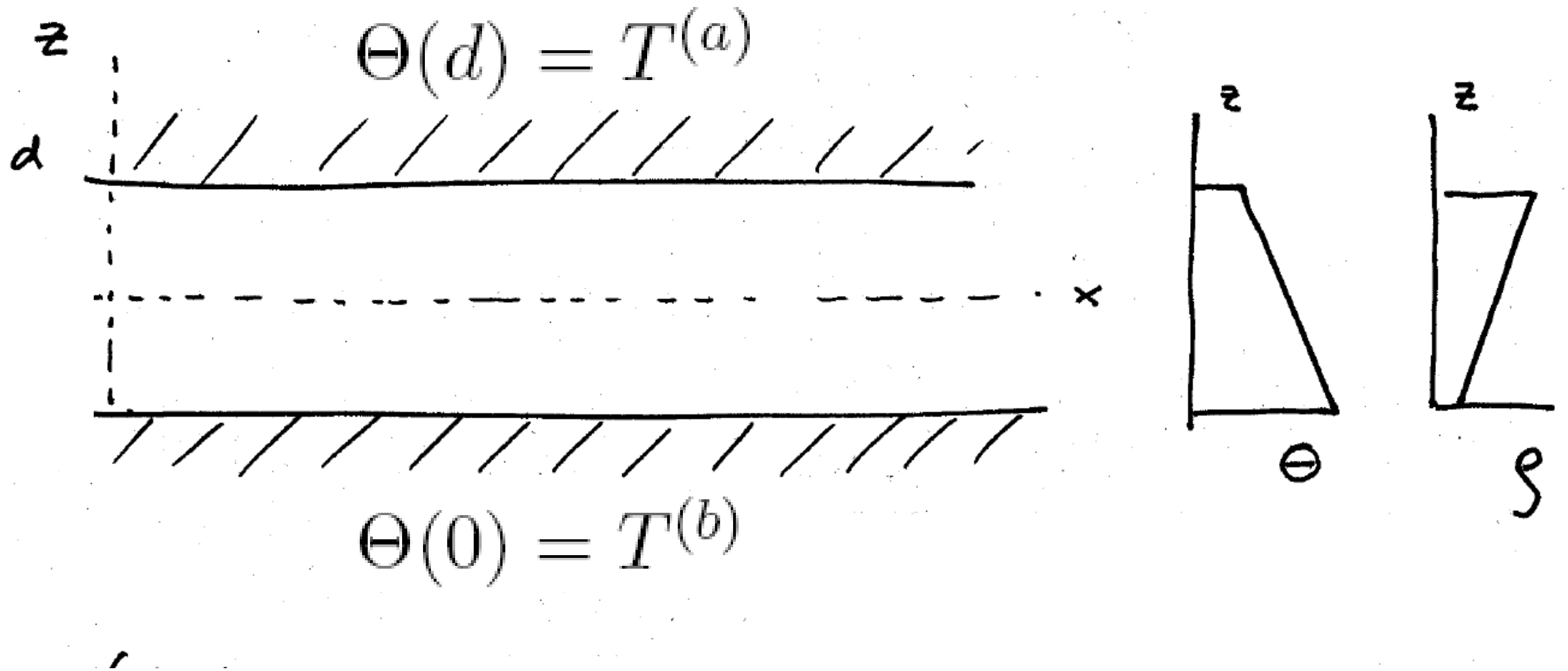
Kelvin-Helmholtz



Instability analysis:

1. **Physical mechanism**
2. **Equations and boundary conditions**
3. **Base state**
4. **Linearized equations**
5. **Normal mode expansion**
6. **Dispersion relation**
7. **Analysis of the dispersion relation**

Rayleigh Benard



Rayleigh Benard

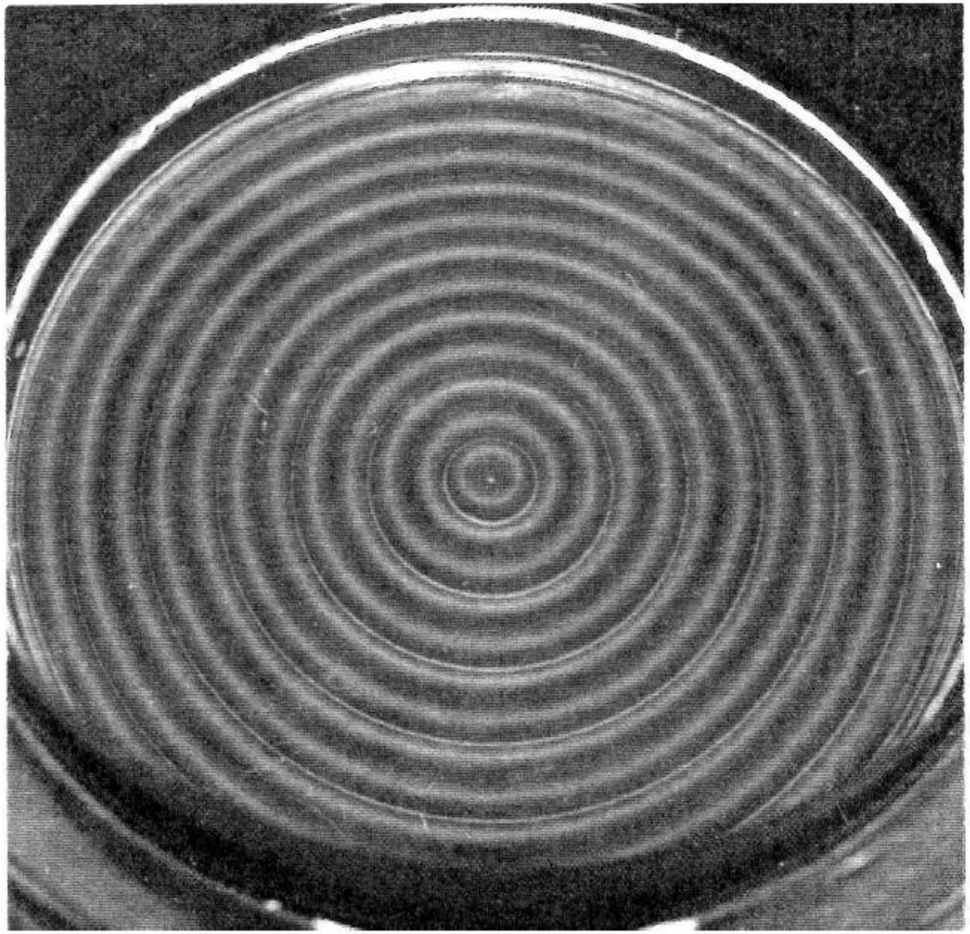
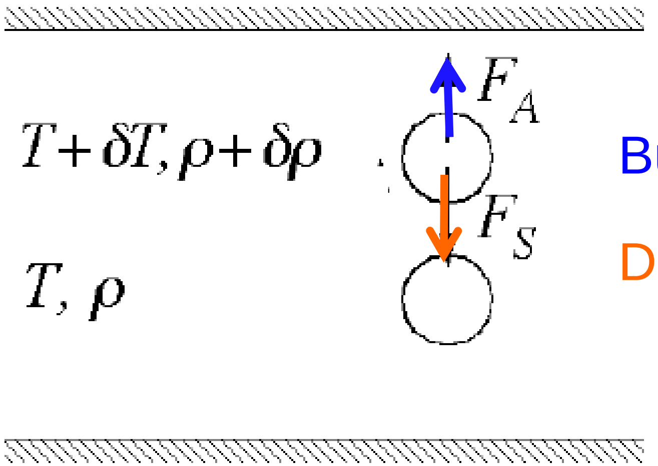


FIG. 2.13 – *Vue de dessus de rouleaux de convection thermique dans une huile silicone, visualisés par de la poudre d'aluminium à travers une paroi supérieure en Plexiglas. La frontière circulaire induit des rouleaux circulaires.* $Ra = 2,9\, Ra_c$. *Cliché Koschmieder (1974) d'après (Van Dyke 1982).*

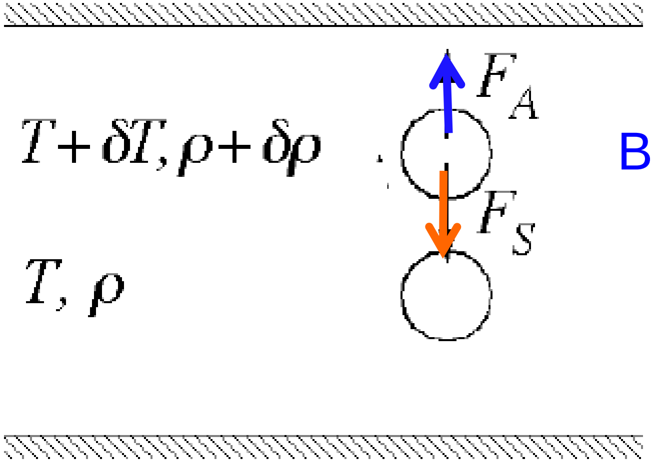
Rayleigh Benard



Buoyancy (Archimedes) force

Drag force

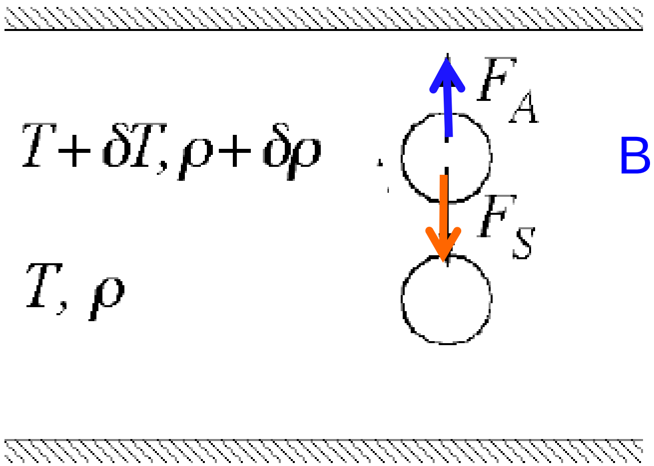
Rayleigh Benard



$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

Rayleigh Benard

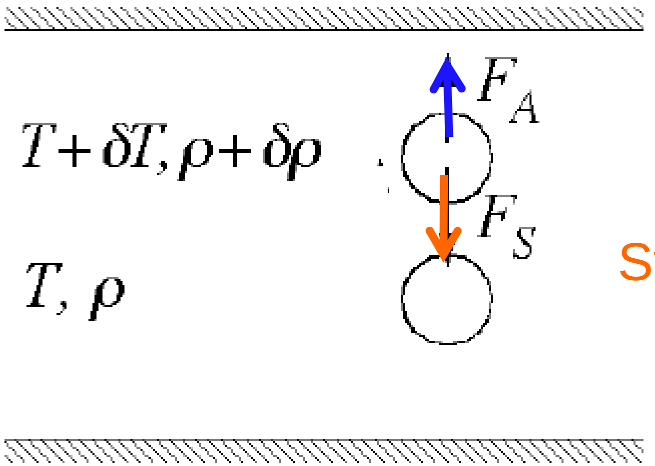


$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

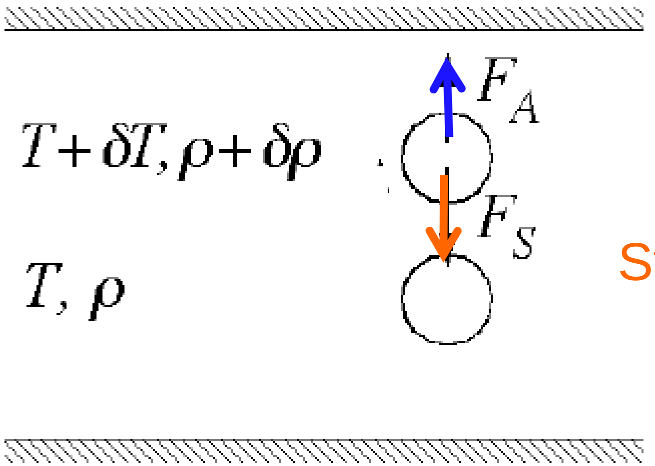
$\delta\rho=\alpha_p\delta T$

Rayleigh Benard



Stokes Drag

Rayleigh Benard



Stokes Drag

$$\boxed{F_S = -6\pi\mu a v}$$

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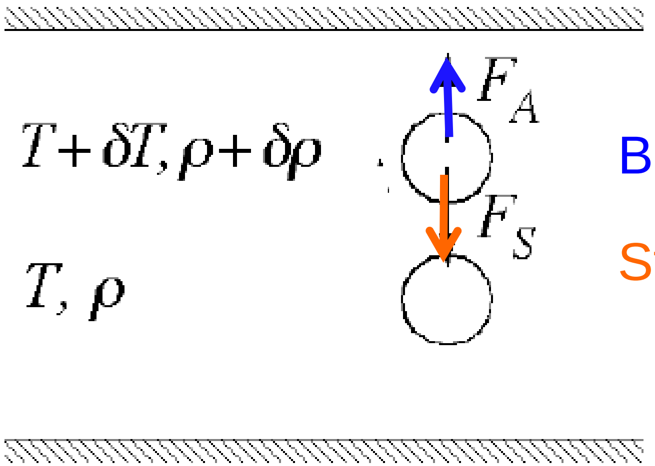
$\delta\rho = \alpha_p \delta T$



$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

Stokes Drag

$$\boxed{F_S = -6\pi\mu a v}$$

$$\boxed{v = \frac{2}{9}\,\frac{\alpha_p g a^2\, \delta T}{\mu}}$$

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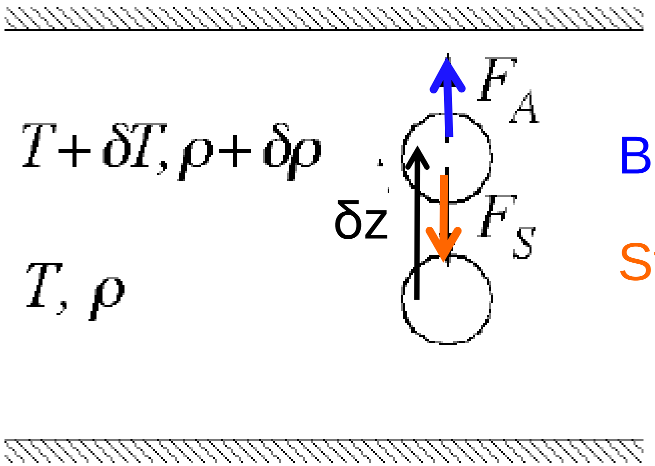
$$v = \frac{2}{9}\frac{\alpha_p g a^2 \, \delta T}{\nu}$$

$$F_A = \frac{4}{3}\pi a^3 \delta\rho g$$



Buoyancy (Archimedes) force

Stokes Drag

$$F_S = -6\pi\mu a v$$

Diffusive time scale
(damps the instability)

a^2/κ

δz/v

Advective time
(regenerates the instability)

$\delta T/\delta z = (T_a - T_b)/d$

$$\frac{a^2}{\kappa} \sim \frac{\delta T \, d}{(T_1 - T_2) v}$$

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Diffusive time scale
(damps the instability)

$$\boxed{\frac{a^2}{\kappa} \sim \frac{\delta T\, d}{(T_1 - T_2)v}}$$

Advective time
(regenerates the instability)

$$\boxed{v = \frac{2}{9}\,\frac{\alpha_p g a^2\, \delta T}{\nu}}$$

$\boxed{a \sim d/2}$

$$Ra \;:=\; \frac{\alpha_p g (T_1 - T_2) d^3}{\nu\kappa} = \frac{\text{Diffusive time scale}}{\text{Advective time scale}}$$

STABLE

UNSTABLE

Ra

Ra_c=72

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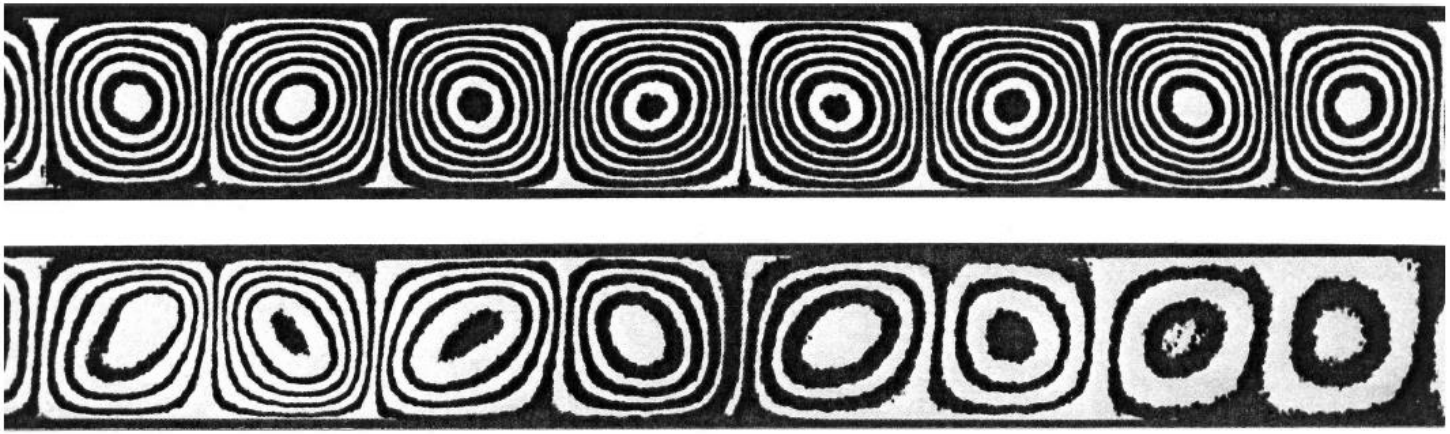


FIG. 2.12 – *Rouleaux de convection thermique dans une huile silicone. Les lignes de courant sont visualisés par interférogrammes. Les dimensions relatives de la boîte sont 10 :4 :1. Haut : rouleaux réguliers orientés parallèlement au petit côté, pour un gradient de température uniforme. Bas : La différence de température et donc l'intensité des rouleaux décroît de la gauche vers la droite. Cliché Oertel & Kirchartz, 1979 (Van Dyke 1982).*

Navier Stokes equations
Oberbeck boussinesq equations

$$
\begin{aligned}
\nabla \cdot \vec{u} &= 0 \\
\rho^* \left[\frac{\partial \vec{u}}{\partial t} + (\nabla \vec{u})\, \vec{u} \right] &= -\rho(T)\, g\, \vec{e}_z - \nabla p + \mu^* \, \nabla^2 \vec{u} \\
C_v^* \left(\frac{\partial T}{\partial t} + \nabla T \cdot \vec{u} \right) &= \chi^* \, \nabla^2 T \;\; \cancel{+2\mu/\rho^* \, D:D}
\end{aligned}
$$

$$
\rho(T) = \rho^* \left[1 - \alpha^* \left(T - T^* \right) \right]
$$

H1: the density is constant except in the buoyancy term
H2: the heat release by viscous dissipation is neglected
H3: viscosity and thermal conductivity are assumed constant

Base flow solution
Pure conduction solution

$$\rho^* \left[1 - \alpha^* \left(\Theta - T^{(b)}\right)\right] g + dP/dz = 0$$
$$d^2\Theta/dz^2 = 0$$

$$\Theta(z) = T^{(b)} - \frac{T^{(b)} - T^{(a)}}{d} z$$

Perturbation expansion

$$\vec{u}(x,y,z;t) = \vec{U} + \vec{u}'(x,y,z;t) = \vec{u}'(x,y,z;t)$$
$$p(x,y,z;t) = P(z) + p'(x,y,z;t)$$
$$T(x,y,z;t) = \Theta(z) + \theta'(x,y,z;t)$$

$$\nu^* = \mu^*/\rho^*$$

$$\nabla \cdot \vec{u}' = 0$$
$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\,\vec{u}' = \alpha^* \theta' g \vec{e}_z - \nabla p'/\rho^* + \nu^* \nabla^2 \vec{u}'$$
$$\frac{\partial \theta'}{\partial t} + \nabla\Theta \cdot \vec{u}' + \nabla\theta' \cdot \vec{u}' = \kappa^* \nabla^2 \theta'$$

$$\kappa^* = \chi^*/C_v^*$$

Perturbation expansion

$$\nabla \cdot \vec{u}' = 0$$

$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\, \vec{u}' = \alpha^* \, \theta' \, g \, \vec{e}_z - \frac{\nabla p'}{\rho^*} + \nu^* \, \nabla^2 \vec{u}'$$

$$\frac{\partial \theta'}{\partial t} - \frac{T^{(b)} - T^{(a)}}{d} \, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' = \kappa^* \, \nabla^2 \theta'$$

Perturbation expansion

$$\boxed{\mathrm{Pr} = \frac{\nu^*}{\kappa^*}} \qquad \boxed{\mathrm{Pr}}$$

$$\nabla \cdot \vec{u}' = 0$$

$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\, \vec{u}' = \alpha^*\, \theta'\, g\, \vec{e}_z - \frac{\nabla p'}{\rho^*} + \nu^*\, \nabla^2 \vec{u}'$$

$$\frac{\partial \theta'}{\partial t} - \frac{T^{(b)} - T^{(a)}}{d}\, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' = \kappa^*\, \nabla^2 \theta'$$

$$\boxed{\mathrm{Ra} = \frac{\alpha^* g \Delta T d^3}{\nu^* \kappa^*}}$$

Non dimensionalisation

$$L_{ref} = d$$

$$t_{ref} = d^2/\kappa^*$$

$$U_{ref} = L_{ref}/t_{ref} = \kappa^*/d$$

$$P_{ref} = \rho^* U_{ref}^2 = \rho^* \kappa_*^2/d^2$$

$$T_{ref} = \kappa^* \nu^* / \alpha^* g d^3$$

Perturbation expansion

$$\nabla \cdot \vec{u}' = 0$$
$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\,\vec{u}' = \alpha^*\,\theta'\,g\,\vec{e}_z - \frac{\nabla p'}{\rho^*} + \nu^*\,\nabla^2 \vec{u}'$$
$$\frac{\partial \theta'}{\partial t} - \frac{T^{(b)} - T^{(a)}}{d}\,\vec{e}_z \cdot \vec{u}' + \nabla\theta' \cdot \vec{u}' = \kappa^*\,\nabla^2\theta'$$

Non dimensionalisation

$$\begin{aligned}
\nabla \cdot \vec{u}' &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\,\vec{u}' &= \mathrm{Pr}\,\theta'\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}' \\
\frac{\partial \theta'}{\partial t} - \mathrm{Ra}\,\vec{e}_z \cdot \vec{u}' + \nabla\theta' \cdot \vec{u}' &= \nabla^2\theta'
\end{aligned}$$

Linearisation

$$\begin{aligned}
\nabla \cdot \vec{u}' &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\,\vec{u}' &= \mathrm{Pr}\,\theta' \vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}' \\
\frac{\partial \theta'}{\partial t} - \mathrm{Ra}\,\vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' &= \nabla^2 \theta'
\end{aligned}$$

Linearisation

$$
\begin{aligned}
\nabla \cdot \vec{u}' &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + \cancel{(\nabla \vec{u}')\,\vec{u}'} &= \mathrm{Pr}\,\theta' \vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}' \\
\frac{\partial \theta'}{\partial t} - \mathrm{Ra}\,\vec{e}_z \cdot \vec{u}' + \cancel{\nabla \theta' \cdot \vec{u}'} &= \nabla^2 \theta'
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\nabla \cdot \vec{u}' = 0 \\
&\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}' \\
&\frac{\partial \theta'}{\partial t} = \mathrm{Ra}\,\vec{e}_z \cdot \vec{u}' + \nabla^2 \theta'
\end{aligned}
$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

$$\nabla\times \Big\Downarrow$$

$$\partial(\nabla \times \vec{u}')/\partial t = \mathrm{Pr}\,\nabla \times (\theta'\,\vec{e}_z) + \mathrm{Pr}\,\nabla^2 (\nabla \times \vec{u}')$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

$$\Big\Downarrow \nabla\times$$

$$\partial(\nabla \times \vec{u}')/\partial t = \mathrm{Pr}\,\nabla \times (\theta'\,\vec{e}_z) + \mathrm{Pr}\,\nabla^2(\nabla \times \vec{u}')$$

$$\Big\Downarrow \nabla\times$$

$$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times \nabla \times \vec{u}' = \nabla(\nabla \cdot \vec{u}') - \nabla^2 \vec{u}'$$

$$\nabla \times \nabla \times \theta' \vec{e}_z = \nabla(\partial\theta'/\partial z) - \nabla^2 \theta' \vec{e}_z$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

$$\Big\Downarrow \nabla\times$$

$$\partial(\nabla \times \vec{u}')/\partial t = \mathrm{Pr}\,\nabla \times (\theta'\,\vec{e}_z) + \mathrm{Pr}\,\nabla^2(\nabla \times \vec{u}')$$

$$\Big\Downarrow \nabla\times$$

$$\nabla^2 \partial \vec{u}'/\partial t = \mathrm{Pr}\,\Big(\nabla^2 \theta' \vec{e}_z - \nabla(\partial\theta'/\partial z)\Big) + \mathrm{Pr}\,\nabla^4 \vec{u}'$$

$$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times \nabla \times \vec{u}' = \nabla(\nabla \cdot \vec{u}') - \nabla^2 \vec{u}'$$

$$\nabla \times \nabla \times \theta' \vec{e}_z = \nabla(\partial\theta'/\partial z) - \nabla^2 \theta' \vec{e}_z$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

$$\nabla\times \Downarrow$$

$$\partial(\nabla\times\vec{u}')/\partial t = \mathrm{Pr}\,\nabla\times(\theta'\,\vec{e}_z) + \mathrm{Pr}\,\nabla^2(\nabla\times\vec{u}')$$

$$\nabla\times \Downarrow$$

$$\nabla^2\partial\vec{u}'/\partial t = \mathrm{Pr}\left(\nabla^2\theta'\vec{e}_z - \nabla(\partial\theta'/\partial z)\right) + \mathrm{Pr}\,\nabla^4\vec{u}'$$

$$\boxed{\nabla^2\frac{\partial u_z'}{\partial t} \quad = \quad \mathrm{Pr}\,\nabla^4 u_z' + \mathrm{Pr}\,\nabla_h^2\theta'} \qquad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Reduction to two equations

$$\begin{aligned}\nabla^2 \frac{\partial u'_z}{\partial t} &= \mathrm{Pr}\,\nabla^4 u'_z + \mathrm{Pr}\,\nabla_h^2 \theta' \\ \frac{\partial \theta'}{\partial t} &= \mathrm{Ra}\, u'_z + \nabla^2 \theta',\end{aligned}$$

$$\boldsymbol{M}\frac{\partial \phi}{\partial t} = \boldsymbol{L}\phi$$

$$\phi = \left\{ \begin{array}{c} u'_z \\ \theta' \end{array} \right\}; \quad \boldsymbol{M} = \left[\begin{array}{cc} \nabla^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \boldsymbol{L} = \left[\begin{array}{cc} \mathrm{Pr}\,\nabla^4 & \mathrm{Pr}\,\nabla_h^2 \\ \mathrm{Ra} & \nabla^2 \end{array} \right]$$

Normal mode expansion

$$\phi = \widehat{\phi}(z) e^{st + i(k_x x + k_y y)}$$

$s = -i\omega$

Normal mode expansion

$$\boldsymbol{M}\frac{\partial \boldsymbol{\phi}}{\partial t} = \boldsymbol{L}\boldsymbol{\phi}$$

$$\boldsymbol{\phi} = \left\{ \begin{array}{c} u'_z \\ \theta' \end{array} \right\}; \quad \boldsymbol{M} = \left[\begin{array}{cc} \nabla^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \boldsymbol{L} = \left[\begin{array}{cc} \mathrm{Pr}\,\nabla^4 & \mathrm{Pr}\,\nabla_h^2 \\ \mathrm{Ra} & \nabla^2 \end{array} \right]$$

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(z)e^{st+i(k_x x+k_y y)}$$ s=-iω $$\vec{k} = \{k_x, k_y\}$$

$$k = |\vec{k}|$$

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \left[\begin{array}{cc} D^2 - k^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \widehat{\boldsymbol{L}} = \left[\begin{array}{cc} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{array} \right]$$

Normal mode expansion

$$\boldsymbol{M}\frac{\partial \boldsymbol{\phi}}{\partial t} = \boldsymbol{L}\boldsymbol{\phi}$$

$$\boldsymbol{\phi} = \left\{ \begin{array}{c} u_z' \\ \theta' \end{array} \right\}; \quad \boldsymbol{M} = \left[\begin{array}{cc} \nabla^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \boldsymbol{L} = \left[\begin{array}{cc} \mathrm{Pr}\,\nabla^4 & \mathrm{Pr}\,\nabla_h^2 \\ \mathrm{Ra} & \nabla^2 \end{array} \right]$$

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(z)e^{st+i(k_x x+k_y y)}$$ *s=-iω* $$\vec{k} = \{k_x, k_y\}$$

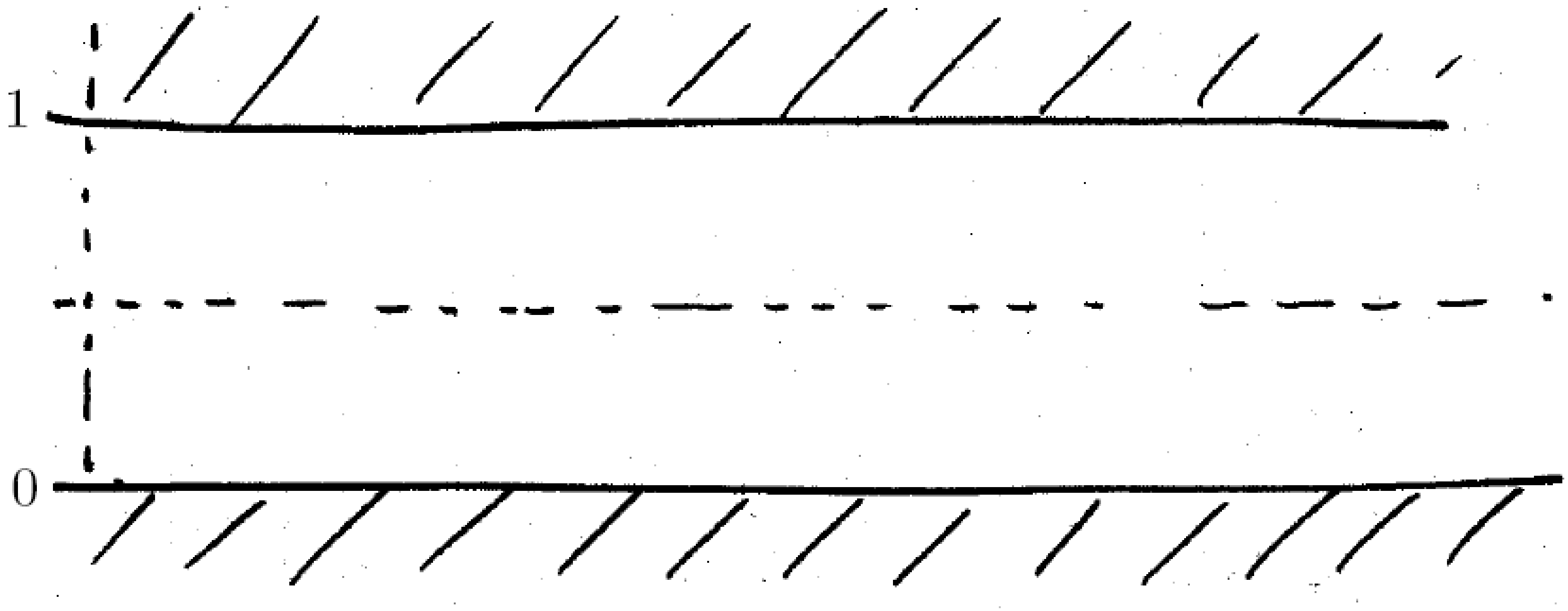
$$k = |\vec{k}|$$

Thermal boundary conditions

$$\widehat{\theta}(z=1)=0$$

$$\theta'(x,y,z=1;t)=0$$

$$T(x,y,z=1;t)=\Theta(z=1)=T^{(a)}$$



$$T(x,y,z=0;t)=\Theta(z=0)=T^{(b)}$$

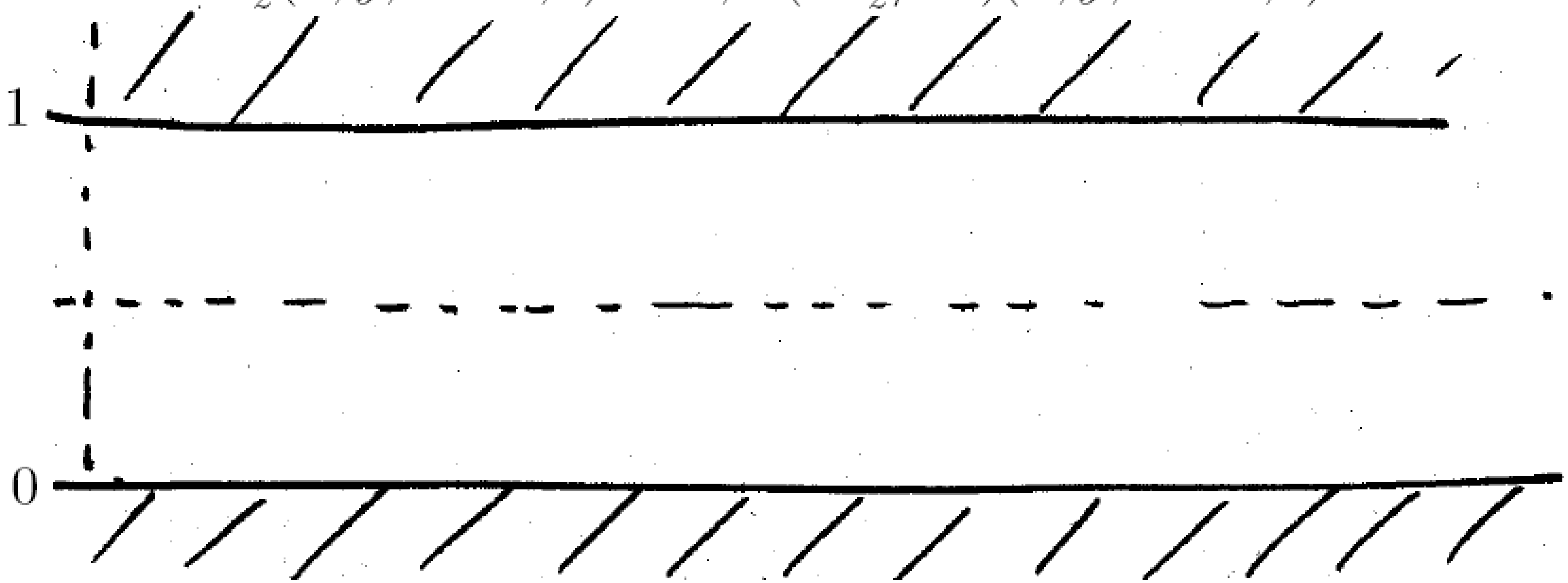
$$\theta'(x,y,z=0;t)=0$$

$$\widehat{\theta}(z=0)=0$$

Rigid boundary conditions

$$\widehat{u}_z(z=1)=0\,,\quad (D\widehat{u}_z)(z=1)=0$$

$$u'_z(x,y,z=1;t)=0\,,\quad (\partial u'_z/\partial z)(x,y,z=1;t)=0$$



$$u'_z(x,y,z=0;t)=0\,,\quad (\partial u'_z/\partial z)(x,y,z=0;t)=0$$

$$\widehat{u}_z(z=0)=0\,,\quad (D\widehat{u}_z)(z=0)=0$$

Stress free boundary conditions

$$\sigma_{xz} = 0$$

$$\mu(\partial u_x/\partial z + \partial u_z/\partial x) = 0$$

$$\boxed{\partial u_x/\partial z = 0}$$

Stress free boundary conditions

$$\sigma_{xz} = 0$$

$$\mu(\partial u_x/\partial z + \partial u_z/\partial x) = 0$$

$$\boxed{\partial u_x/\partial z = 0}$$

$$\sigma_{yz} = 0$$

$$\boxed{\partial u_y/\partial z = 0}$$

Stress free boundary conditions

$$\sigma_{xz} = 0$$

$$\sigma_{yz} = 0$$

$$\mu(\partial u_x/\partial z + \partial u_z/\partial x) = 0$$

$$\boxed{\partial u_x/\partial z = 0}$$

$$\boxed{\partial u_y/\partial z = 0}$$

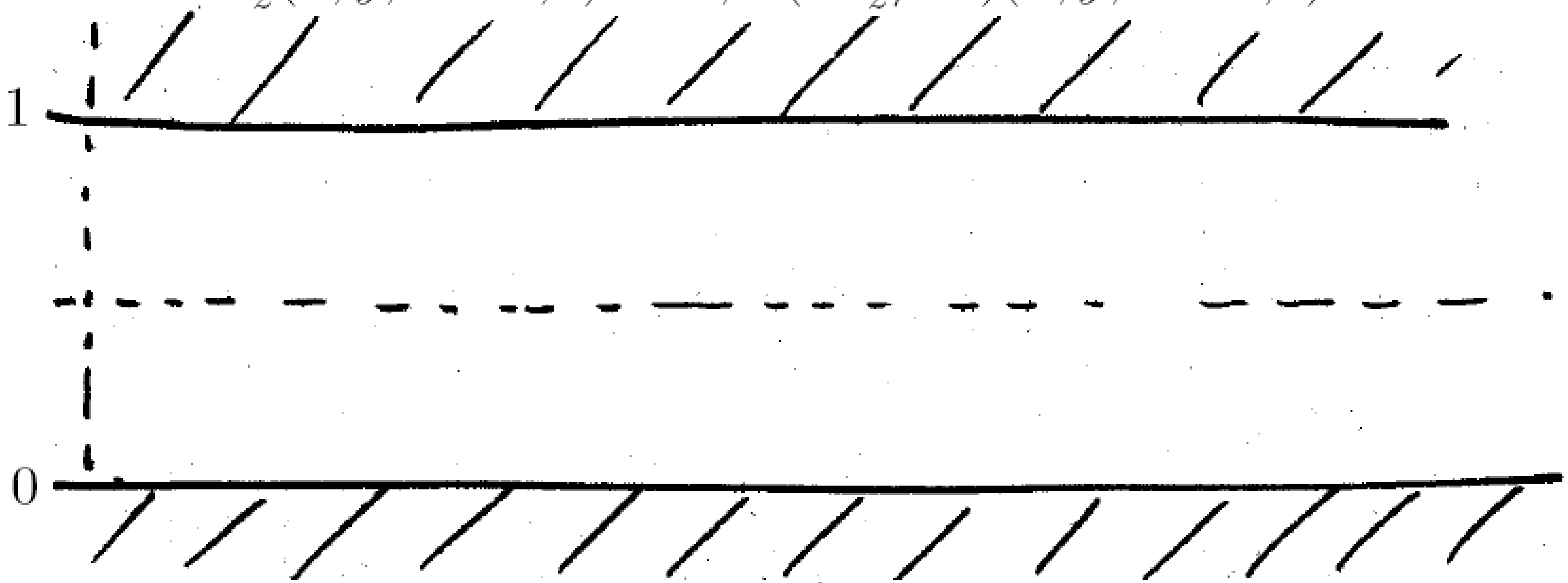
$$\partial/\partial z \quad \partial u_x/\partial x + \partial u_y/\partial y + \partial u_z/\partial z = 0$$

$$\boxed{\partial^2 u_z/\partial z^2 = 0}$$

Free stress boundary conditions

$$\widehat{u}_z(z=1)=0\,,\quad (D^2\widehat{u}_z)(z=1)=0$$

$$u'_z(x,y,z=1;t)=0\,,\quad (\partial^2 u'_z/\partial z^2)(x,y,z=1;t)=0$$



$$u'_z(x,y,z=0;t)=0\,,\quad (\partial^2 u'_z/\partial z^2)(x,y,z=0;t)=0$$

$$\widehat{u}_z(z=0)=0\,,\quad (D^2\widehat{u}_z)(z=0)=0$$

Solution ansatz (which satisfies the b.c.)

$$\widehat{\boldsymbol{\phi}} = \widetilde{\boldsymbol{\phi}_n} \sin(n\pi z)$$

$$s\widetilde{\boldsymbol{M}}\widetilde{\boldsymbol{\phi}} = \widetilde{\boldsymbol{L}}\widetilde{\boldsymbol{\phi}}$$

$$\widetilde{\boldsymbol{M}} = \begin{bmatrix} -k_n^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widetilde{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\, k_n^4 & -\mathrm{Pr}\, k^2 \\ \mathrm{Ra} & -k_n^2 \end{bmatrix}$$

$$k_n^2 := k^2 + (n\pi)^2$$

Solution ansatz (which satisfies the b.c.)

$$\widehat{\phi} = \widetilde{\phi}_n \sin(n\pi z)$$

$$s\widetilde{\boldsymbol{M}}\widetilde{\phi} = \widetilde{\boldsymbol{L}}\widetilde{\phi}$$

$$\widetilde{\boldsymbol{M}} = \begin{bmatrix} -k_n^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widetilde{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\, k_n^4 & -\mathrm{Pr}\, k^2 \\ \mathrm{Ra} & -k_n^2 \end{bmatrix}$$

$$k_n^2 := k^2 + (n\pi)^2$$

$$\det\left(s\widetilde{\boldsymbol{M}} - \widetilde{\boldsymbol{L}}\right) = 0$$

$$\boxed{s^2 + (1+\mathrm{Pr})\, k_n^2 s + \mathrm{Pr}\left(k_n^4 - \mathrm{Ra}\frac{k^2}{k_n^2}\right) = 0}$$

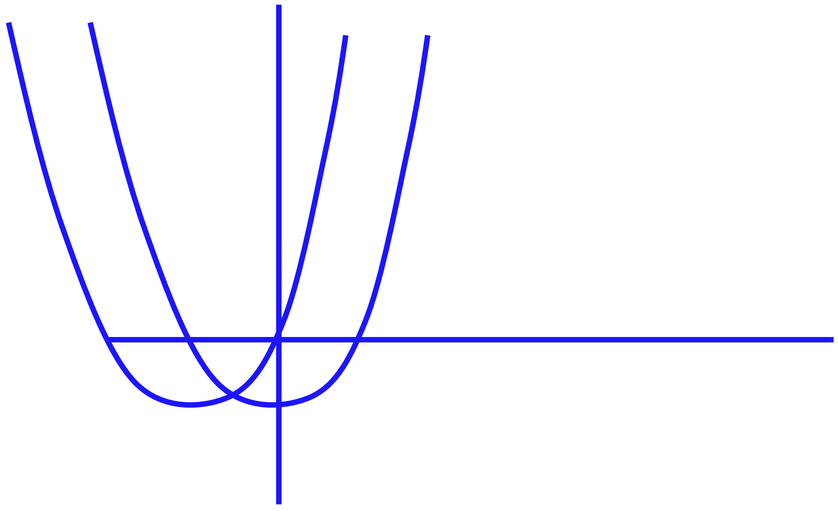
Dispersion relation

$$s^2 + (1 + \mathrm{Pr})\, k_n^2 s + \mathrm{Pr}\left(k_n^4 - \mathrm{Ra}\frac{k^2}{k_n^2}\right) = 0$$

<0

Condition for instability

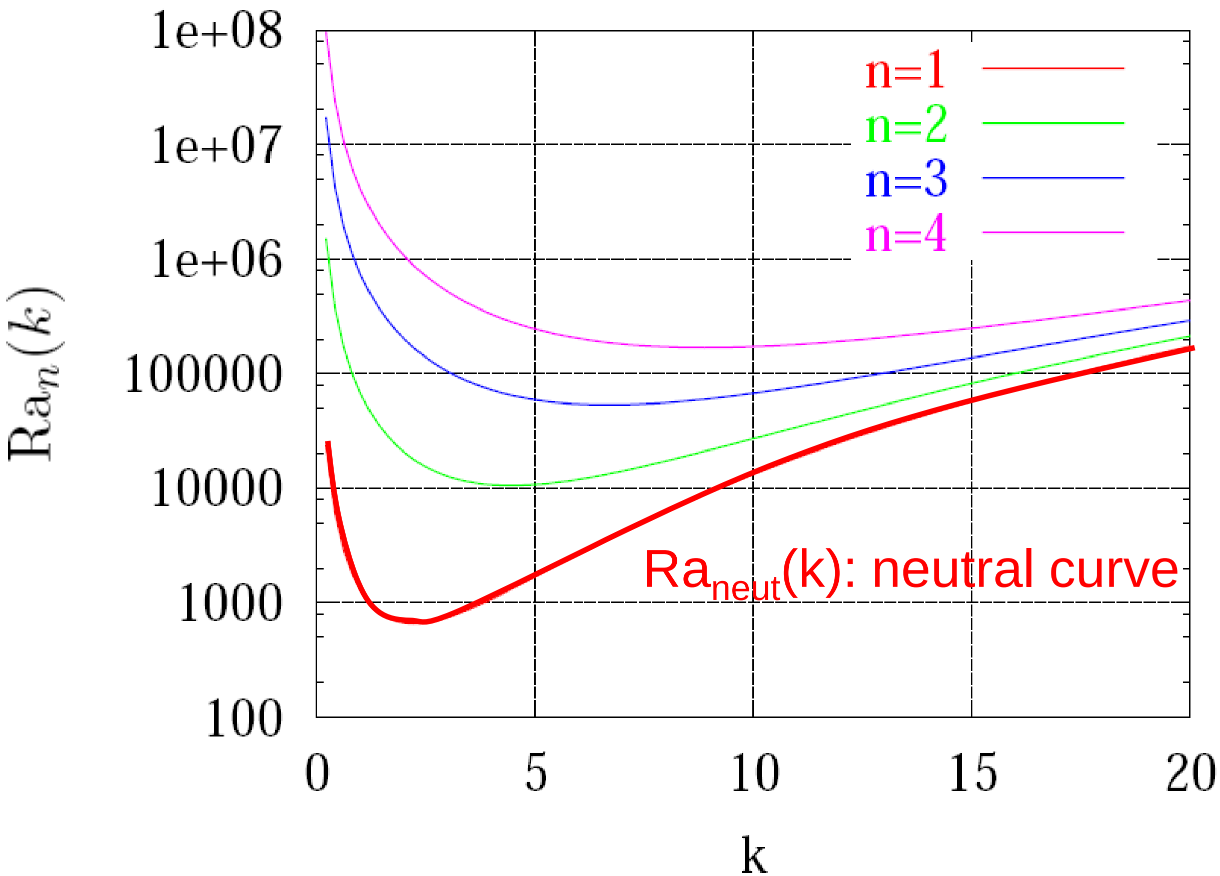
$$\mathrm{Ra} > \frac{(k^2 + n^2\pi^2)^3}{k^2} = \mathrm{Ra}_n(k)$$

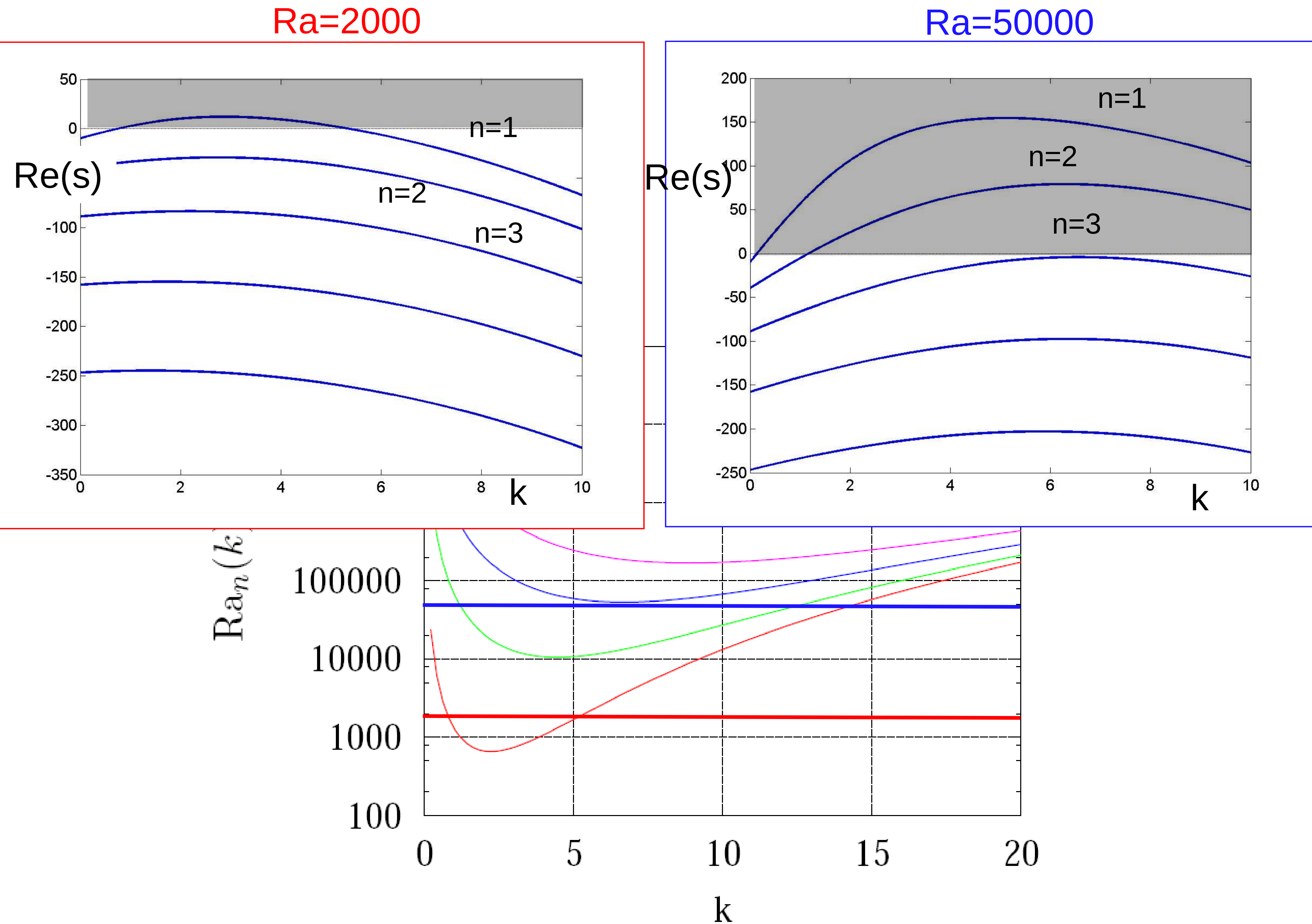


The parabola points upwards
The sum of the roots is negative

Condition for instability

$$\mathrm{Ra} > \frac{(k^2 + n^2\pi^2)^3}{k^2} = \mathrm{Ra}_n(k)$$





Ra=2000
Ra=50000
Re(s)
n=1
n=2
n=3
k
Ra$_n$(k)
100
1000
10000
100000
0
5
10
15
20

Marginal curve for instability

$$n = 1 \qquad \mathrm{Ra}_{neut}(k) = \frac{(k^2 + \pi^2)^3}{k^2}$$

Minimum minimorum

$$\mathrm{Ra}_c = \min_k \mathrm{Ra}_{neut}(k)$$

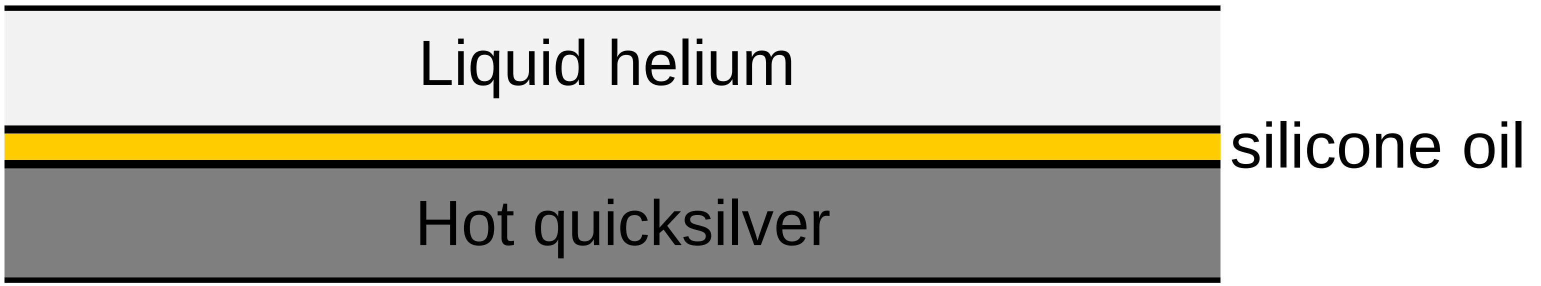
$$d\mathrm{Ra}_{neut}/dk = 0$$

$$\mathrm{Ra}_c = 27\pi^4/4 \sim 657.5 \,, \quad k = \pi/\sqrt{2} \sim 2.22$$

Stress-free boundary conditions

$$\mathrm{Ra}_c = 27\pi^4/4 \sim 657.5\ , \quad k = \pi/\sqrt{2} \sim 2.22$$

Experimentally verified only in 69 by Goldstein and Graham



Rigid boundary conditions

$$\mathrm{Ra}_c = 1707.76 \text{ and } k_c = 3.163.$$

Calculated in ‘40 by Pellew and Southwell

1. Solve Rayleigh-Benard eigenvalue problem

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \left[\begin{array}{cc} D^2 - k^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \widehat{\boldsymbol{L}} = \left[\begin{array}{cc} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{array} \right]$$

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(z)e^{st+i(k_x x + k_y y)}$$ $s=-i\omega$ $$\vec{k} = \{k_x, k_y\}$$

$$k = |\vec{k}|$$

With rigid boundary conditions

$$\widehat{\theta}(z=0) = 0$$

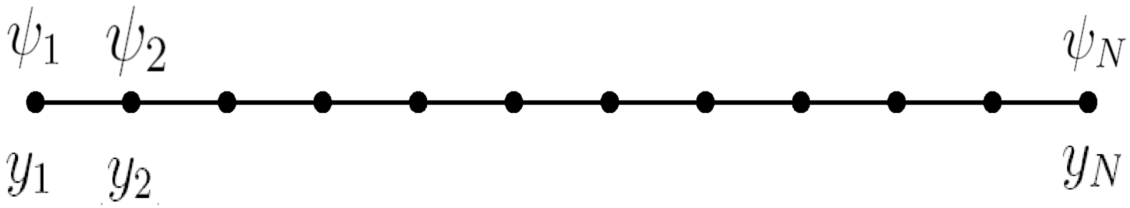
$$\widehat{u}_z(z=0) = 0\,, \quad (D\widehat{u}_z)(z=0) = 0$$

$$\widehat{\theta}(z=1) = 0$$

$$\widehat{u}_z(z=1) = 0\,, \quad (D\widehat{u}_z)(z=1) = 0$$

How to solve Rayleigh-Benard's equation?

Method 1: Finite differences of order 1.



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \qquad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \end{pmatrix}$$

How to solve Rayleigh-Benard's equation?

Method 1: Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

Finite difference scheme

Central finite difference scheme second order

Finite difference formulas for first derivatives

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \dots$$

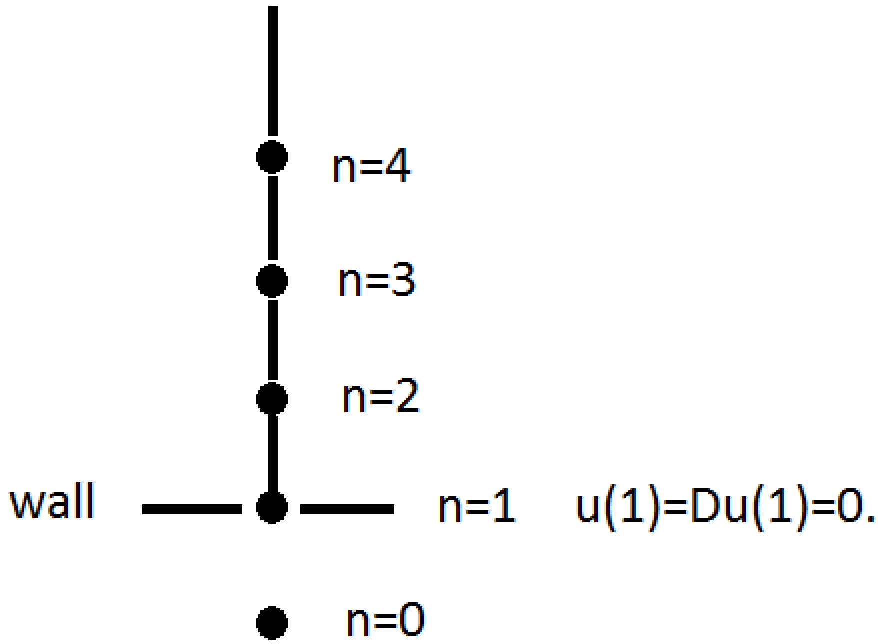
Finite difference formulas for second derivatives

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12} \left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} + \dots$$

Finite difference formulas for fourth derivatives

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{\Delta x^4} - \frac{\Delta x^2}{6} \left.\frac{\partial^6 u}{\partial x^6}\right|_{x_i} + \dots$$

Rigid boundary condition



Boundary condition

$$u(1) = 0.$$

$$u(2) = u(0)$$

$$\frac{\partial^2 u(2)}{\partial x^2} = \frac{u(1) - 2u(2) + u(3)}{dx^2} = \frac{-2u(2) + u(3)}{dx^2}$$

$$\frac{\partial^4 u(2)}{\partial x^4} = \frac{u(0) - 4u(1) + 6u(2) - 4u(3) + u(4)}{dx^4} = \frac{7u(2) - 4u(3) + u(4)}{dx^4}$$

D2 =

-2	1	0	0	0
1	-2	1	0	0
0	1	-2	1	0
0	0	1	-2	1
0	0	0	1	-2

D4 =

7	-4	1	0	0
-4	6	-4	1	0
1	-4	6	-4	1
0	1	-4	6	-4
0	0	1	-4	7

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widehat{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{bmatrix}$$

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widehat{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{bmatrix}$$

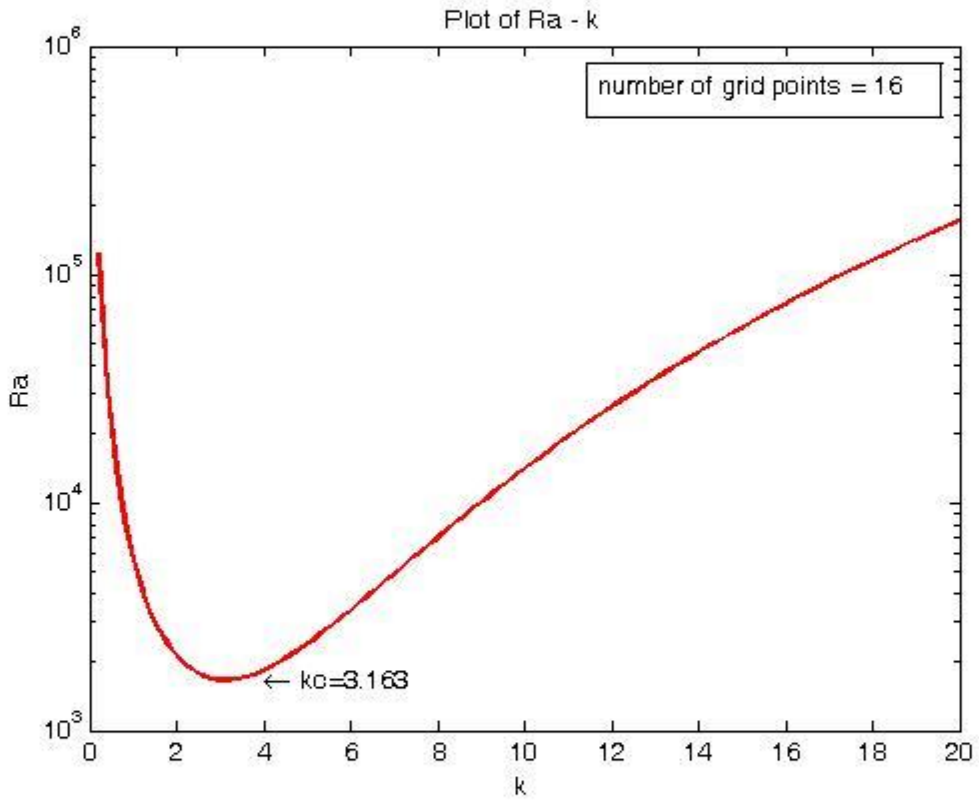
$$s\hat{M}\hat{\phi} = \hat{L}\hat{\phi}$$

$$eig(M^{-1}L) \text{ or } det\left(s\hat{M} - \hat{L}\right) = 0.$$

$$f(s) = det\left(s\hat{M} - \hat{L}\right)$$

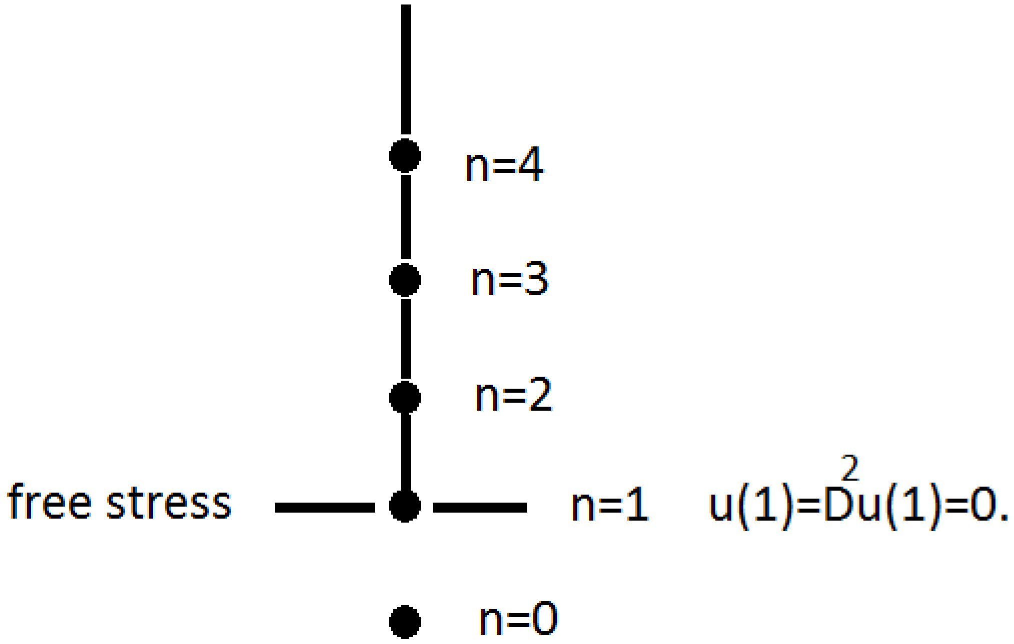
order of polynomial = 2 * number of grid point

Rigid boundary condition



$$\mathrm{Ra}_c = 1707.76 \text{ and } k_c = 3.163.$$

Free stress boundary condition



Boundary condition

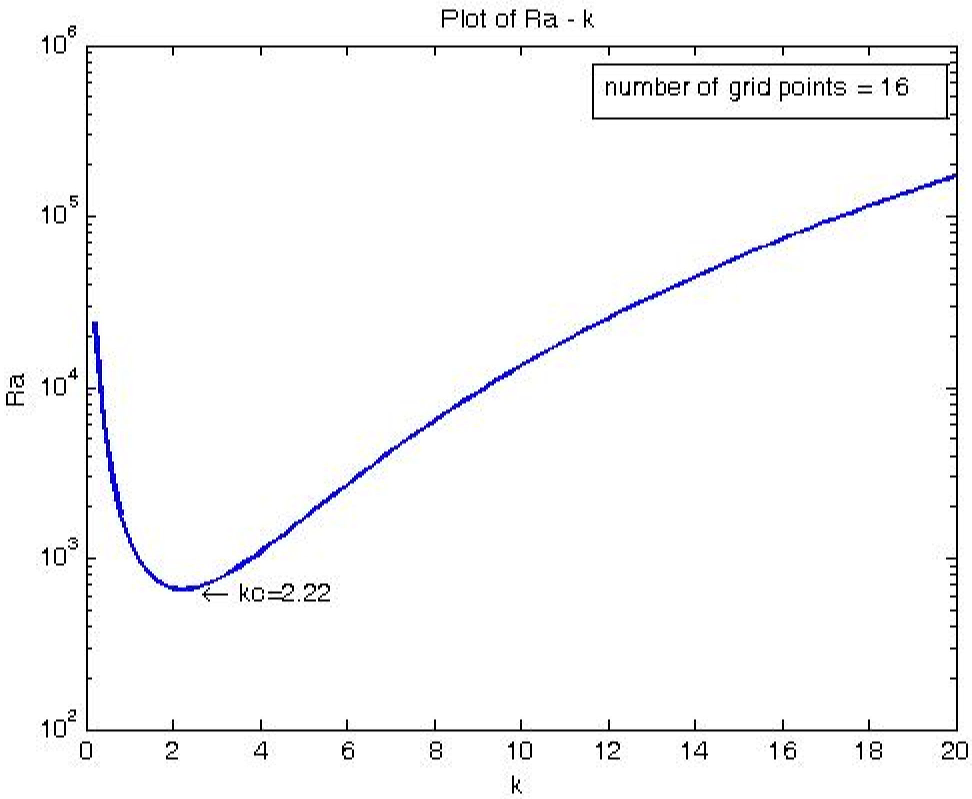
$$u(1) = 0.$$

$$u(2) = -u(0)$$

$$\frac{\partial^2 u(2)}{\partial x^2} = \frac{u(1) - 2u(2) + u(3)}{dx^2} = \frac{-2u(2) + u(3)}{dx^2}$$

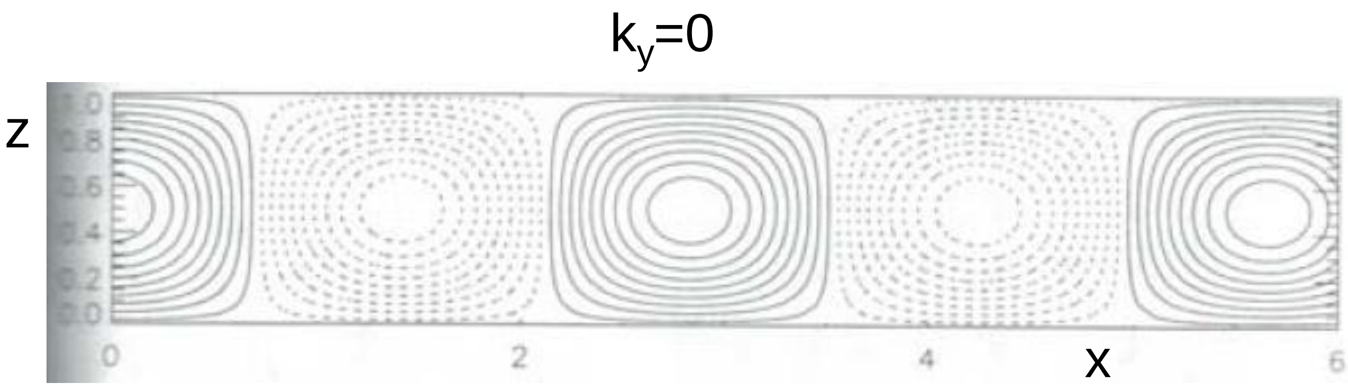
$$\frac{\partial^4 u(2)}{\partial x^4} = \frac{u(0) - 4u(1) + 6u(2) - 4u(3) + u(4)}{dx^4} = \frac{5u(2) - 4u(3) + u(4)}{dx^4}$$

Free stress boundary condition



$$\mathrm{Ra}_c = 27\pi^4/4 \sim 657.5\ , \quad k = \pi/\sqrt{2} \sim 2.22$$

Typical structures : looking at eigenmodes



Rolls

Squares and hexagons

$k_x = k_y = k_c ; A_x = A_y$

$A\sin(\pi z)[\cos(k_c x)+\cos(k_c y)]$ $A\sin(\pi z)[\cos(k_c x)+\cos(k_c(-x/2+3y/2))+\cos(k_c(x/2+3y/2)$

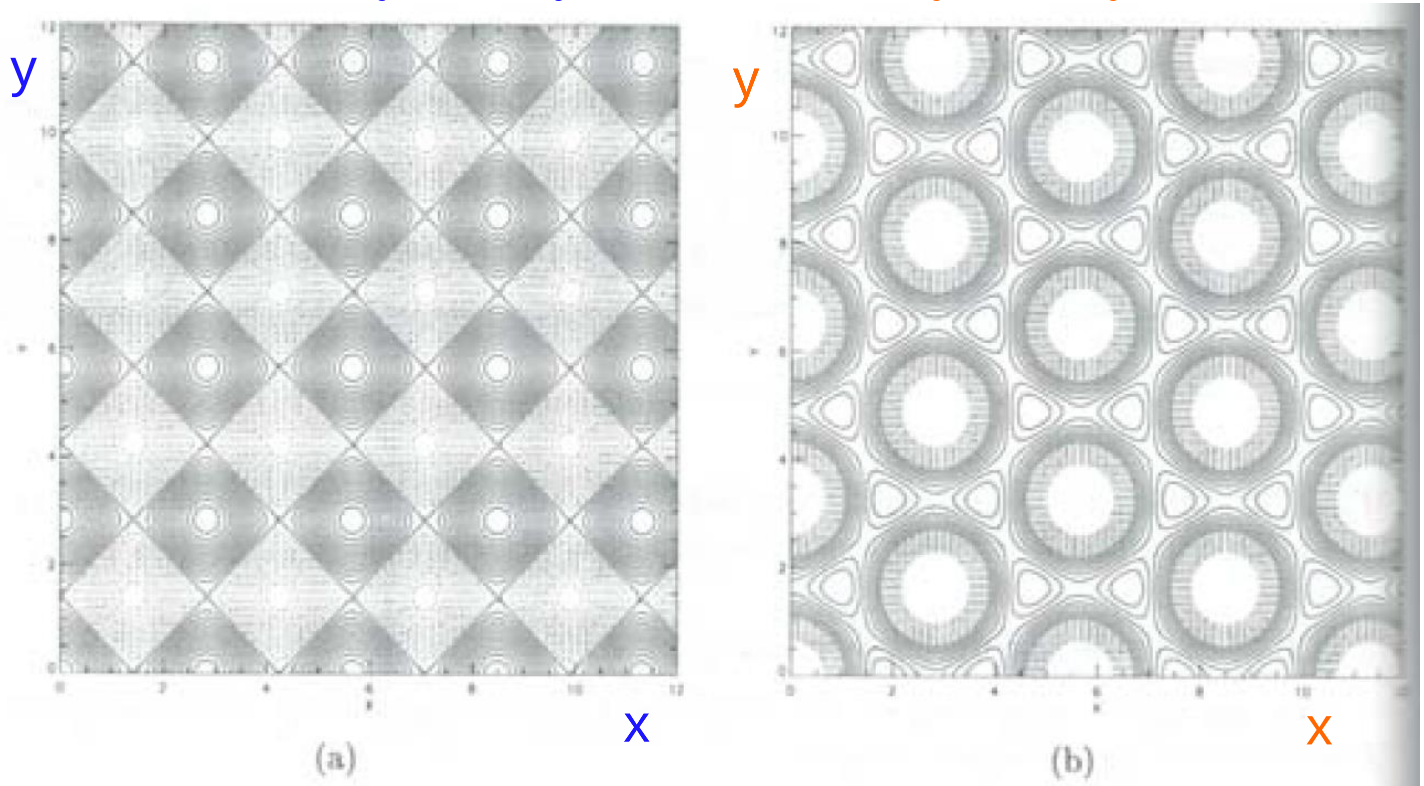


FIG. 7.5 - *Figures de convection en carrés (a) ou en hexagones (b). On a tracé les courbes isothermes : en pointillés $\delta T < 0$, en trait pleins $\delta T > 0$.*

Typical structures : looking at eigenmodes

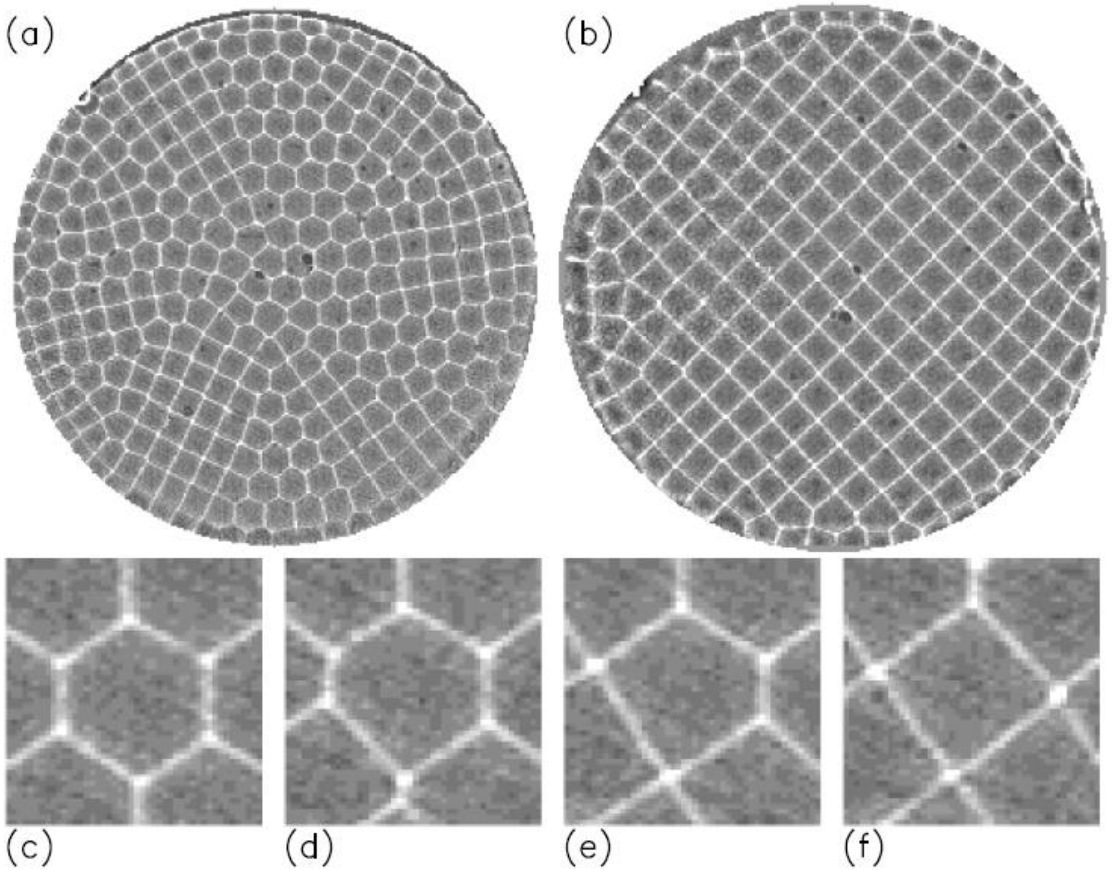


Figure 7 Transition to square patterns in Marangoni convection for $d = 0.072$ cm. (*a* At $\epsilon = 3.9$, a stationary pattern of mixed hexagonal, pentagonal, and square symmetry i observed. (*b*) At $\epsilon = 7.23$, a stationary, nearly ideal square pattern arises. Close observatio of a few cells demonstrates local changes in topology from threefold to fourfold vertices a ϵ is gradually increased from (*c*) 2.67, through (*d*) 2.98, and (*e*) 3.3, to (*f*) 3.89.

Marangoni effect

2 movies

Marangoni effect

Normal stress $$\mathbf{n} \cdot \mathbf{T}^1 \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{T}^2 \cdot \mathbf{n} = \sigma (\nabla \cdot \mathbf{n})$$

Marangoni effect

Normal stress

$$\mathbf{n} \cdot \mathbf{T}^1 \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{T}^2 \cdot \mathbf{n} = \sigma (\nabla \cdot \mathbf{n})$$

Stress tensor

Curvature

Inviscid flow

Marangoni effect

Normal stress

$$\mathbf{n}\cdot\mathbf{T}^1\cdot\mathbf{n} - \mathbf{n}\cdot\mathbf{T}^2\cdot\mathbf{n} = \sigma(\nabla\cdot\mathbf{n})$$

Stress tensor

Curvature

Inviscid flow

$$\mathbf{T} = -p\mathbf{I}$$

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} = -p$$

$$p^2 - p^1 = \sigma\,\nabla\cdot\mathbf{n}$$

Marangoni effect

Normal stress

$$\mathbf{n} \cdot \mathbf{T}^1 \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{T}^2 \cdot \mathbf{n} = \sigma (\nabla \cdot \mathbf{n})$$

Stress tensor

Curvature

Inviscid flow

$$\mathbf{T} = -p\mathbf{I}$$

$$\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = -p$$

Laplace law

$$p^2 - p^1 = \sigma \nabla \cdot \mathbf{n}$$

Marangoni effect

Normal stress $\mathbf{n} \cdot \mathbf{T}^1 \cdot \mathbf{n} - \mathbf{n} \cdot \hat{\mathbf{T}} \cdot \mathbf{n} = \sigma \, (\nabla \cdot \mathbf{n})$

Tangential stress $\mathbf{n} \cdot \mathbf{T}^1 \cdot \mathbf{t} - \mathbf{n} \cdot \mathbf{T}^2 \cdot \mathbf{t} = \nabla\sigma \cdot \mathbf{t}$

Rayleigh Benard flow with stress free B.C. (constant σ)

$$\mathbf{T}_{xz} = 0$$
$$\mathbf{T}_{yz} = 0$$

When the surface tension σ varies in space: Marangoni effect

Benard-Marangoni instability

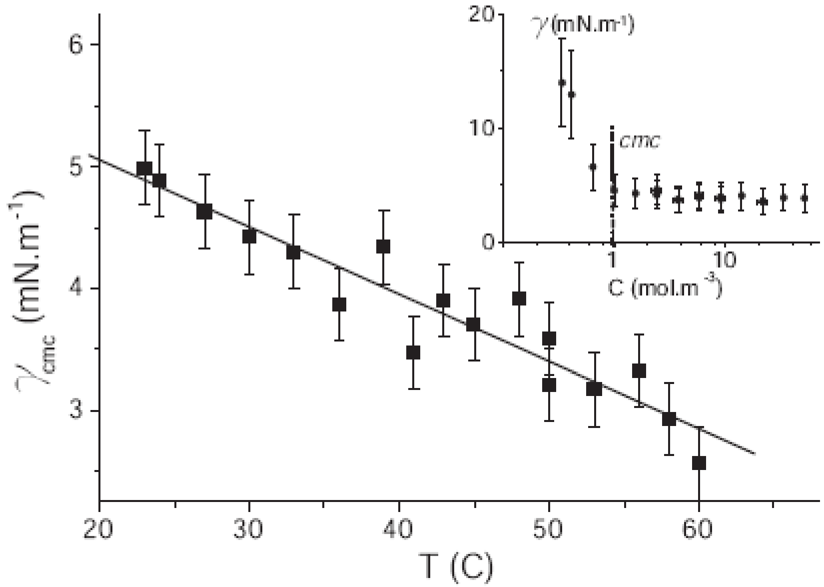


Figure 2: Surface tension of a water in hexadecane drop above the *cmc* ($C = 36cmc$) as a function of the bulk oil temperature, fitted with a linear law (solid line). Inset: Surface tension as a function of the concentration C of surfactant at T=28 °C. The *cmc* is estimated at 1 mol.m^{-3}.

Marangoni effect
(following J.W. Bush, MIT)

Figure 2: The 'footprint' of a whale, caused by the whales sweeping biomaterial to the sea surface. The biomaterial acts as a surfactant in locally suppressing the capillary waves evident elsewhere on the sea surface. Observed in the wake of a whale on a whale watch from Boston Harbour.

Marangoni effect (following J.W. Bush, MIT)

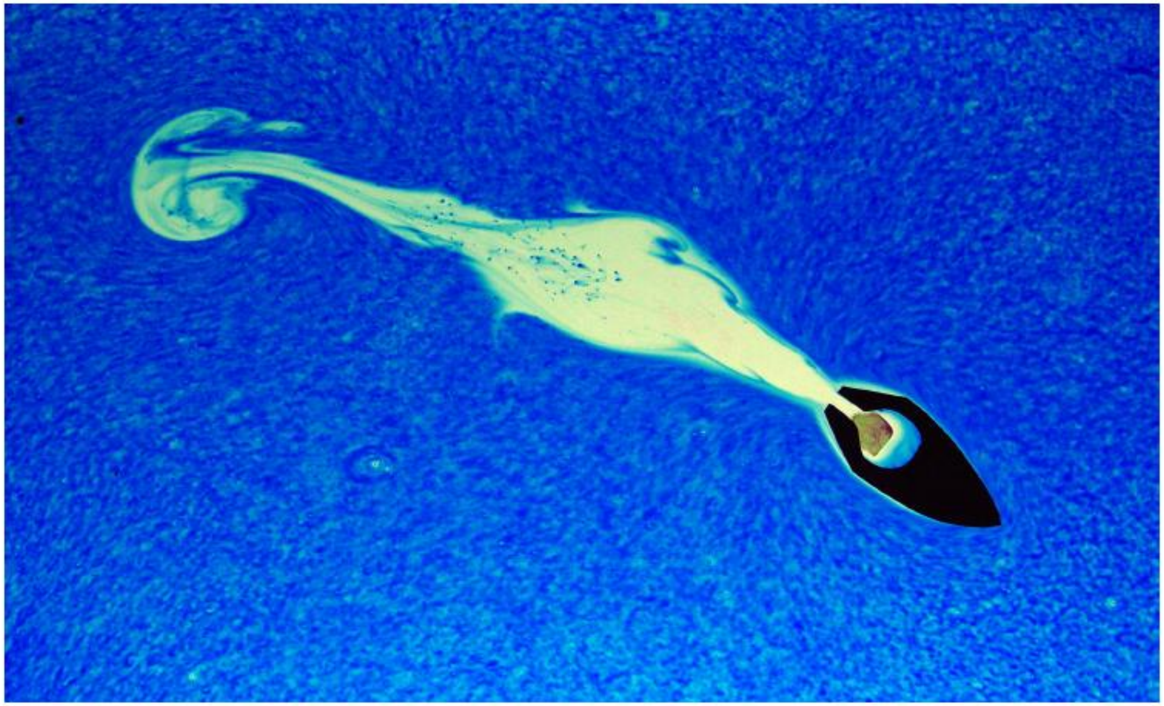


Figure 3: The soap boat. A floating body (length 2.5cm) contains a small volume of soap, which serves as its fuel in propelling it across the free surface.The soap exits the rear of the boat, decreasing the local surface tension. The resulting fore-to-aft surface tension gradient propels the boat forward. The water surface is covered with Thymol blue, which parts owing to the presence of soap, which is visible as a white streak.

Marangoni effect
(following J.W. Bush, MIT)

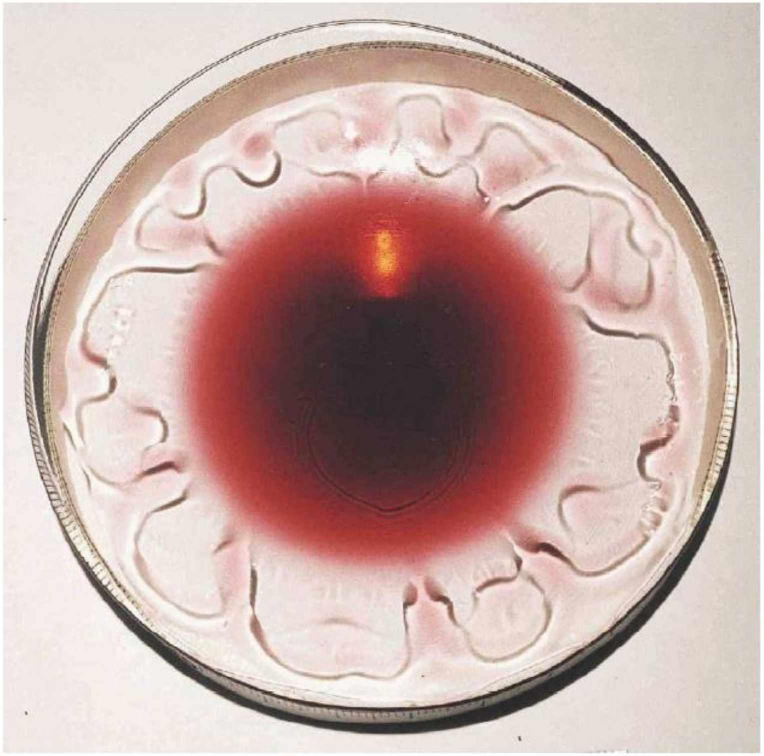


Figure 1: The tears of wine. Fluid is drawn from the bulk up the thin film adjoining the walls of the glass by Marangoni stresses induced by evaporation of alcohol from the free surface.

Benard-Marangoni instability

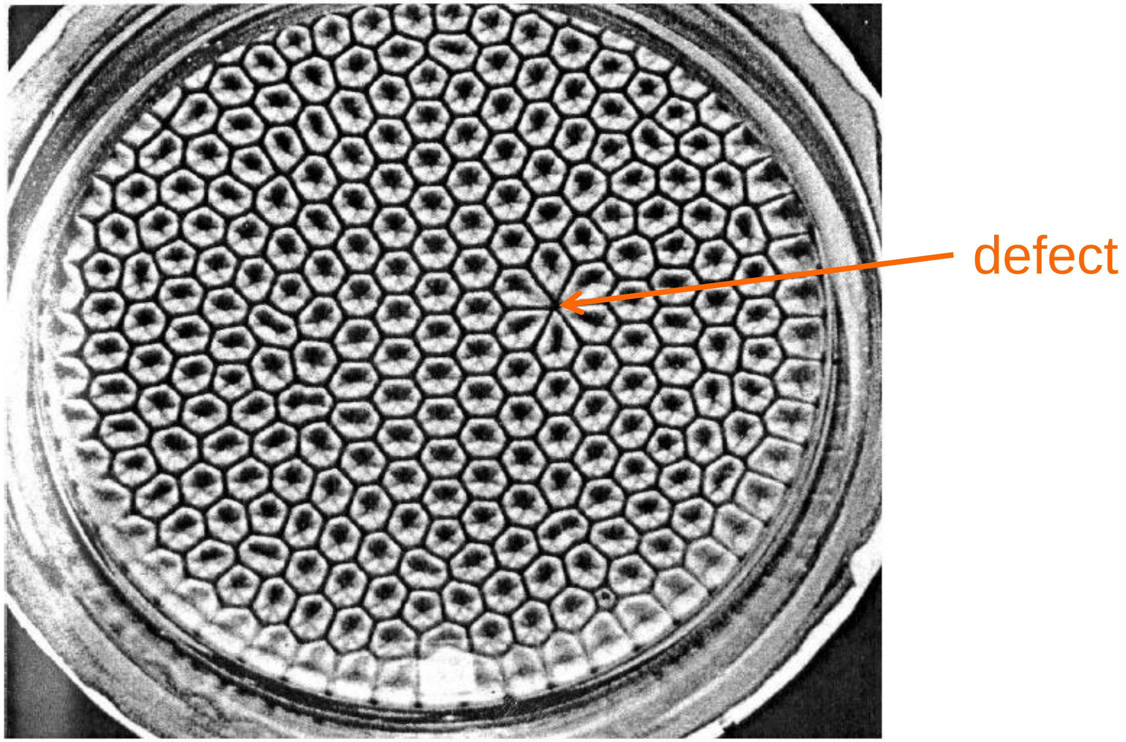


FIG. 2.16 – *Cellules de convection thermo-capillaire dans une fine couche d'huile silicone de viscosité 0,5 cm^2/s. Photo Koschmieder, 1974 (Van Dyke 1982).*

Benard-Marangoni instability

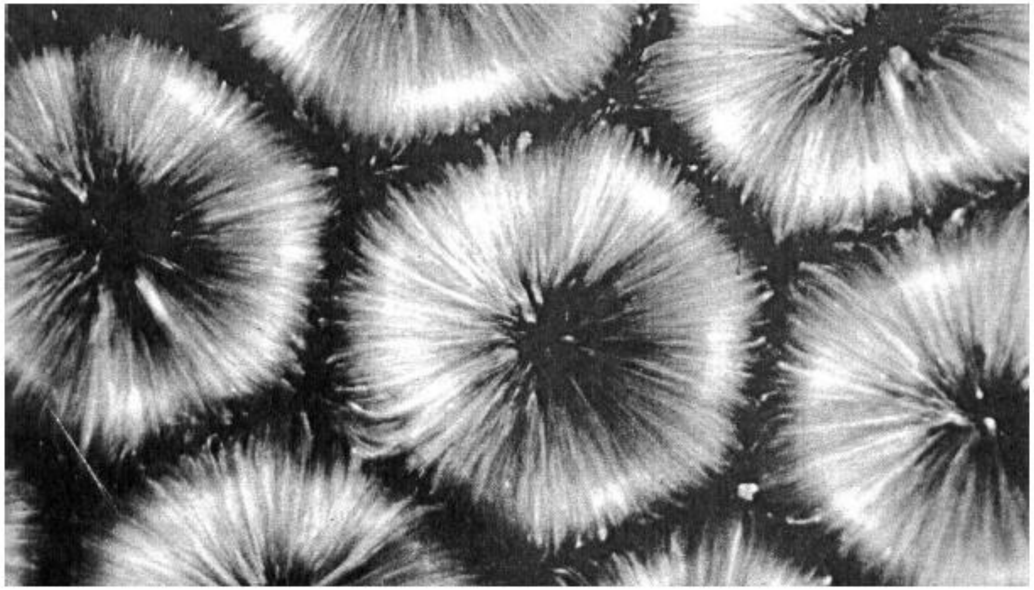
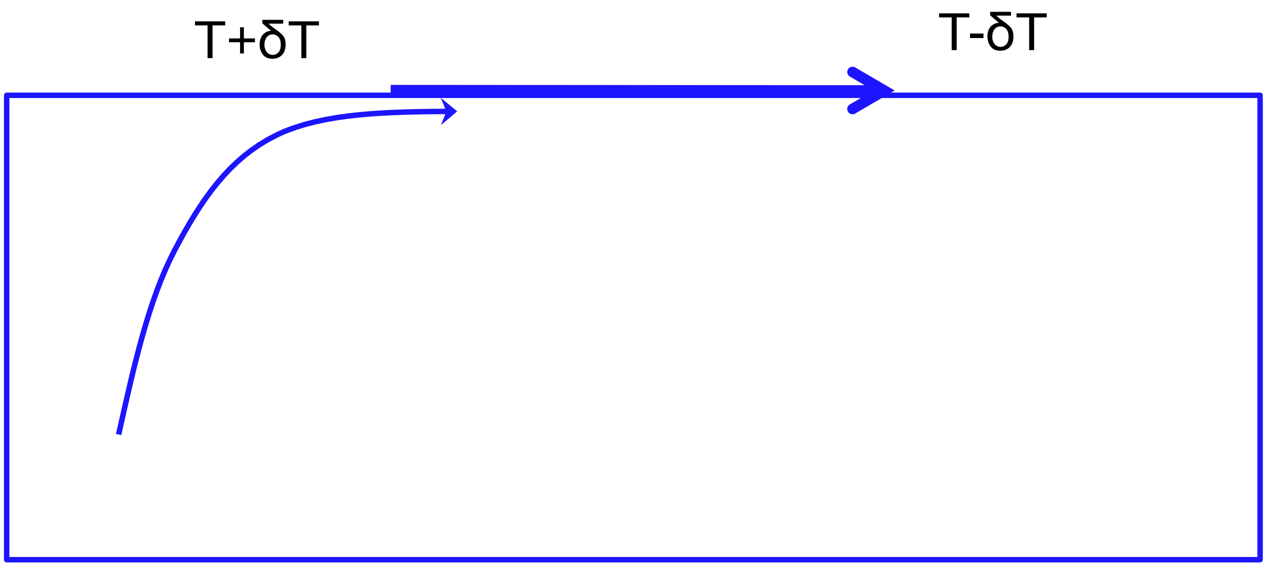
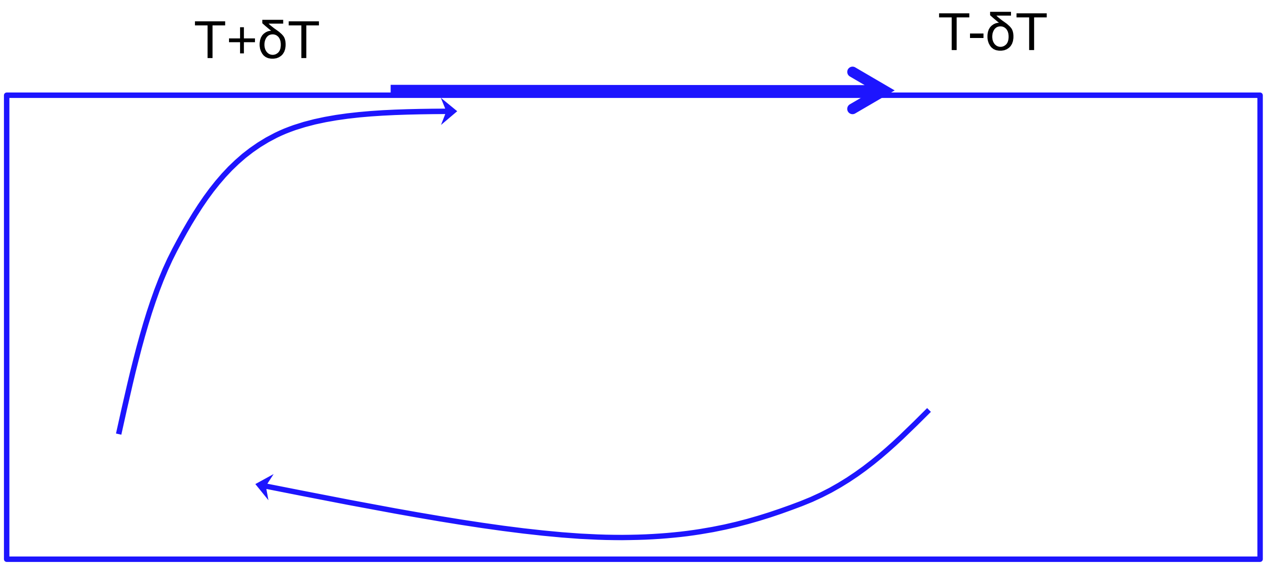


FIG. 2.17 – *Vue de dessus de cellules de convection thermo-capillaire dans une couche d'huile silicone de 1mm d'épaisseur, surmontée par de l'air. La lumière réfléchie par des particules d'aluminium montre le fluide montant au coeur de la cellule hexagonale et descendant sur les bords. Le temps d'exposition est de 10 secondes, et le fluide traverse la cellule du centre au bord en 2 secondes. La dimension des cellules est 3,6 mm. Cliché Velarde, Yuste & Salan (Van Dyke 1982).*

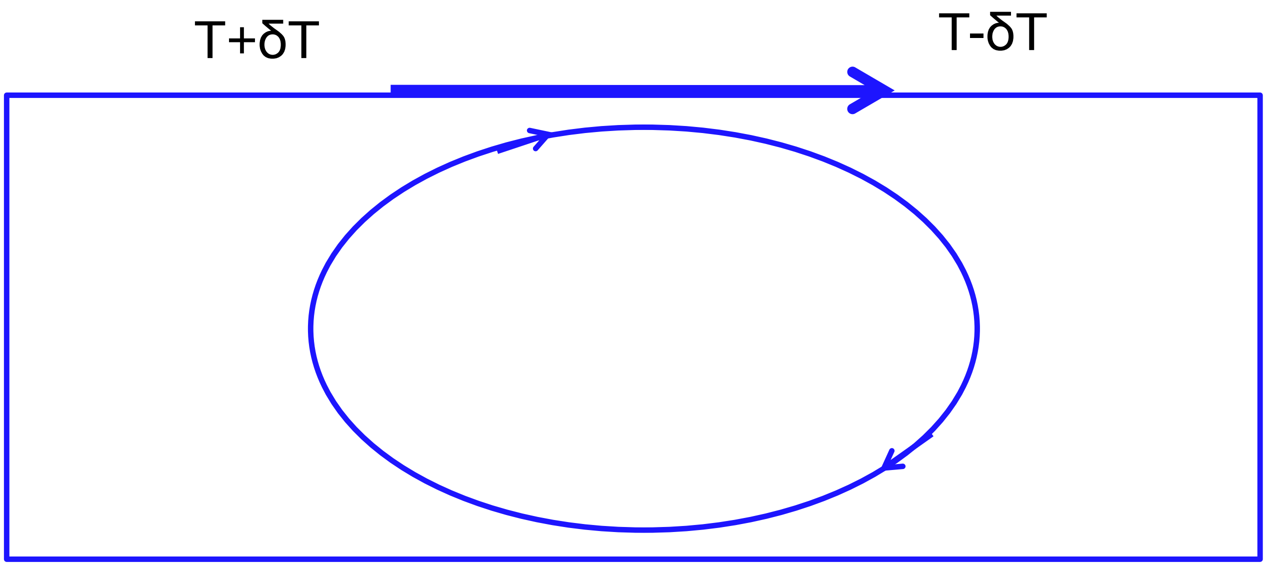
Benard-Marangoni instability



Benard-Marangoni instability



Benard-Marangoni instability



Benard-Marangoni instability

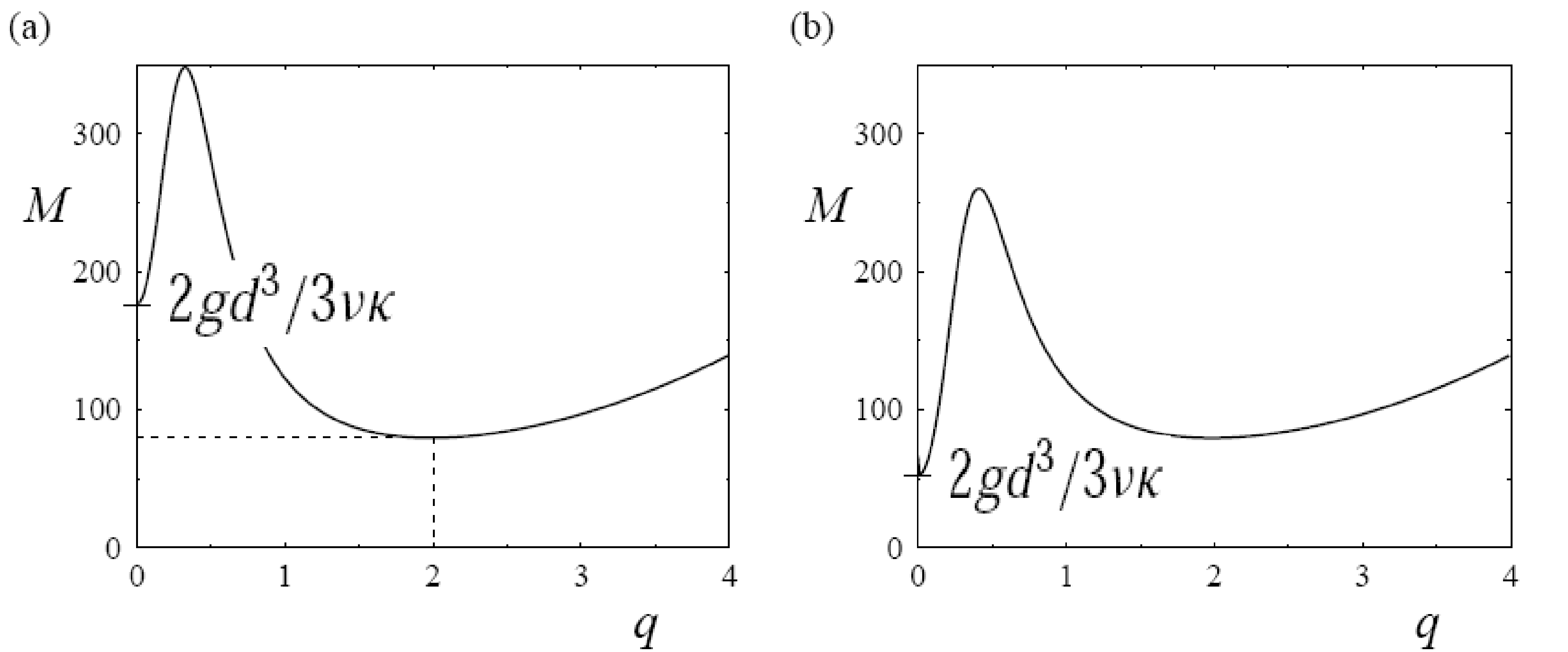


Figure 2 Marginal stability curves for Marangoni convection indicate two different types of primary instabilities can typically arise. The stability curves are computed for $\nu = 0.10\ \mathrm{cm}^2\ \mathrm{s}^{-1}$ silicone oil under terrestrial gravity with $d \ll d_g$. (*a*) Short-wavelength modes ($q \neq 0$) are primary for thick oil layers; instability of these modes leads to the classic cellular flows studied by Bénard (1900). (*b*) For sufficiently thin oil layers ($d \lesssim 0.03$ cm), the global minimum is at $q = 0$, and modes with long-wavelength deformation of the interface become primary.

$$\boxed{M = \sigma_T \Delta T d/\rho\nu\kappa}$$ Marangoni number