Most flows are unstable...

Vortex shedding

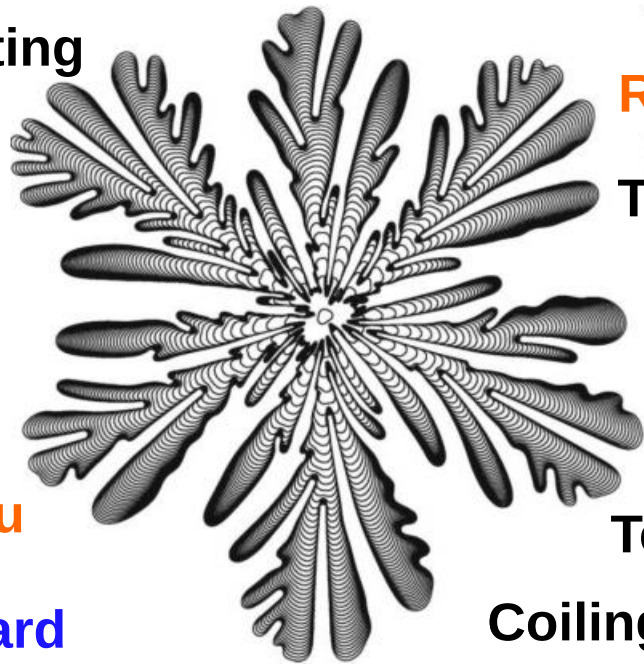
Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

Lift-up and Streaks



Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

Tearing instability

Rayleigh-Benard

Coiling instability

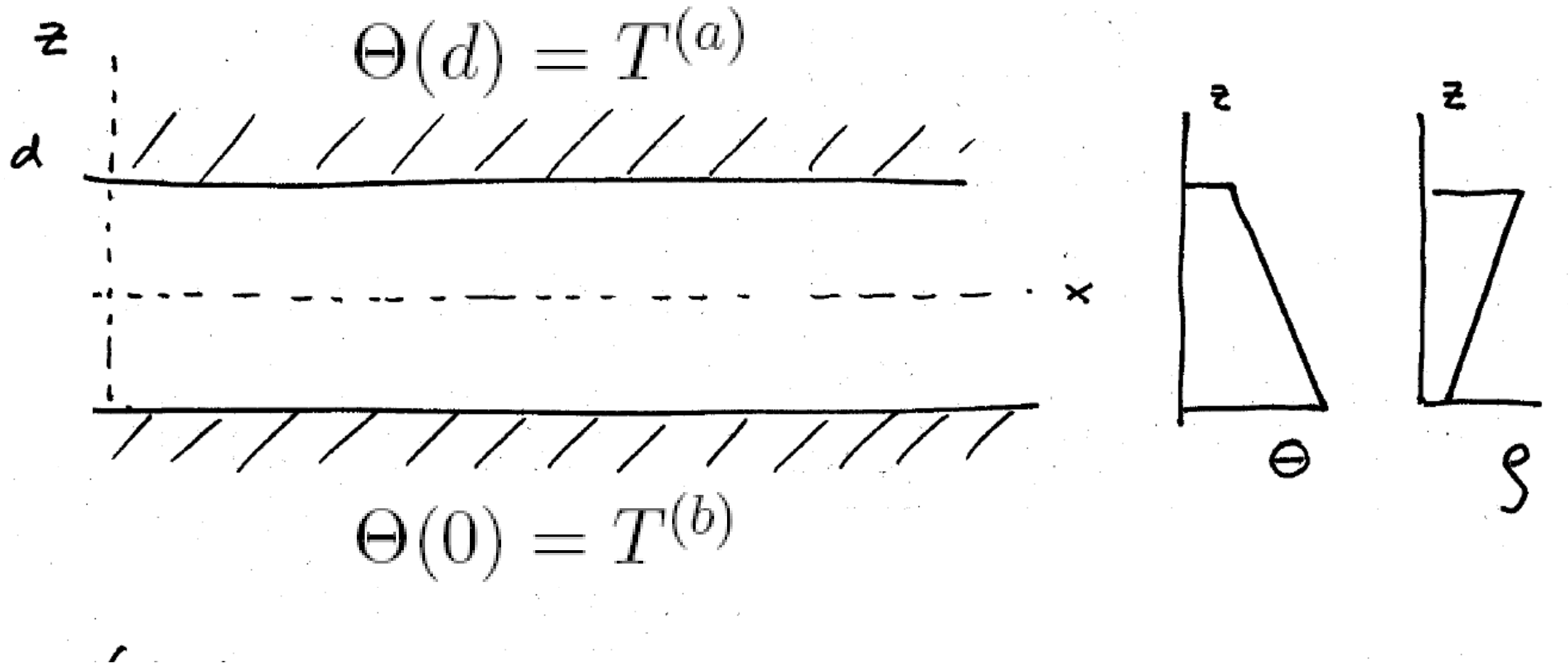
Benard-Marangoni

Kelvin-Helmholtz

Instability analysis:

1. **Physical mechanism**
2. **Equations and boundary conditions**
3. **Base state**
4. **Linearized equations**
5. **Normal mode expansion**
6. **Dispersion relation**
7. **Analysis of the dispersion relation**

Rayleigh Benard



Rayleigh Benard

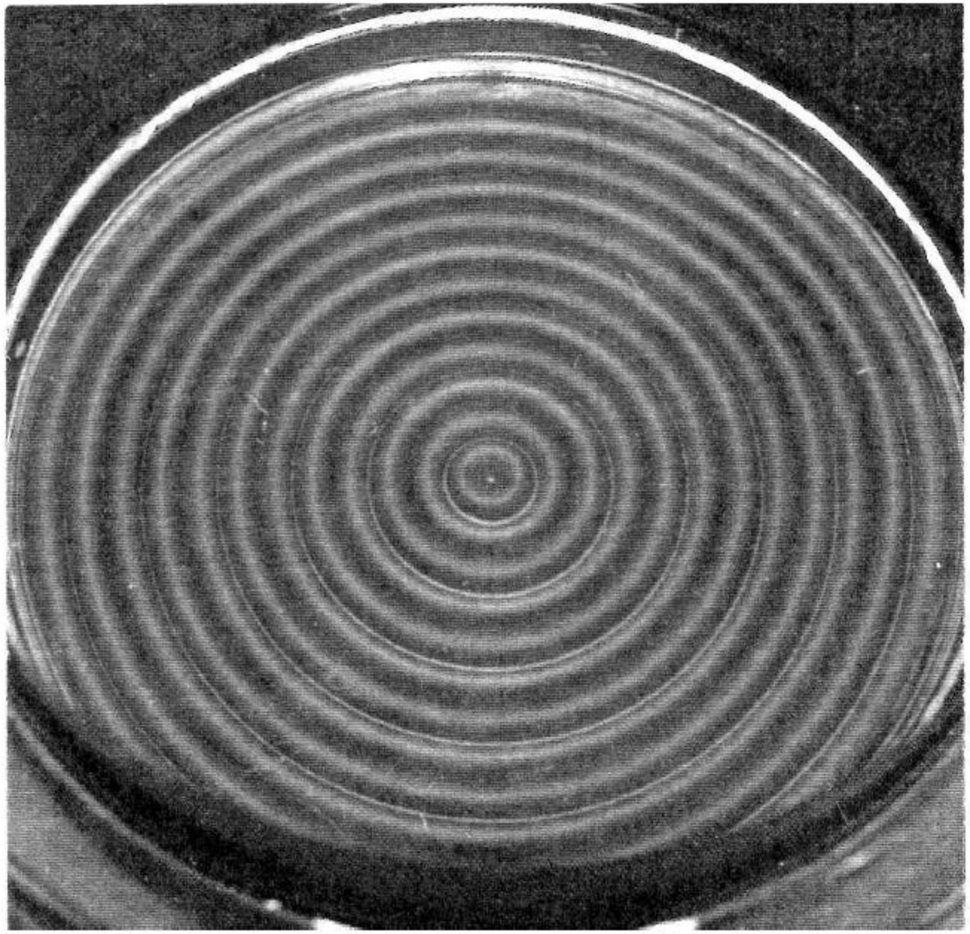
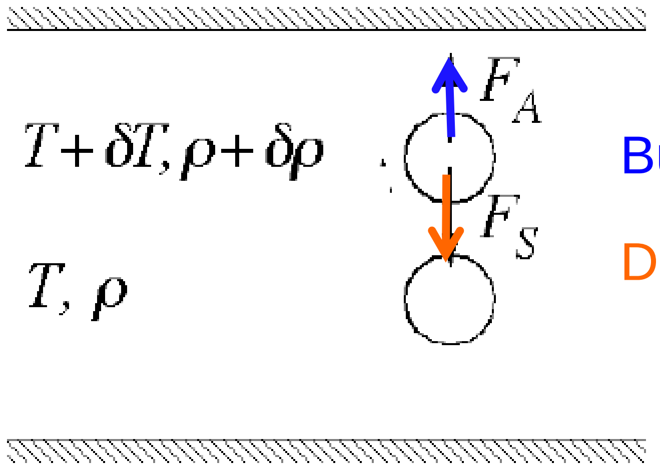


FIG. 2.13 – *Vue de dessus de rouleaux de convection thermique dans une huile silicone, visualisés par de la poudre d'aluminium à travers une paroi supérieure en Plexiglas. La frontière circulaire induit des rouleaux circulaires. $Ra = 2,9\, Ra_c$. Cliché Koschmieder (1974) d'après (Van Dyke 1982).*

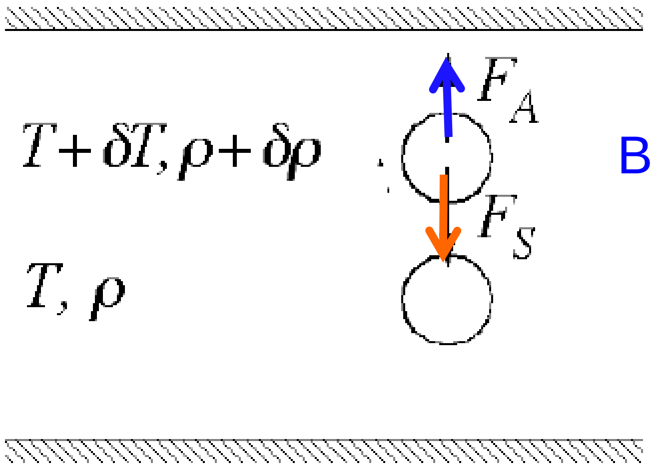
Rayleigh Benard



Buoyancy (Archimedes) force

Drag force

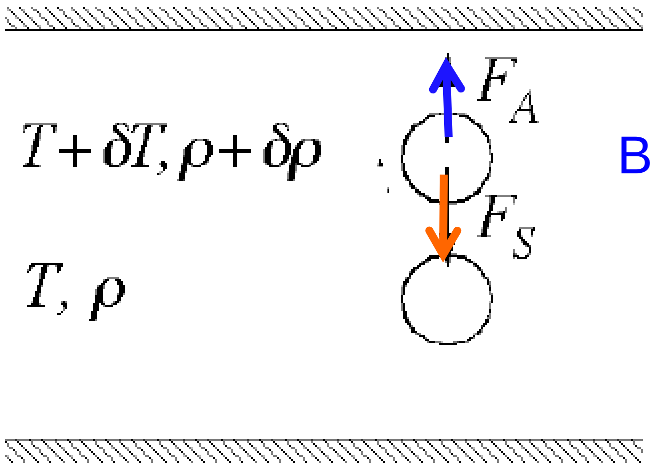
Rayleigh Benard



$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

Rayleigh Benard

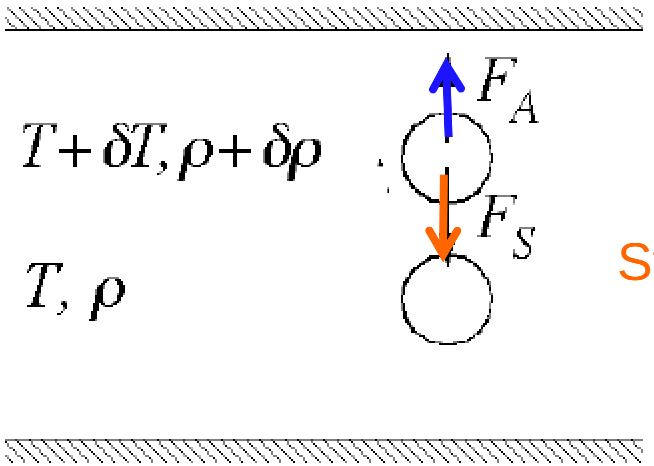


$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

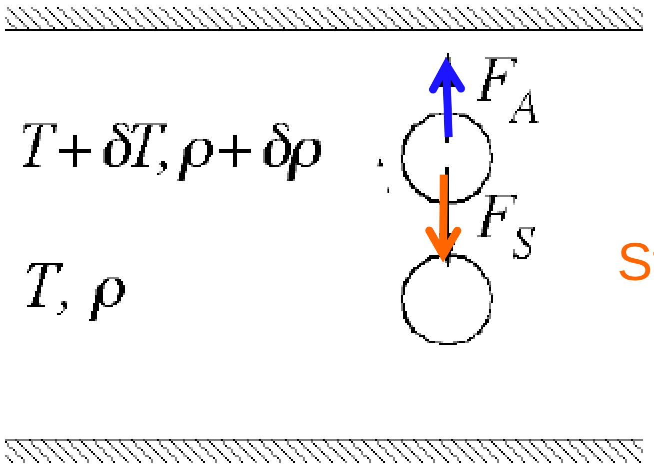
$\delta\rho=\alpha_p\delta T$

Rayleigh Benard



Stokes Drag

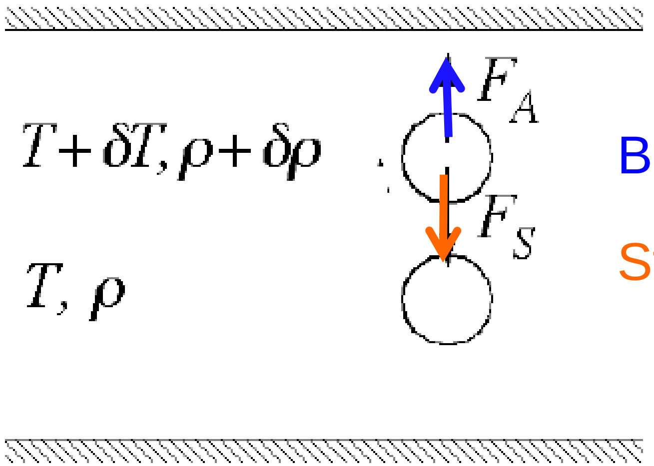
Rayleigh Benard



Stokes Drag

$$\boxed{F_S = -6\pi\mu a v}$$

Rayleigh Benard



$\delta\rho=\alpha_p\delta T$

$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

Stokes Drag

$$\boxed{F_S = -6\pi\mu a v}$$

$$\boxed{v = \frac{2}{9}\,\frac{\alpha_p g a^2\,\delta T}{\mu}}$$

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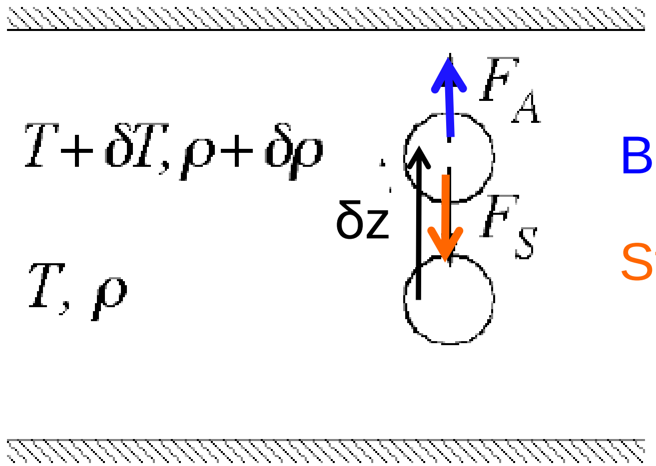
$$\boxed{v = \frac{2}{9}\frac{\alpha_p g a^2\,\delta T}{\nu}}$$



$$\boxed{F_A = \frac{4}{3}\pi a^3 \delta\rho g}$$

Buoyancy (Archimedes) force

Stokes Drag

$$\boxed{F_S = -6\pi\mu a v}$$

Diffusive time scale
(damps the instability)

$\boxed{a^2/\kappa}$

$\boxed{\delta z/v}$

Advective time
(regenerates the instability)

$\delta T/\delta z = (T_a - T_b)/d$

$$\boxed{\frac{a^2}{\kappa} \sim \frac{\delta T\, d}{(T_1 - T_2)v}}$$

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Diffusive time scale
(damps the instability)

$$\boxed{\frac{a^2}{\kappa} \sim \frac{\delta T\, d}{(T_1 - T_2) v}}$$

Advective time
(regenerates the instability)

$$\boxed{v = \frac{2}{9} \frac{\alpha_p g a^2\, \delta T}{\nu}}$$

$\boxed{\text{a~d/2}}$

$$Ra \;:=\; \frac{\alpha_p g (T_1 - T_2) d^3}{\nu \kappa} = \frac{\text{Diffusive time scale}}{\text{Advective time scale}}$$

STABLE

UNSTABLE

Ra

Ra_c=72

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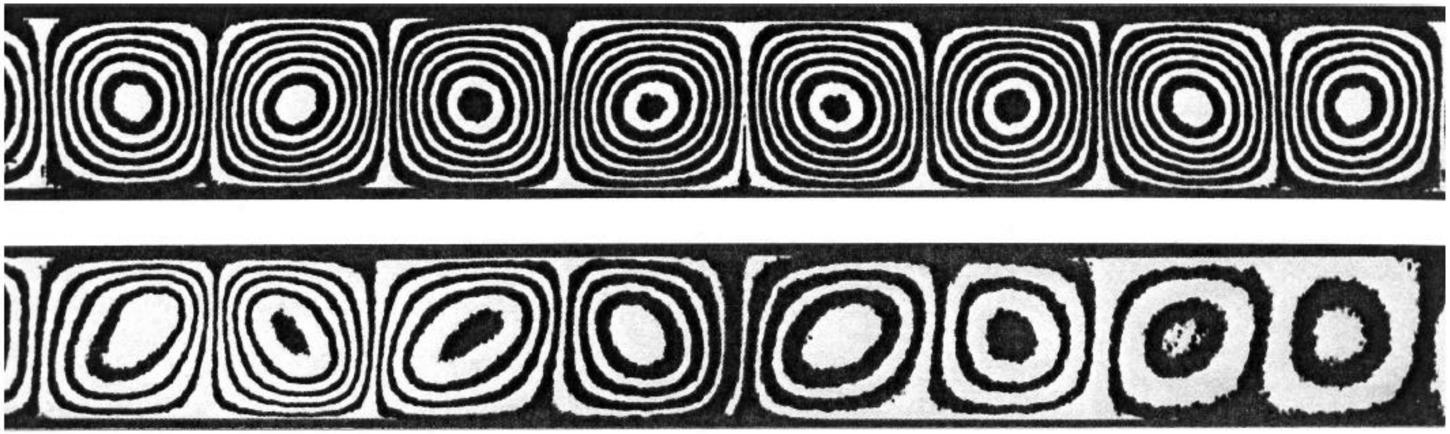


FIG. 2.12 – *Rouleaux de convection thermique dans une huile silicone. Les lignes de courant sont visualisés par interférogrammes. Les dimensions relatives de la boîte sont 10 :4 :1. Haut : rouleaux réguliers orientés parallèlement au petit côté, pour un gradient de température uniforme. Bas : La différence de température et donc l'intensité des rouleaux décroît de la gauche vers la droite. Cliché Oertel & Kirchartz, 1979 (Van Dyke 1982).*

Navier Stokes equations
Oberbeck boussinesq equations

$$
\begin{aligned}
\nabla \cdot \vec{u} &= 0 \\
\rho^* \left[\frac{\partial \vec{u}}{\partial t} + (\nabla \vec{u})\, \vec{u} \right] &= -\rho(T)\, g\, \vec{e}_z - \nabla p + \mu^* \, \nabla^2 \vec{u} \\
C_v^* \left(\frac{\partial T}{\partial t} + \nabla T \cdot \vec{u} \right) &= \chi^* \, \nabla^2 T \quad +2\mu/\rho^* \; D{:}D
\end{aligned}
$$

$$\rho(T) = \rho^* \left[1 - \alpha^* \left(T - T^*\right)\right]$$

H1: the density is constant except in the buoyancy term
H2: the heat release by viscous dissipation is neglected
H3: viscosity and thermal conductivity are assumed constant

Base flow solution
Pure conduction solution

$$\rho^* \left[1 - \alpha^* \left(\Theta - T^{(b)}\right)\right] g + dP/dz = 0$$

$$d^2\Theta/dz^2 = 0$$

$$\Theta(z) = T^{(b)} - \frac{T^{(b)} - T^{(a)}}{d}\, z$$

Perturbation expansion

$$\vec{u}(x,y,z;t) = \vec{U} + \vec{u}'(x,y,z;t) = \vec{u}'(x,y,z;t)$$
$$p(x,y,z;t) = P(z) + p'(x,y,z;t)$$
$$T(x,y,z;t) = \Theta(z) + \theta'(x,y,z;t)$$

$$\nu^* = \mu^*/\rho^*$$

$$\nabla \cdot \vec{u}' = 0$$
$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\,\vec{u}' = \alpha^* \theta' g \vec{e}_z - \nabla p'/\rho^* + \nu^* \nabla^2 \vec{u}'$$
$$\frac{\partial \theta'}{\partial t} + \nabla \Theta \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' = \kappa^* \nabla^2 \theta'$$

$$\kappa^* = \chi^*/C_v^*$$

Perturbation expansion

$$\nabla \cdot \vec{u}' = 0$$

$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\,\vec{u}' = \alpha^* \, \theta' \, g \, \vec{e}_z - \frac{\nabla p'}{\rho^*} + \nu^* \, \nabla^2 \vec{u}'$$

$$\frac{\partial \theta'}{\partial t} - \frac{T^{(b)} - T^{(a)}}{d} \, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' = \kappa^* \, \nabla^2 \theta'$$

Perturbation expansion

$$\boxed{\mathrm{Pr} = \frac{\nu^*}{\kappa^*}}$$

$$\boxed{\mathrm{Pr}}$$

$$\nabla \cdot \vec{u}' = 0$$

$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\, \vec{u}' = \alpha^*\, \theta'\, g\, \vec{e}_z - \frac{\nabla p'}{\rho^*} + \nu^*\, \nabla^2 \vec{u}'$$

$$\frac{\partial \theta'}{\partial t} - \frac{T^{(b)} - T^{(a)}}{d}\, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' = \kappa^*\, \nabla^2 \theta'$$

$$\boxed{\mathrm{Ra} = \frac{\alpha^* g \Delta T d^3}{\nu^* \kappa^*}}$$

Non dimensionalisation

$$L_{ref} = d$$

$$t_{ref} = d^2/\kappa^*$$

$$U_{ref} = L_{ref}/t_{ref} = \kappa^*/d$$

$$P_{ref} = \rho^* U_{ref}^2 = \rho^* \kappa_*^2/d^2$$

$$T_{ref} = \kappa^* \nu^*/\alpha^* g d^3$$

Perturbation expansion

$$\nabla \cdot \vec{u}' = 0$$
$$\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\, \vec{u}' = \alpha^*\, \theta'\, g\, \vec{e}_z - \frac{\nabla p'}{\rho^*} + \nu^*\, \nabla^2 \vec{u}'$$
$$\frac{\partial \theta'}{\partial t} - \frac{T^{(b)} - T^{(a)}}{d}\, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' = \kappa^*\, \nabla^2 \theta'$$

Non dimensionalisation

$$\begin{aligned}
\nabla \cdot \vec{u}' &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\, \vec{u}' &= \mathrm{Pr}\, \theta' \vec{e}_z - \nabla p' + \mathrm{Pr}\, \nabla^2 \vec{u}' \\
\frac{\partial \theta'}{\partial t} - \mathrm{Ra}\, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' &= \nabla^2 \theta'
\end{aligned}$$

Linearisation

$$\begin{aligned}
\nabla \cdot \vec{u}' &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + (\nabla \vec{u}')\, \vec{u}' &= \mathrm{Pr}\, \theta' \vec{e}_z - \nabla p' + \mathrm{Pr}\, \nabla^2 \vec{u}' \\
\frac{\partial \theta'}{\partial t} - \mathrm{Ra}\, \vec{e}_z \cdot \vec{u}' + \nabla \theta' \cdot \vec{u}' &= \nabla^2 \theta'
\end{aligned}$$

Linearisation

$$
\begin{aligned}
\nabla \cdot \vec{u}' &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + \cancel{(\nabla \vec{u}')\,\vec{u}'} &= \mathrm{Pr}\,\theta' \vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}' \\
\frac{\partial \theta'}{\partial t} - \mathrm{Ra}\,\vec{e}_z \cdot \vec{u}' + \cancel{\nabla \theta' \cdot \vec{u}'} &= \nabla^2 \theta'
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\nabla \cdot \vec{u}' = 0 \\
&\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}' \\
&\frac{\partial \theta'}{\partial t} = \mathrm{Ra}\,\vec{e}_z \cdot \vec{u}' + \nabla^2 \theta'
\end{aligned}
$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \Pr \theta' \, \vec{e}_z - \nabla p' + \Pr \nabla^2 \vec{u}'$$

$$\nabla \times \Downarrow$$

$$\partial(\nabla \times \vec{u}')/\partial t = \Pr \nabla \times (\theta' \, \vec{e}_z) + \Pr \nabla^2 (\nabla \times \vec{u}')$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

$$\nabla \times \Big\Downarrow$$

$$\partial(\nabla \times \vec{u}')/\partial t = \mathrm{Pr}\,\nabla \times (\theta'\,\vec{e}_z) + \mathrm{Pr}\,\nabla^2 (\nabla \times \vec{u}')$$

$$\nabla \times \Big\Downarrow$$

$$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times \nabla \times \vec{u}' = \nabla(\nabla \cdot \vec{u}') - \nabla^2 \vec{u}'$$

$$\nabla \times \nabla \times \theta' \vec{e}_z = \nabla(\partial \theta'/\partial z) - \nabla^2 \theta' \vec{e}_z$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\, \theta'\, \vec{e}_z - \nabla p' + \mathrm{Pr}\, \nabla^2 \vec{u}'$$

$$\nabla \times \Downarrow$$

$$\partial(\nabla \times \vec{u}')/\partial t = \mathrm{Pr}\, \nabla \times (\theta'\, \vec{e}_z) + \mathrm{Pr}\, \nabla^2 (\nabla \times \vec{u}')$$

$$\nabla \times \Downarrow$$

$$\nabla^2 \partial \vec{u}' / \partial t = \mathrm{Pr}\, \Big(\nabla^2 \theta' \vec{e}_z - \nabla(\partial \theta' / \partial z) \Big) + \mathrm{Pr}\, \nabla^4 \vec{u}'$$

$$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times \nabla \times \vec{u}' = \nabla(\nabla \cdot \vec{u}') - \nabla^2 \vec{u}'$$

$$\nabla \times \nabla \times \theta' \vec{e}_z = \nabla(\partial \theta' / \partial z) - \nabla^2 \theta' \vec{e}_z$$

Change of variables

$$\frac{\partial \vec{u}'}{\partial t} = \mathrm{Pr}\,\theta'\,\vec{e}_z - \nabla p' + \mathrm{Pr}\,\nabla^2 \vec{u}'$$

$$\nabla\times \Big\Downarrow$$

$$\partial(\nabla \times \vec{u}')/\partial t = \mathrm{Pr}\,\nabla \times (\theta'\,\vec{e}_z) + \mathrm{Pr}\,\nabla^2(\nabla \times \vec{u}')$$

$$\nabla\times \Big\Downarrow$$

$$\nabla^2 \partial \vec{u}'/\partial t = \mathrm{Pr}\,\Big(\nabla^2 \theta' \vec{e}_z - \nabla(\partial\theta'/\partial z)\Big) + \mathrm{Pr}\,\nabla^4 \vec{u}'$$

$$\boxed{\nabla^2 \frac{\partial u'_z}{\partial t} = \mathrm{Pr}\,\nabla^4 u'_z + \mathrm{Pr}\,\nabla_h^2 \theta'} \qquad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Reduction to two equations

$$\begin{aligned}\nabla^2 \frac{\partial u'_z}{\partial t} &= \mathrm{Pr}\,\nabla^4 u'_z + \mathrm{Pr}\,\nabla_h^2 \theta' \\ \frac{\partial \theta'}{\partial t} &= \mathrm{Ra}\,u'_z + \nabla^2 \theta',\end{aligned}$$

$$\boldsymbol{M}\frac{\partial \phi}{\partial t} = \boldsymbol{L}\phi$$

$$\phi = \left\{ \begin{array}{c} u'_z \\ \theta' \end{array} \right\}; \quad \boldsymbol{M} = \left[\begin{array}{cc} \nabla^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \boldsymbol{L} = \left[\begin{array}{cc} \mathrm{Pr}\,\nabla^4 & \mathrm{Pr}\,\nabla_h^2 \\ \mathrm{Ra} & \nabla^2 \end{array} \right]$$

Normal mode expansion

$$\phi = \widehat{\phi}(z) e^{st + i(k_x x + k_y y)}$$

s=-iω

Normal mode expansion

$$\boldsymbol{M}\frac{\partial \boldsymbol{\phi}}{\partial t} = \boldsymbol{L}\boldsymbol{\phi}$$

$$\phi = \left\{ \begin{array}{c} u'_z \\ \theta' \end{array} \right\}; \quad \boldsymbol{M} = \left[\begin{array}{cc} \nabla^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \boldsymbol{L} = \left[\begin{array}{cc} \mathrm{Pr}\,\nabla^4 & \mathrm{Pr}\,\nabla_h^2 \\ \mathrm{Ra} & \nabla^2 \end{array} \right]$$

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(z)e^{st+i(k_x x + k_y y)}$$

$s=-i\omega$

$$\vec{k} = \{k_x, k_y\}$$

$$k = |\vec{k}|$$

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \left[\begin{array}{cc} D^2 - k^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \widehat{\boldsymbol{L}} = \left[\begin{array}{cc} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{array} \right]$$

Normal mode expansion

$$\boldsymbol{M}\frac{\partial \boldsymbol{\phi}}{\partial t} = \boldsymbol{L}\phi$$

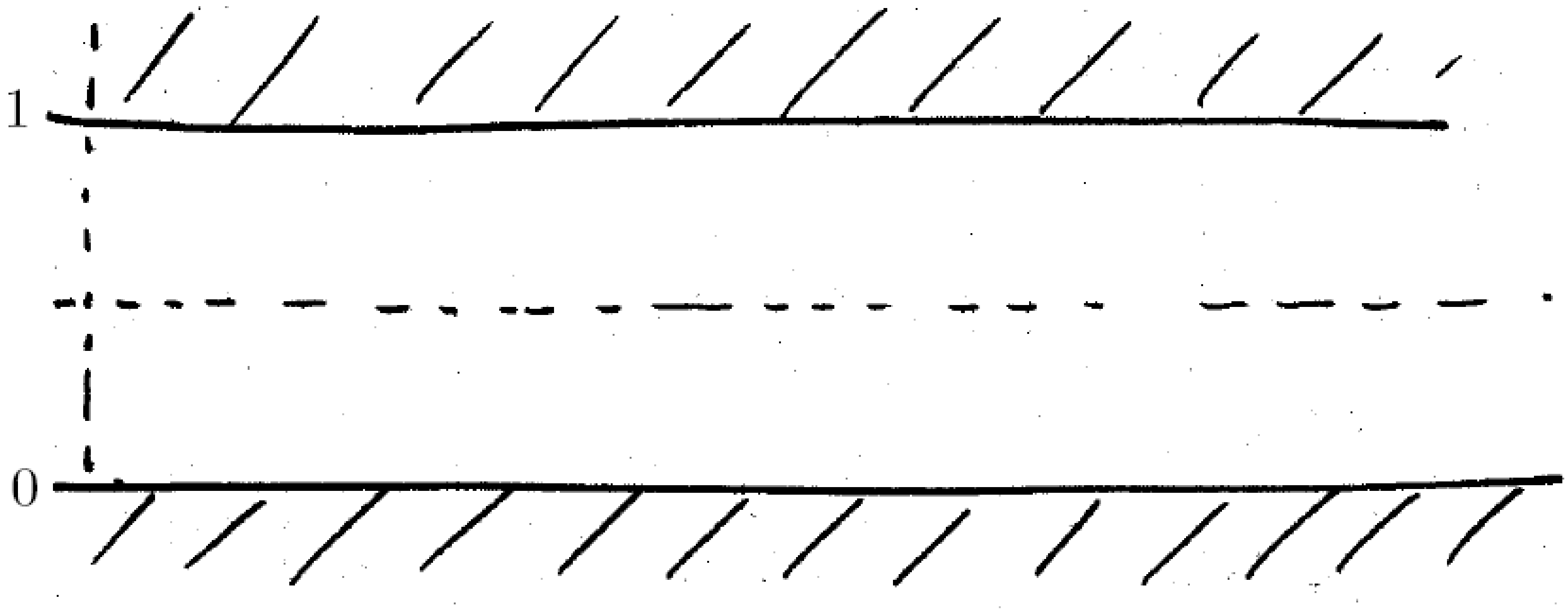
$$\phi = \left\{ \begin{array}{c} u'_z \\ \theta' \end{array} \right\}; \quad \boldsymbol{M} = \left[\begin{array}{cc} \nabla^2 & 0 \\ 0 & 1 \end{array} \right]; \quad \boldsymbol{L} = \left[\begin{array}{cc} \mathrm{Pr}\,\nabla^4 & \mathrm{Pr}\,\nabla_h^2 \\ \mathrm{Ra} & \nabla^2 \end{array} \right]$$

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(z)e^{st+i(k_x x+k_y y)}$$ *s*=-*iω* $$\vec{k} = \{k_x, k_y\}$$

$$k = |\vec{k}|$$

Thermal boundary conditions

$$\widehat{\theta}(z=1)=0$$
$$\theta'(x,y,z=1;t)=0$$
$$T(x,y,z=1;t)=\Theta(z=1)=T^{(a)}$$

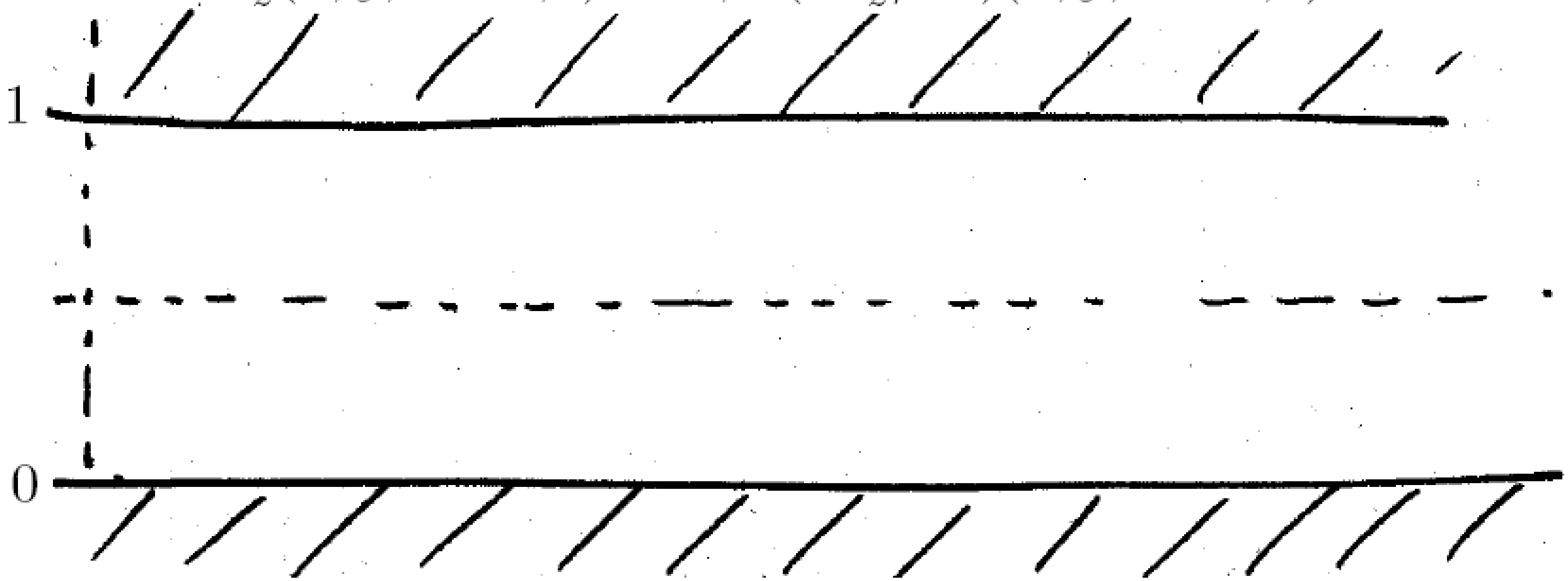


$$T(x,y,z=0;t)=\Theta(z=0)=T^{(b)}$$
$$\theta'(x,y,z=0;t)=0$$
$$\widehat{\theta}(z=0)=0$$

Rigid boundary conditions

$$\widehat{u}_z(z=1)=0\,, \quad (D\widehat{u}_z)(z=1)=0$$

$$u_z'(x,y,z=1;t)=0\,, \quad (\partial u_z'/\partial z)(x,y,z=1;t)=0$$



$$u_z'(x,y,z=0;t)=0\,, \quad (\partial u_z'/\partial z)(x,y,z=0;t)=0$$

$$\widehat{u}_z(z=0)=0\,, \quad (D\widehat{u}_z)(z=0)=0$$

Stress free boundary conditions

$$\sigma_{xz} = 0$$

$$\mu(\partial u_x/\partial z + \partial u_z/\partial x) = 0$$

$$\boxed{\partial u_x/\partial z = 0}$$

Stress free boundary conditions

$$\sigma_{xz} = 0$$

$$\mu(\partial u_x/\partial z + \partial u_z/\partial x) = 0$$

$$\boxed{\partial u_x/\partial z = 0}$$

$$\sigma_{yz} = 0$$

$$\boxed{\partial u_y/\partial z = 0}$$

Stress free boundary conditions

$$\sigma_{xz} = 0$$

$$\mu(\partial u_x/\partial z + \partial u_z/\partial x) = 0$$

$$\boxed{\partial u_x/\partial z = 0}$$

$$\sigma_{yz} = 0$$

$$\boxed{\partial u_y/\partial z = 0}$$

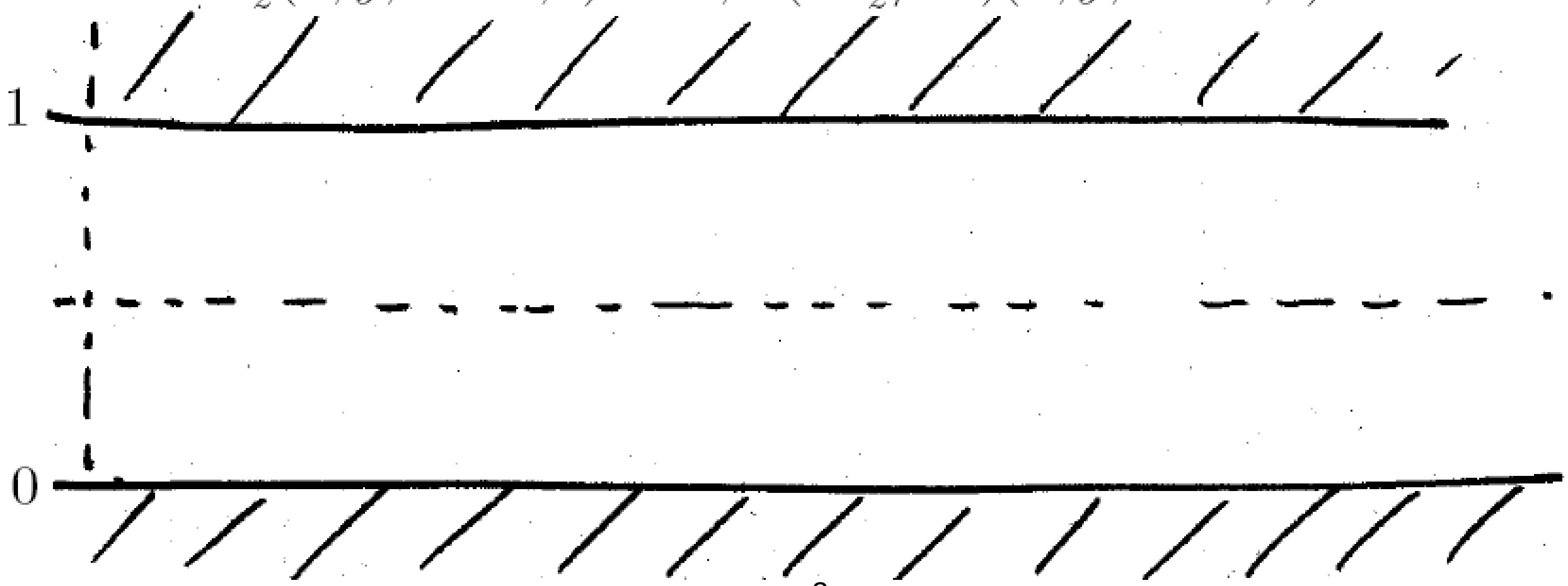
$$\partial/\partial z \quad \partial u_x/\partial x + \partial u_y/\partial y + \partial u_z/\partial z = 0$$

$$\boxed{\partial^2 u_z/\partial z^2 = 0}$$

Free stress boundary conditions

$$\widehat{u}_z(z=1)=0\,, \quad (D^2\widehat{u}_z)(z=1)=0$$

$$u'_z(x,y,z=1;t)=0\,, \quad (\partial^2 u'_z/\partial z^2)(x,y,z=1;t)=0$$



$$u'_z(x,y,z=0;t)=0\,, \quad (\partial^2 u'_z/\partial z^2)(x,y,z=0;t)=0$$

$$\widehat{u}_z(z=0)=0\,, \quad (D^2\widehat{u}_z)(z=0)=0$$

Solution ansatz (which satisfies the b.c.)

$$\widehat{\phi} = \widetilde{\phi_n} \sin(n\pi z)$$

$$s\widetilde{\boldsymbol{M}}\widetilde{\phi} = \widetilde{\boldsymbol{L}}\widetilde{\phi}$$

$$\widetilde{\boldsymbol{M}} = \begin{bmatrix} -k_n^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widetilde{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\, k_n^4 & -\mathrm{Pr}\, k^2 \\ \mathrm{Ra} & -k_n^2 \end{bmatrix}$$

$$k_n^2 := k^2 + (n\pi)^2$$

Solution ansatz (which satisfies the b.c.)

$$\widehat{\phi} = \widetilde{\phi}_n \sin(n\pi z)$$

$$s\widetilde{\boldsymbol{M}}\widetilde{\phi} = \widetilde{\boldsymbol{L}}\widetilde{\phi}$$

$$\widetilde{\boldsymbol{M}} = \begin{bmatrix} -k_n^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widetilde{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\, k_n^4 & -\mathrm{Pr}\, k^2 \\ \mathrm{Ra} & -k_n^2 \end{bmatrix}$$

$$k_n^2 := k^2 + (n\pi)^2$$

$$\det\left(s\widetilde{\boldsymbol{M}} - \widetilde{\boldsymbol{L}}\right) = 0$$

$$\boxed{s^2 + (1 + \mathrm{Pr})\, k_n^2 s + \mathrm{Pr}\left(k_n^4 - \mathrm{Ra}\frac{k^2}{k_n^2}\right) = 0}$$

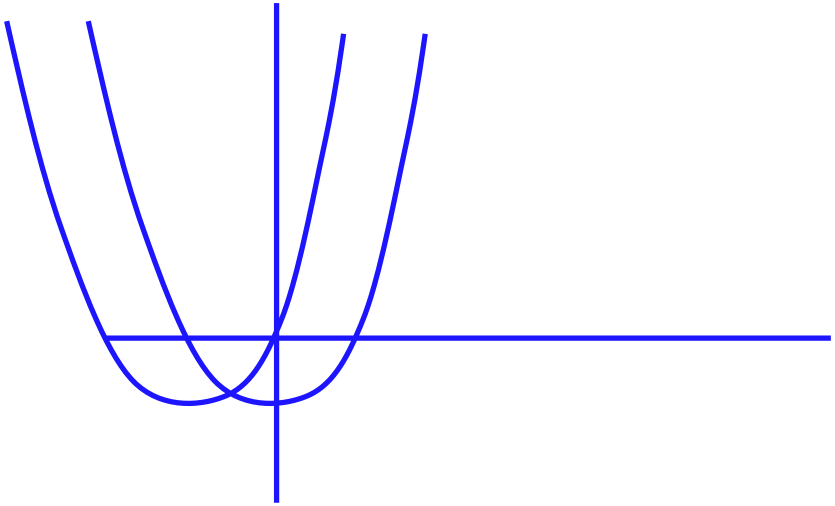
Dispersion relation

$$s^2 + (1 + \mathrm{Pr})\, k_n^2 s + \mathrm{Pr}\left(k_n^4 - \mathrm{Ra}\frac{k^2}{k_n^2}\right) = 0$$

<0

Condition for instability

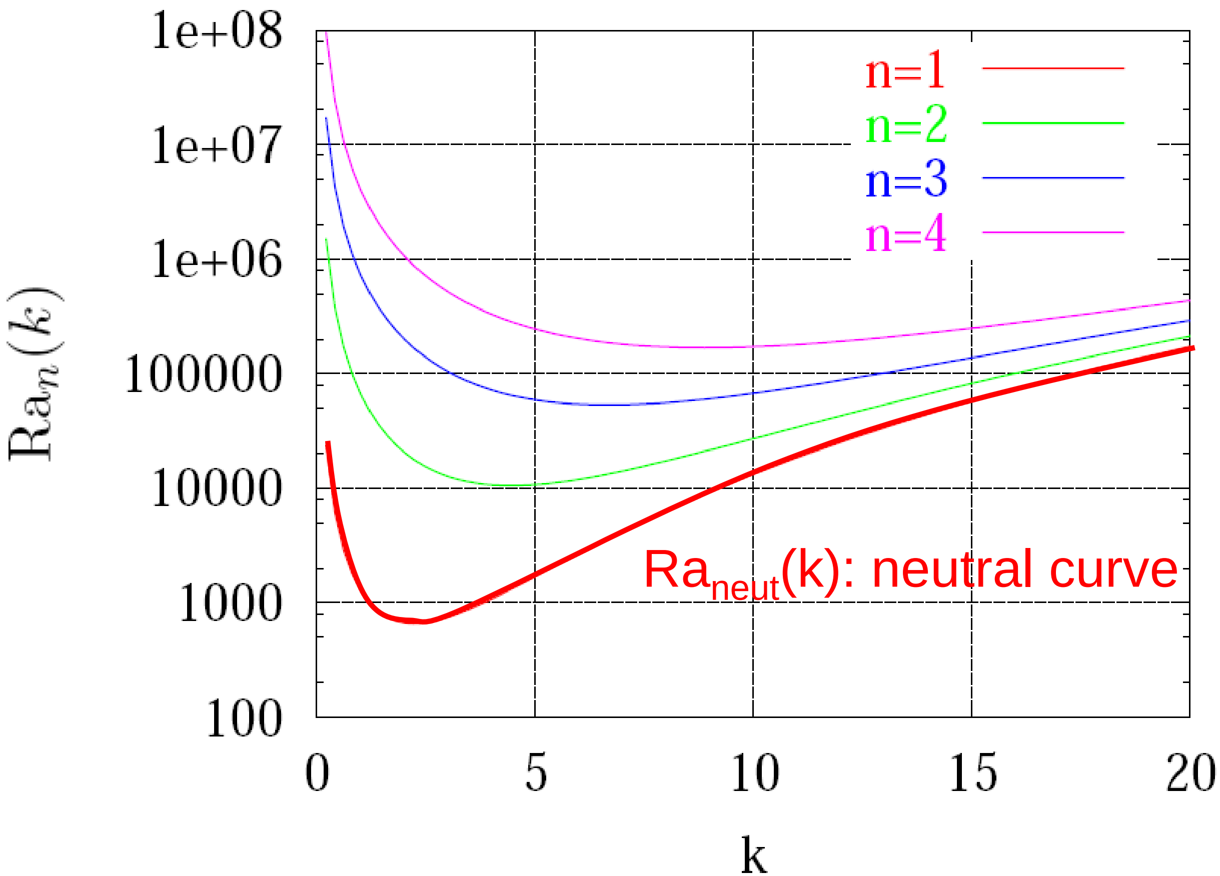
$$\mathrm{Ra} > \frac{(k^2 + n^2\pi^2)^3}{k^2} = \mathrm{Ra}_n(k)$$

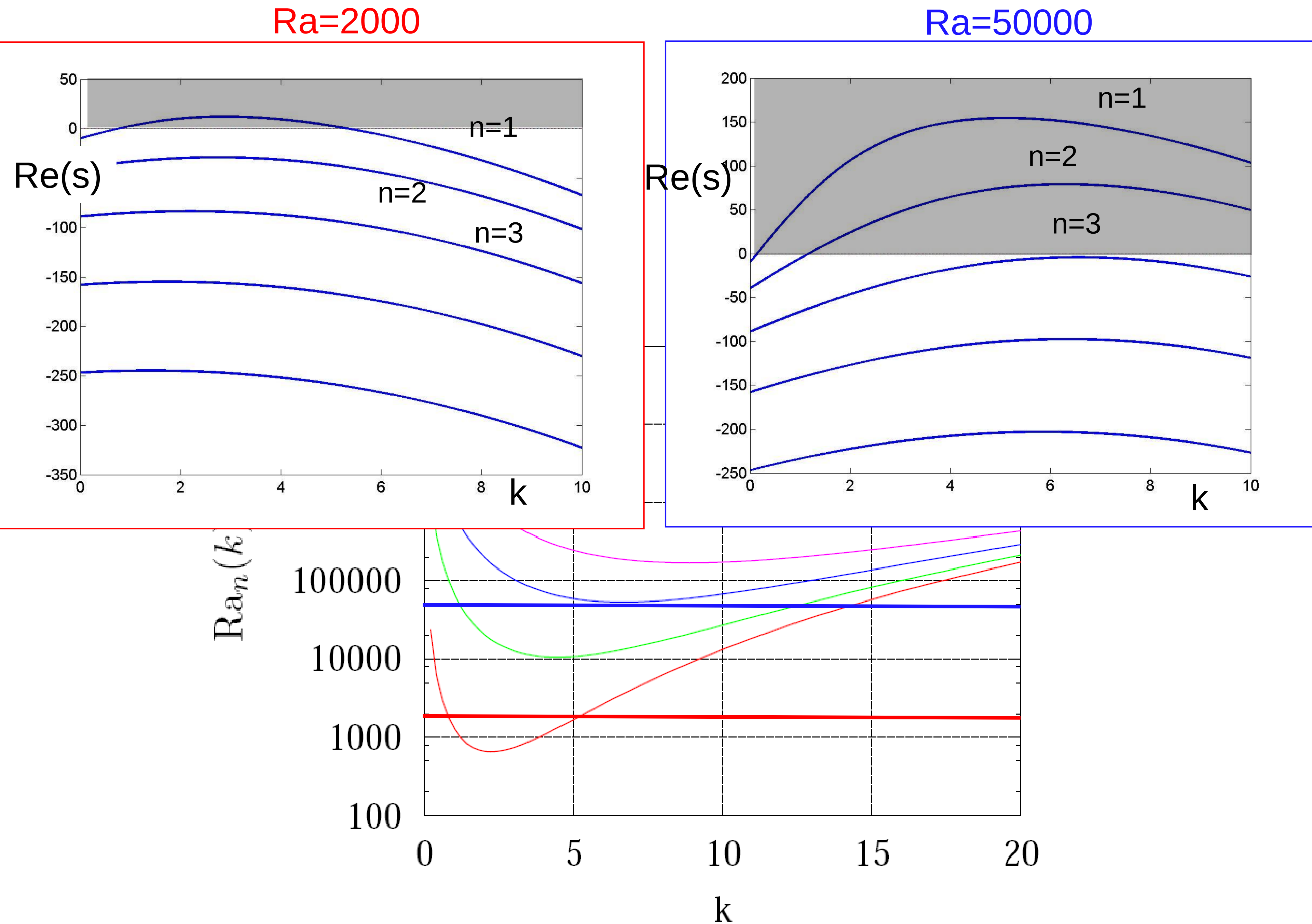


The parabola points upwards
The sum of the roots is negative

Condition for instability

$$\mathrm{Ra} > \frac{(k^2 + n^2\pi^2)^3}{k^2} = \mathrm{Ra}_n(k)$$





Ra=2000
Ra=50000
Re(s)
n=1
n=2
n=3
k
Re(s)
n=1
n=2
n=3
k
$\mathrm{Ra}_n(k)$
k

Marginal curve for instability

$$n = 1 \qquad \mathrm{Ra}_{neut}(k) = \frac{(k^2 + \pi^2)^3}{k^2}$$

Minimum minimorum

$$\mathrm{Ra}_c = \min_k \mathrm{Ra}_{neut}(k)$$

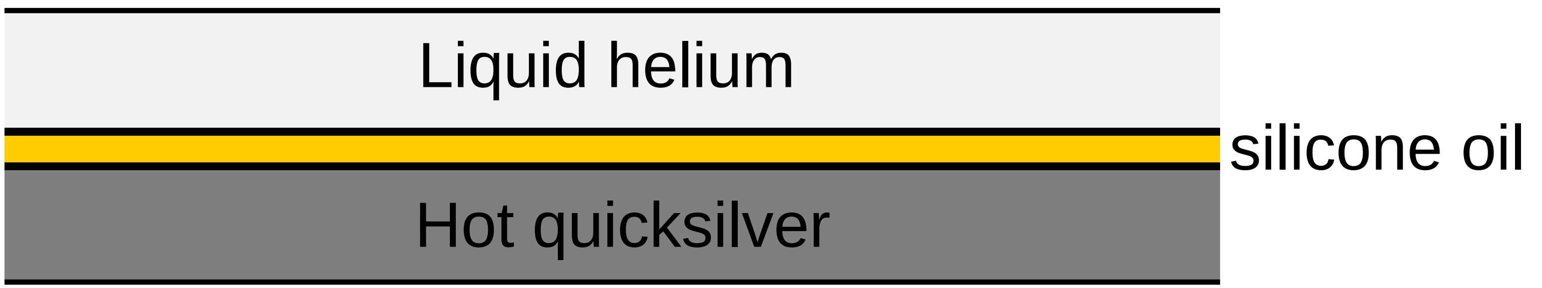
$$d\mathrm{Ra}_{neut}/dk = 0$$

$$\mathrm{Ra}_c = 27\pi^4/4 \sim 657.5\ , \quad k = \pi/\sqrt{2} \sim 2.22$$

Stress-free boundary conditions

$$\mathrm{Ra}_c = 27\pi^4/4 \sim 657.5\ , \quad k = \pi/\sqrt{2} \sim 2.22$$

Experimentally verified only in 69 by Goldstein and Graham



Rigid boundary conditions

$$\mathrm{Ra}_c = 1707.76 \text{ and } k_c = 3.163.$$

Calculated in ‘40 by Pellew and Southwell

1. Solve Rayleigh-Benard eigenvalue problem

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \left[\begin{array}{cc} D^2 - k^2 & 0 \\ 0 & 1 \end{array}\right]; \quad \widehat{\boldsymbol{L}} = \left[\begin{array}{cc} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{array}\right]$$

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(z)e^{st+i(k_x x + k_y y)}$$ s=-iω $$\vec{k} = \{k_x, k_y\}$$ $$k = |\vec{k}|$$

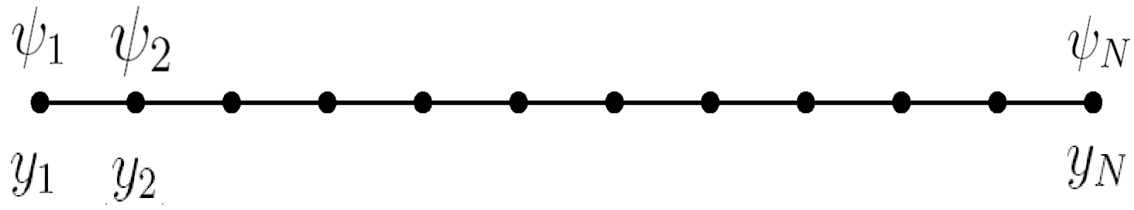
With rigid boundary conditions

$$\widehat{\theta}(z=0) = 0$$
$$\widehat{u}_z(z=0) = 0\,, \quad (D\widehat{u}_z)(z=0) = 0$$

$$\widehat{\theta}(z=1) = 0$$
$$\widehat{u}_z(z=1) = 0\,, \quad (D\widehat{u}_z)(z=1) = 0$$

How to solve Rayleigh-Benard's equation?

Method 1: Finite differences of order 1.



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \qquad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \end{pmatrix}$$

How to solve Rayleigh-Benard's equation?

Method 1: Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

Finite difference scheme

Central finite difference scheme second order

Finite difference formulas for first derivatives

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \dots$$

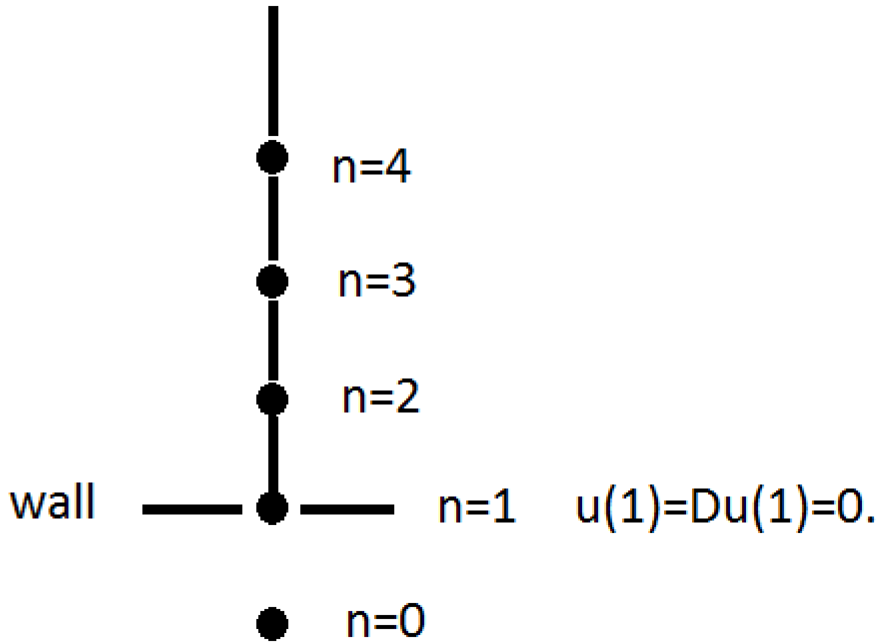
Finite difference formulas for second derivatives

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} + \dots$$

Finite difference formulas for fourth derivatives

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{\Delta x^4} - \frac{\Delta x^2}{6}\left.\frac{\partial^6 u}{\partial x^6}\right|_{x_i} + \dots$$

Rigid boundary condition



Boundary condition

$$u(1) = 0.$$
$$u(2) = u(0)$$

$$\frac{\partial^2 u(2)}{\partial x^2} = \frac{u(1) - 2u(2) + u(3)}{dx^2} = \frac{-2u(2) + u(3)}{dx^2}$$

$$\frac{\partial^4 u(2)}{\partial x^4} = \frac{u(0) - 4u(1) + 6u(2) - 4u(3) + u(4)}{dx^4} = \frac{7u(2) - 4u(3) + u(4)}{dx^4}$$

D2 =

$$\begin{matrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{matrix}$$

D4 =

$$\begin{matrix} 7 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 7 \end{matrix}$$

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widehat{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{bmatrix}$$

$$s\widehat{\boldsymbol{M}}\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{L}}\widehat{\boldsymbol{\phi}}$$

$$\widehat{\boldsymbol{M}} = \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \widehat{\boldsymbol{L}} = \begin{bmatrix} \mathrm{Pr}\,(D^2 - k^2)^2 & -\mathrm{Pr}\,k^2 \\ \mathrm{Ra} & D^2 - k^2 \end{bmatrix}$$

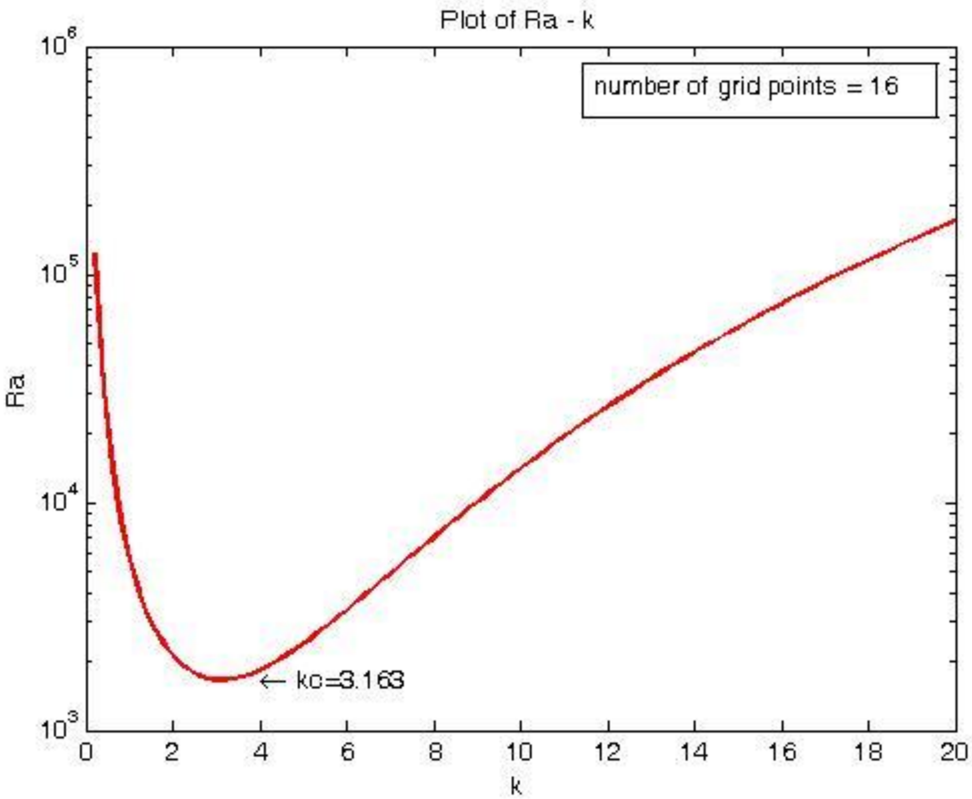
$$s\hat{M}\hat{\phi} = \hat{L}\hat{\phi}$$

$$eig(M^{-1}L) \text{ or } det\left(s\hat{M} - \hat{L}\right) = 0.$$

$$f(s) = det\left(s\hat{M} - \hat{L}\right)$$

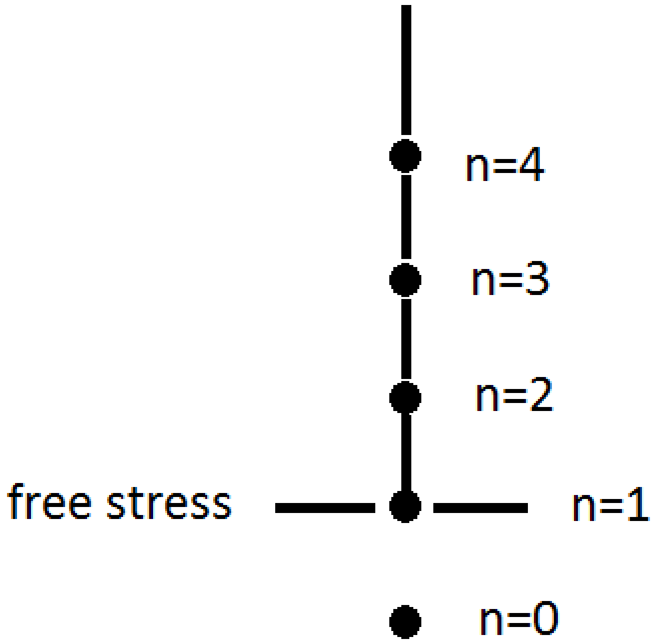
order of polynomial = 2 * number of grid point

Rigid boundary condition



$$\mathrm{Ra}_c = 1707.76 \text{ and } k_c = 3.163.$$

Free stress boundary condition



$u(1) = D^2 u(1) = 0.$

Boundary condition

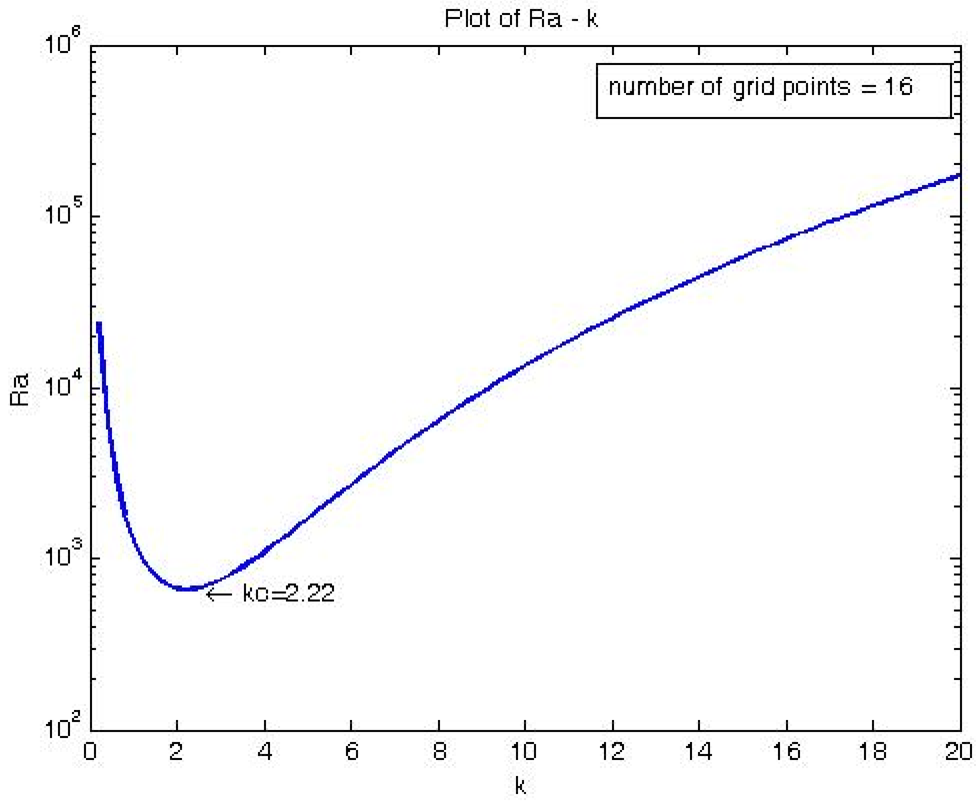
$u(1) = 0.$

$u(2) = -u(0)$

$$\frac{\partial^2 u(2)}{\partial x^2} = \frac{u(1) - 2u(2) + u(3)}{dx^2} = \frac{-2u(2) + u(3)}{dx^2}$$

$$\frac{\partial^4 u(2)}{\partial x^4} = \frac{u(0) - 4u(1) + 6u(2) - 4u(3) + u(4)}{dx^4} = \frac{5u(2) - 4u(3) + u(4)}{dx^4}$$

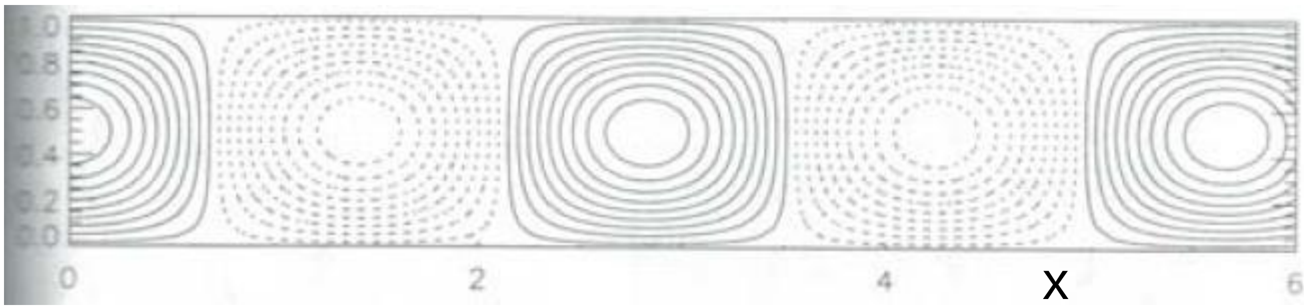
Free stress boundary condition



$$\mathrm{Ra}_c = 27\pi^4/4 \sim 657.5\ , \quad k = \pi/\sqrt{2} \sim 2.22$$

Typical structures : looking at eigenmodes

k_y=0



Rolls

Squares and hexagons

$k_x = k_y = k_c ; A_x = A_y$

$A\sin(\pi z)[\cos(k_c x)+\cos(k_c y)]$ $A\sin(\pi z)[\cos(k_c x)+\cos(k_c(-x/2+3y/2))+\cos(k_c(x/2+3y/2)$

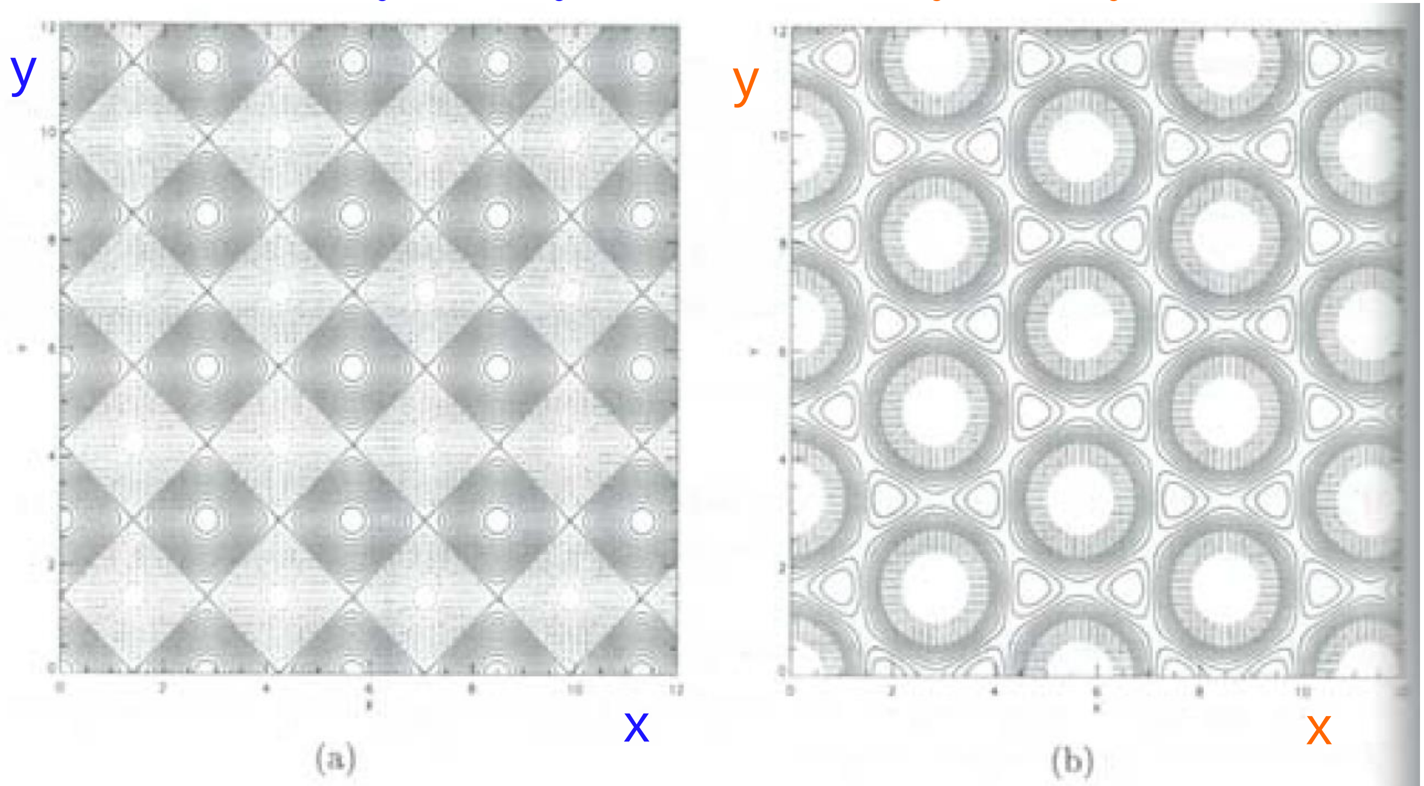


FIG. 7.5 - *Figures de convection en carrés (a) ou en hexagones (b). On a tracé les courbes isothermes : en pointillés $\delta T < 0$, en trait pleins $\delta T > 0$.*

Typical structures : looking at eigenmodes

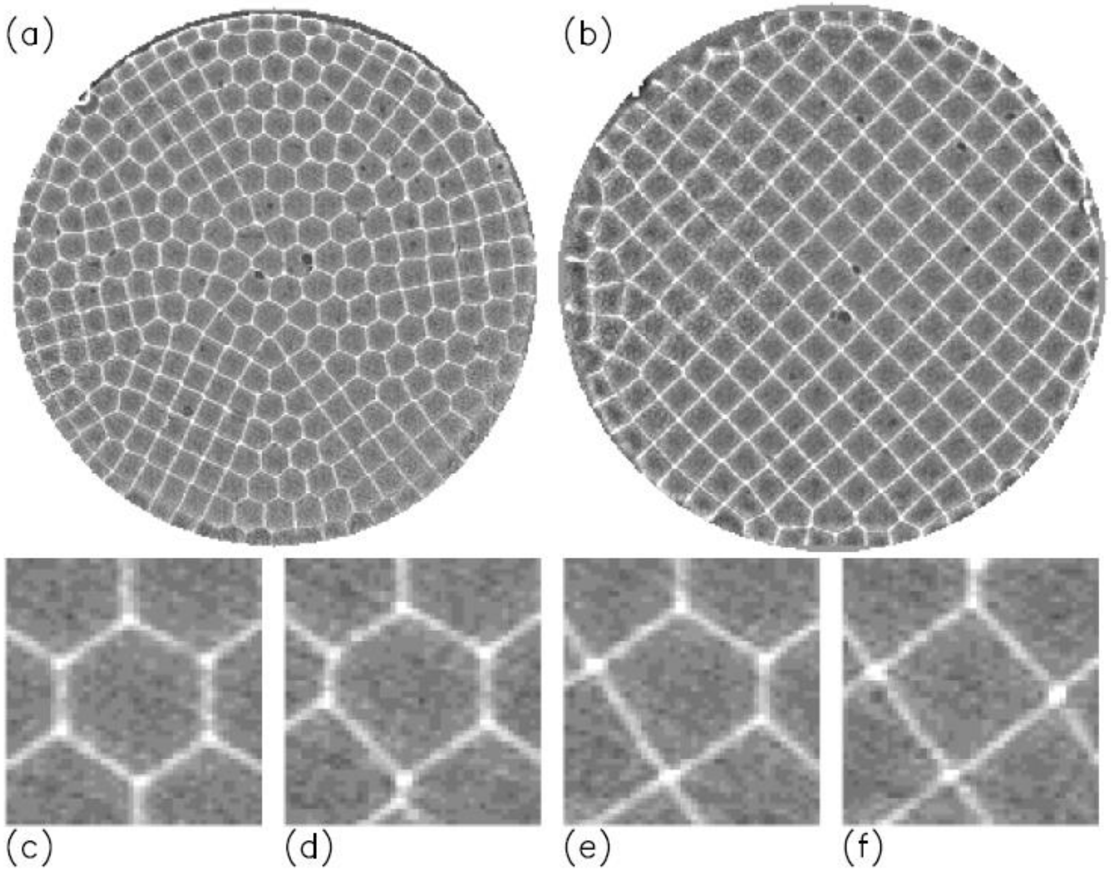


Figure 7 Transition to square patterns in Marangoni convection for $d = 0.072$ cm. (*a*) At $\epsilon = 3.9$, a stationary pattern of mixed hexagonal, pentagonal, and square symmetry is observed. (*b*) At $\epsilon = 7.23$, a stationary, nearly ideal square pattern arises. Close observation of a few cells demonstrates local changes in topology from threefold to fourfold vertices as ϵ is gradually increased from (*c*) 2.67, through (*d*) 2.98, and (*e*) 3.3, to (*f*) 3.89.