Most flows are unstable...

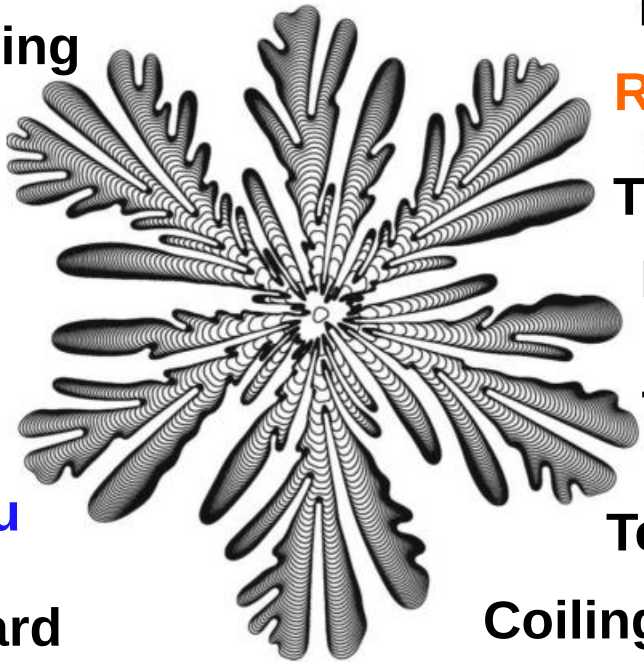
Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor



Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

Tearing instability

Rayleigh-Benard

Coiling instability

Benard-Marangoni

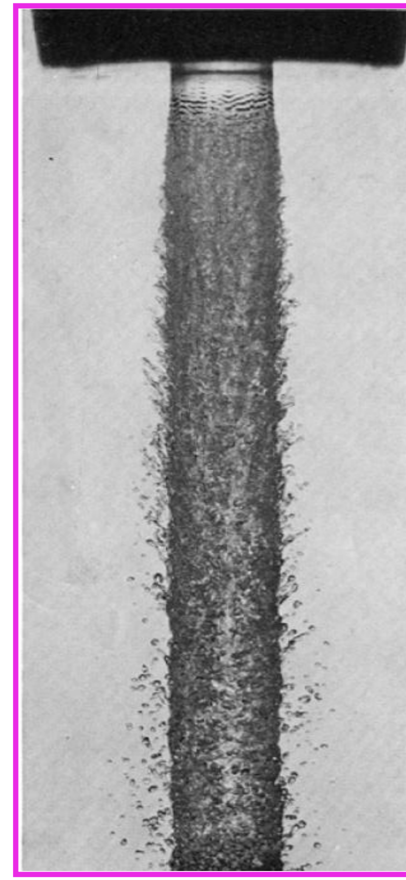
Kelvin-Helmholtz

Laminar ⇨ Instability ⇨ Pattern ⇨ Turbulence

Marmottant and Villermaux (2004)

Rayleigh (1891)

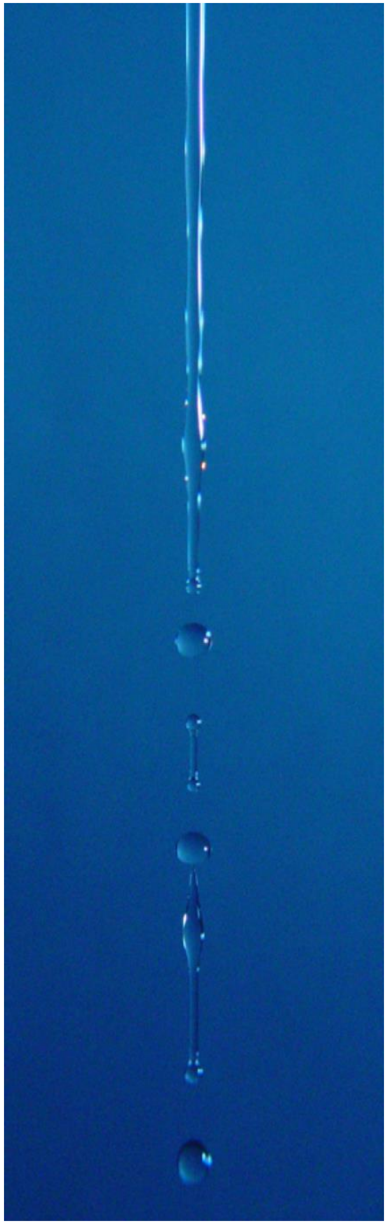
Marmottant and Villermaux (2004)



Hoyt and Taylor (1977)

no fluctuation	⇨	ε	⇨	1	⇨	fluctuation > average
R	⇨	R, λ	⇨	R, λ_1, λ_2	⇨	full spectrum of lengscales
steady	⇨	ω	⇨	ω_1, ω_2	⇨	full frequency spectrum

Rayleigh Plateau Savart instability



Jetting

Jetting

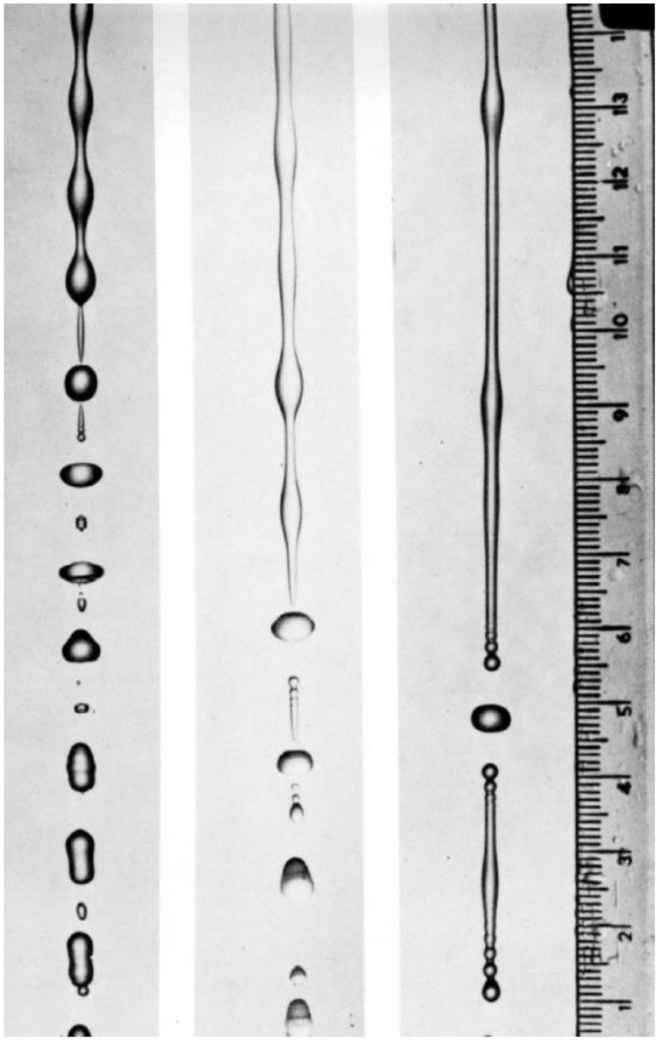
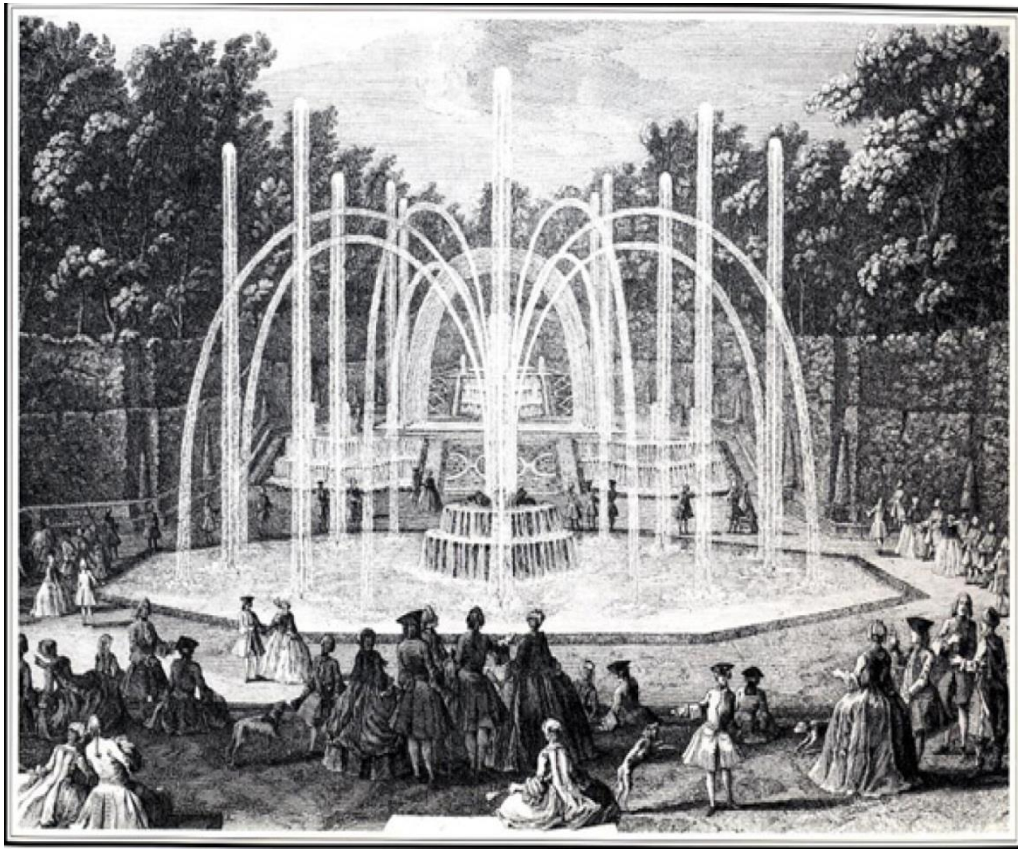


FIG. 2.8 – *Un filet d'eau issu d'un tube de 4 mm de diamètre est perturbé à différentes fréquences par un haut-parleur. Les "longueurs d'onde" du chapelet de gouttes sont de 4,6, 12,5 et 42 diamètres. Cliché Rutland et Jameson 1971 (Van Dyke 1982).*

Dripping

Geneva’s jet

Fascinating…

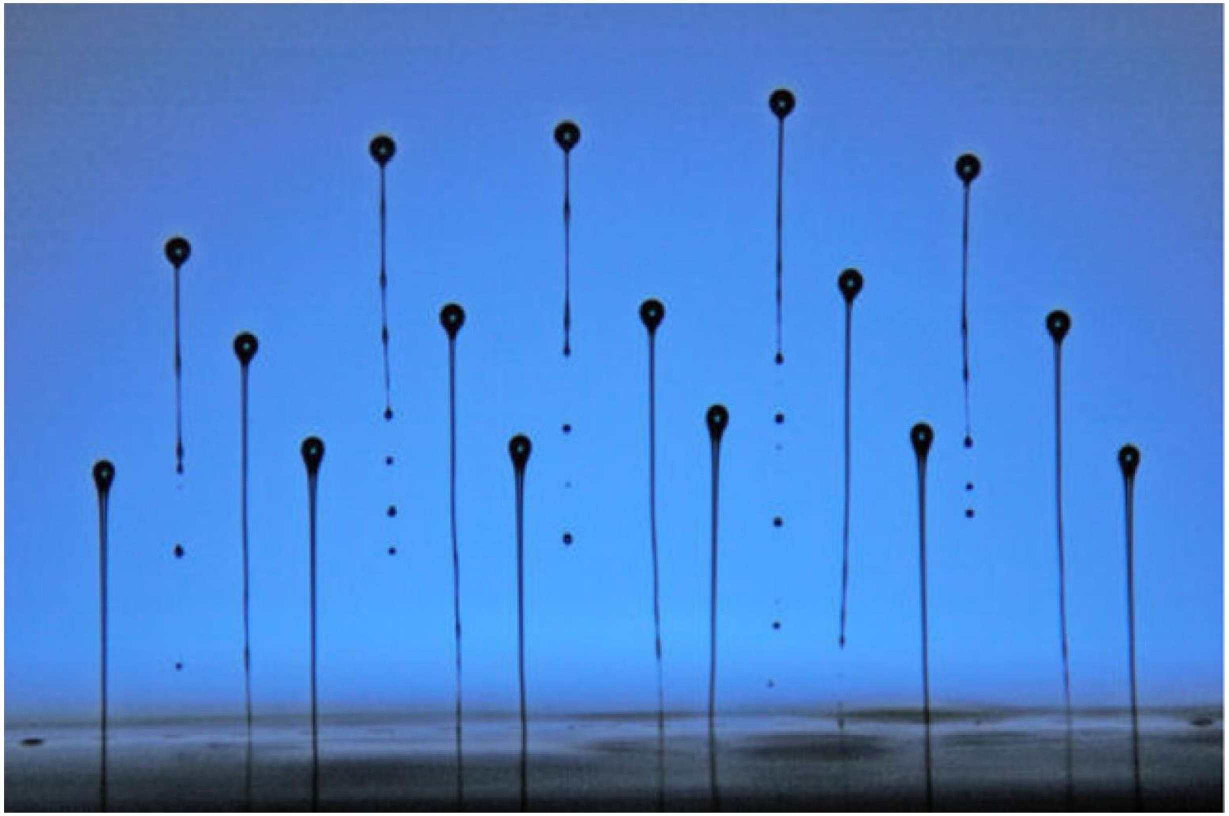


Versailles’ Grandes Eaux (Eggers and Villermaux 2008)

Useful to determine the drop size distribution

Eggers and Villermaux 2008

Useful to determine the drop size distribution



Ink jet printer (Eggers and Villermaux 2008)

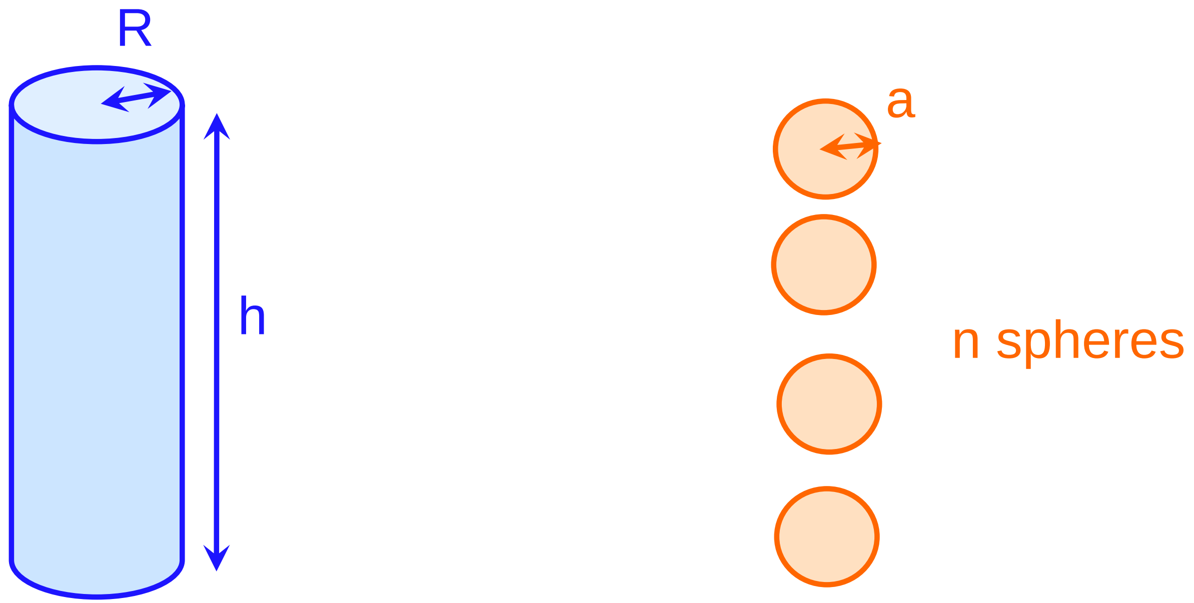
A success story for hydrodynamic linear instability theory

In 1873, Joseph Plateau found experimentally that a vertically falling stream of water will break up into drops if its length is greater than about 3.13 to 3.18 times its diameter.

Later, Lord Rayleigh showed theoretically that a vertically falling column of non-viscous liquid with a circular cross-section should break up into drops if its length exceeded its circumference, or π times its diameter.

Physical mechanism

Surface tension aims at reducing the exposed surface.
For a given volume, compare a cylinder and a collection of spheres



Physical mechanism

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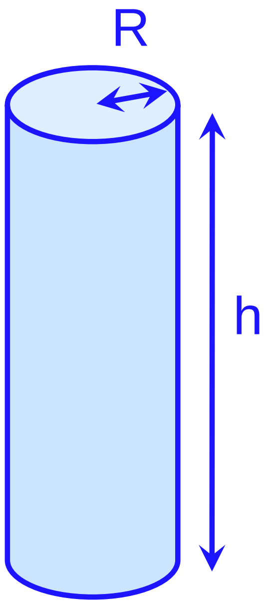
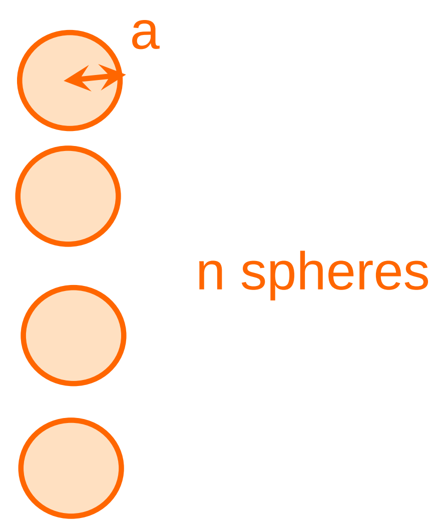
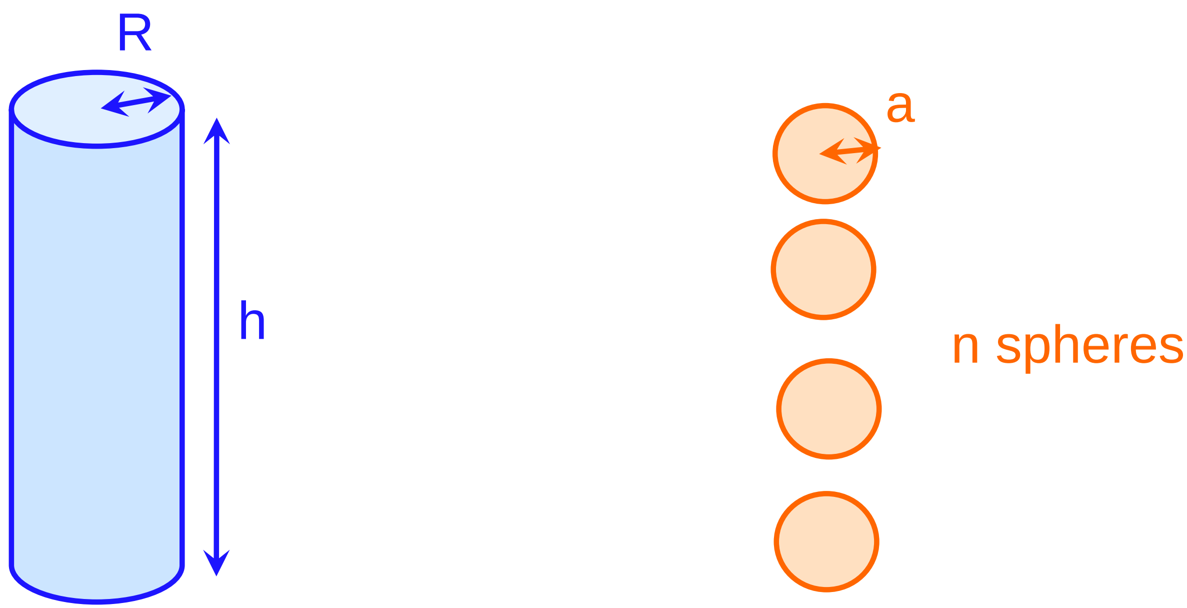




Volume	$\pi R^2 h$	$4/3\pi a^3 n$

Physical mechanism

Surface tension aims at reducing the exposed surface.
For a given volume, compare a cylinder and a collection of spheres

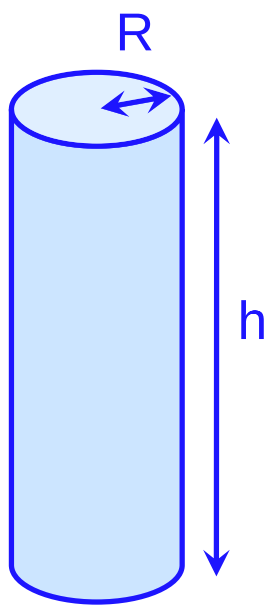


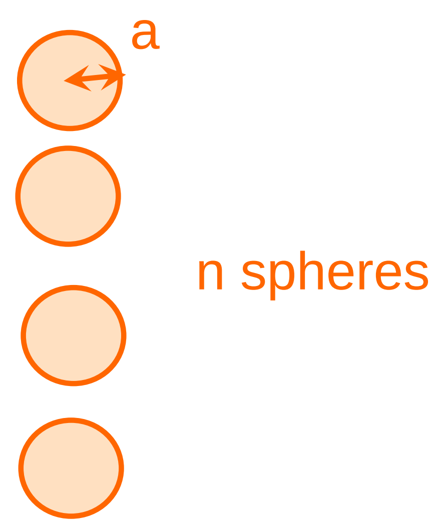
Volume $\pi R^2 h$ $4/3\pi a^3\, n$ $\boxed{n=3R^2h/4a^3}$

Surface?

Physical mechanism

Surface tension aims at reducing the exposed surface.
For a given volume, compare a cylinder and a collection of spheres

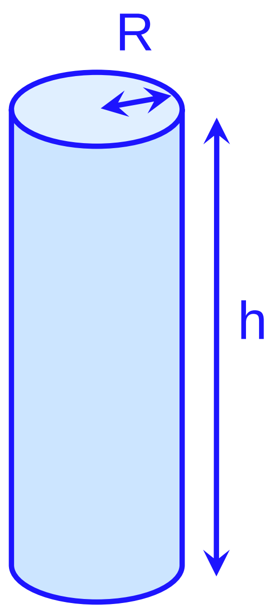


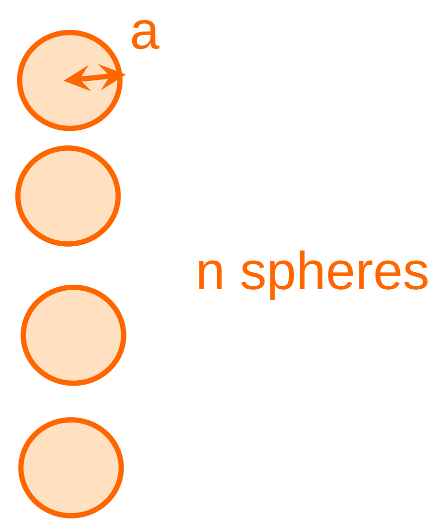


Surface	$2\pi R^2 + 2\pi Rh$	$4\pi a^2 n$	$n=3R^2h/4a^3$

Physical mechanism

Surface tension aims at reducing the exposed surface.
For a given volume, compare a cylinder and a collection of spheres



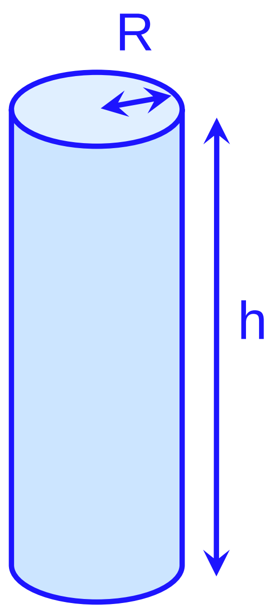


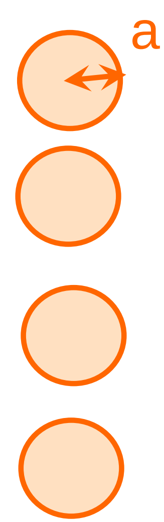
Surface $\cancel{2\pi R^2} + 2\pi Rh$ $3\pi a^2 R^2 h/a^3$

Surface ratio?

Physical mechanism

Surface tension aims at reducing the exposed surface.
For a given volume, compare a cylinder and a collection of spheres



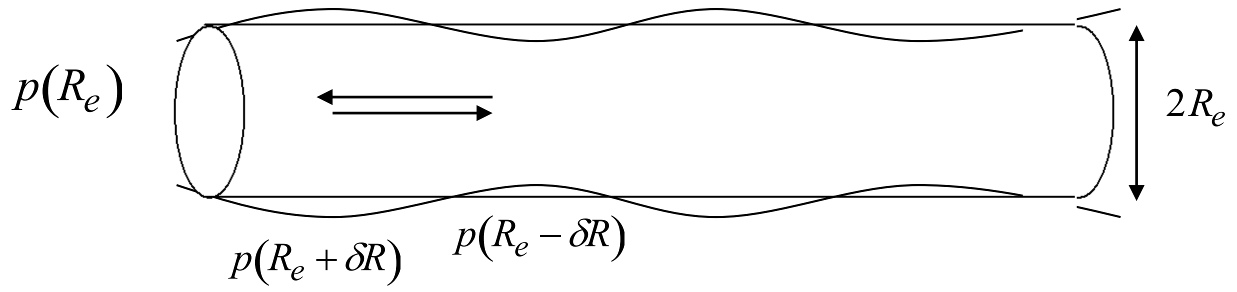


n spheres

Surface	$\cancel{2\pi R^2} + 2\pi R h$	$3\pi a^2 R^2 h / a^3$
Surface ratio	$3R/2a<1$	$\boxed{a>3R/2}$

Should produce relatively large droplets

Hand waving arguments



$$p(R_e+\delta R)-p(R_e-\delta R)=2\delta R\left.\frac{dp}{dR}\right|_{R_e}$$

>0 : stable

<0 : unstable

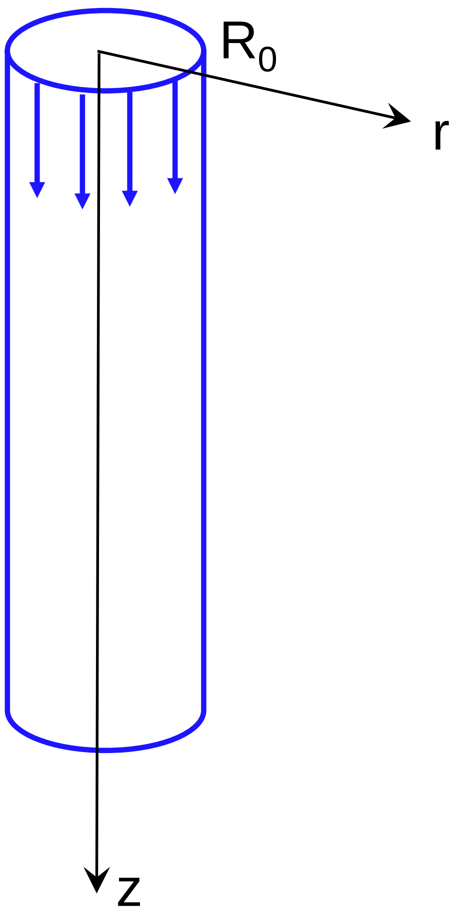
p(R)=γ/R

dp/dR=-γ/R^2 <0 !

Instability analysis:

1. **Equations and boundary conditions**
2. **Base state**
3. **Linearized equations**
4. **Normal mode expansion**
5. **Dispersion relation**
6. **Analysis of the dispersion relation**

1. Liquid jet



1. Equations

Navier Stokes equation in cylindrical geometry

$$r: \quad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right] + \rho g_r$$

$$\phi: \quad \rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2}\right] + \rho g_\phi$$

$$z: \quad \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho g_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0.$$

1. Equations

Navier Stokes equation in cylindrical geometry

Assume axisymmetry and no swirl

$$r: \quad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \cancel{\frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi}} + u_z\frac{\partial u_r}{\partial z} - \cancel{\frac{u_\phi^2}{r}}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2}} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \cancel{\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}}\right] + \rho g_r$$

$$\phi: \quad \cancel{\rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2}\right] + \rho g_\phi}$$

$$z: \quad \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \cancel{\frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi}} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2}} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho g_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \cancel{\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}} + \frac{\partial u_z}{\partial z} = 0.$$

1. Equations

Euler equation in cylindrical geometry

•**Radial momentum**

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\cancel{\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right]}+\cancel{\rho g_r}$$

•**Axial momentum**

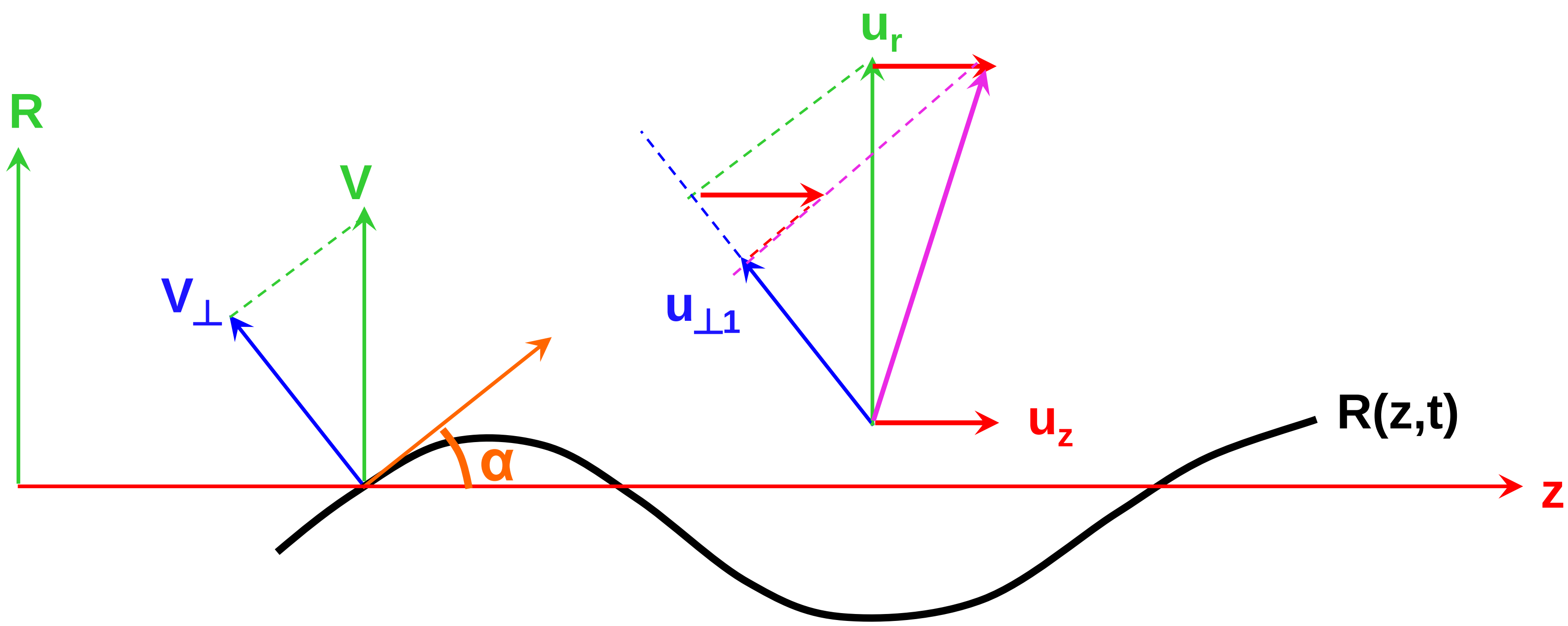
$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\cancel{\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}\right]}+\cancel{\rho g_z}$$

•**Continuity equation**

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r)+\frac{\partial u_z}{\partial z}=0.$$

•**Neglect gravity, viscosity, outer fluid**
•**Axisymmetric assumption**

1. Kinematic boundary condition



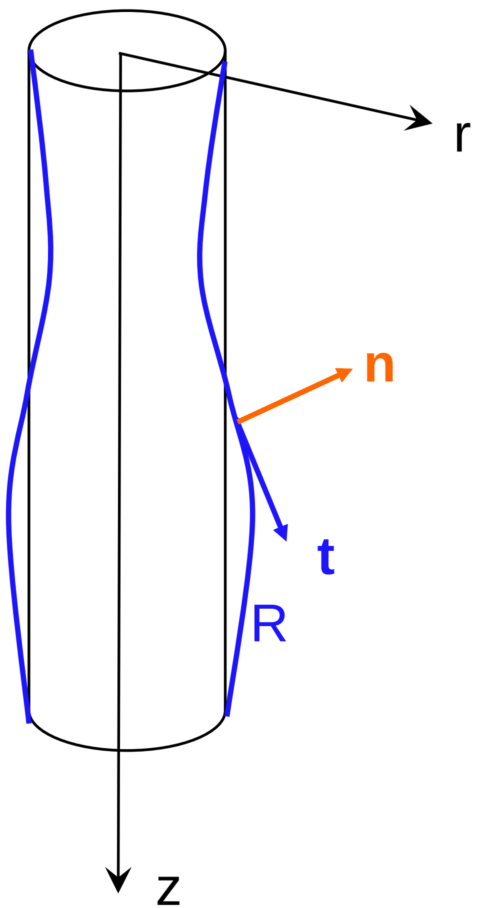
Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$\left.\begin{aligned} V_\perp &= \partial R/\partial t \cos(\alpha) \\ u_{\perp 1} &= u_r \cos(\alpha) - u_z \sin(\alpha) \end{aligned}\right\} \partial R/\partial t = u_r - u_z \tan(\alpha) \quad \Rightarrow \quad \boxed{\partial R/\partial t = u_r - u_z \partial R/\partial z}$$

1. Boundary conditions

$$P - P_{atm} = \gamma \mathcal{C}$$



$$\mathbf{t} = \frac{\left(\frac{\partial R}{\partial z}, 0, 1\right)}{\sqrt{1 + \frac{\partial R}{\partial z}^2}}$$

$$\mathbf{n} = \frac{\left(1, 0, -\frac{\partial R}{\partial z}\right)}{\sqrt{1 + \frac{\partial R}{\partial z}^2}}$$

$$\mathcal{C} = \nabla . \mathbf{n}$$

1. Dynamic boundary conditions

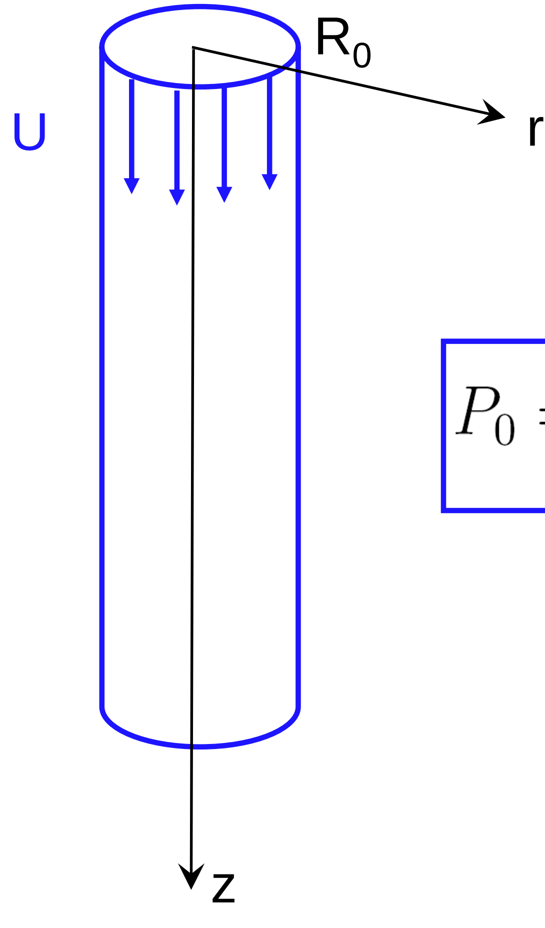
$$\mathcal{C} = \nabla.\mathbf{n}$$

$$\nabla.(f_r, f_\theta, f_z) = \frac{1}{r}\frac{\partial(r f_r)}{\partial r} + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z}$$

$$\mathcal{C} = \frac{1}{R}\frac{\partial\left(r\frac{1}{\sqrt{1+\frac{\partial R}{\partial z}^2}}\right)}{\partial r} - \frac{\partial^2 R}{\partial z^2}\frac{1}{\sqrt{1+\frac{\partial R}{\partial z}^2}^{\,3}}$$

$$\boxed{\mathcal{C} = \frac{1}{R\sqrt{1+\frac{\partial R}{\partial z}^2}} - \frac{\partial^2 R}{\partial z^2}\frac{1}{\sqrt{1+\frac{\partial R}{\partial z}^2}^{\,3}}}$$

2. Base state



$$\boxed{P_0 = P_{air} + \frac{\gamma}{R_0}}$$

3. Perturb and linearize
perturbation expansion

$$
\begin{aligned}
\boxed{U_r} &= 0 &+ \boxed{\epsilon u_r(z,r,t)} \\
U_z &= U &+ \epsilon u_z(z,r,t) \\
P &= P_0 &+ \epsilon p(z,r,t) \\
R &= R_0 &+ \epsilon \eta(z,t)
\end{aligned}
$$

$\epsilon \ll 1$

Variables **Base state** **Small perturbation**

3. Linearized equations

Linearized Euler equations

$$\frac{\partial u_r}{\partial t} + U\frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial u_z}{\partial t} + U\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0$$

3. Linearized equations

Laplace equation for pressure

$$\boxed{\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = 0}$$

3. linearized kinematic boundary conditions

$$\boxed{\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial z} = u_r}$$

This holds at $r=R_0$ (flattening)

3. Perturbed dynamic boundary conditions

$$R = R_0 + \epsilon\eta$$

$$\mathcal{C} = \frac{1}{R_0}\left(1 - \frac{1}{2}\epsilon^2 \frac{\partial\eta}{\partial z}^2\right)\left(1 - \epsilon\frac{\eta}{R_0}\right) - \epsilon\frac{\partial^2\eta}{\partial z^2} \cdots$$

$$\mathcal{C} = \frac{1}{R_0} - \epsilon\frac{\eta}{R_0^2} - \epsilon\frac{\partial^2\eta}{\partial z^2}$$

3. Perturbed and linearized dynamic boundary condition

$$P_0 + \epsilon p\big|_R = P_{air} + \gamma \left(\frac{1}{R_0} - \epsilon \frac{\eta}{R_0^2} - \epsilon \frac{\partial^2 \eta}{\partial z^2} \right)$$

$$\Big\Downarrow \quad P_0 = P_{air} + \frac{\gamma}{R_0}$$

$$\boxed{p\big|_R = -\gamma \left(\frac{\eta}{R_0^2} + \frac{\partial^2 \eta}{\partial z^2} \right)}$$

4. Normal mode expansion

Fourier transform in z and t

$$\boxed{\begin{aligned} u_r &= \hat{u}_r(r)\exp(i(kz-\omega t)), \\ u_z &= \hat{u}_z(r)\exp(i(kz-\omega t)), \\ p &= \hat{p}(r)\exp(i(kz-\omega t)), \\ \eta &= B\exp(i(kx-\omega t)) \end{aligned}}$$

k is the wavenumber and ω the frequency (in rad/s)

$$\boxed{\lambda = 2\pi/k} \qquad \boxed{T = 2\pi/\omega}$$

$$\boxed{f = \omega/(2\pi)}$$

4. Normal mode expansion

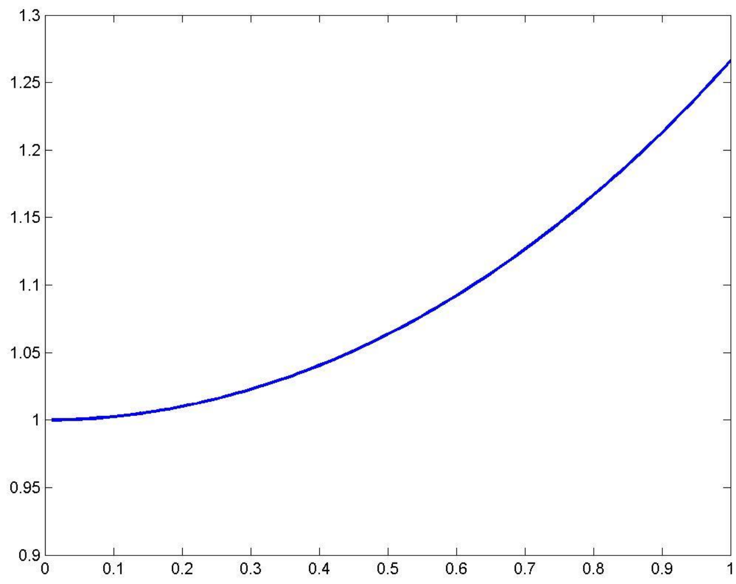
Solution to Laplace equation:

$$\frac{\partial^2 \hat{p}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{p}}{\partial r} - k^2 \hat{p} = 0$$

4. Normal mode expansion

Solution to Laplace equation:

$$\hat{p}(r) = AI_0(kr)$$



Bessel function of order 0

4. Normal mode expansion

Retrieve radial velocity:

$$-i\omega\hat{u}_r + ikU\hat{u}_r = -\frac{1}{\rho}AkI_0'(kr)$$

$$\boxed{\hat{u}_r = -\frac{i}{\rho(\omega - kU)}AkI_0'(kr)}$$

4. Kinematic boundary condition

$$\frac{\partial \hat{\eta}}{\partial t} + U\frac{\partial \hat{\eta}}{\partial z} = \hat{u}_r$$

$$\Downarrow$$

$$\boxed{-i\omega B + ikUB = -\frac{i}{\rho(\omega - Uk)} A k I_0'(kr)}$$

4. Dynamic boundary condition

$$p(R_0) = -\gamma \left(\frac{B}{R_0^2} - k^2 B \right)$$

$$\Downarrow$$

$$\boxed{A I_0(k R_0) = -\gamma \left(\frac{B}{R_0^2} - k^2 B \right)}$$

4. Dispersion relation

$$\begin{pmatrix} I_0(kR_0) & \gamma\left(\frac{1}{R_0^2} - k^2\right) \\ \frac{k^2 I_0'(kR_0)}{\rho(\omega - Uk)} & -(\omega - Uk) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\Downarrow$$

$$\boxed{\omega = Uk \pm \sqrt{\frac{\gamma k^2}{\rho}\left(k^2 - \frac{1}{R_0^2}\right)\frac{I_0'(kR)}{I_0(kR)}}}$$

5. Dispersion relation

$$\boxed{\omega = Uk \pm \sqrt{\frac{\gamma k^2}{\rho}\left(k^2 - \frac{1}{R_0^2}\right)\frac{I_0'(kR)}{I_0(kR)}}}$$

- **Unstable if there exists one ω, Im(ω)>0** **at $k<1/R_0$**
- **Neutral if for all ω, Im(ω)=0** **at $k>1/R_0$**
- **Stable (or damped) if for all ω, Im(ω)<0:**

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

Destabilisation d'un jet

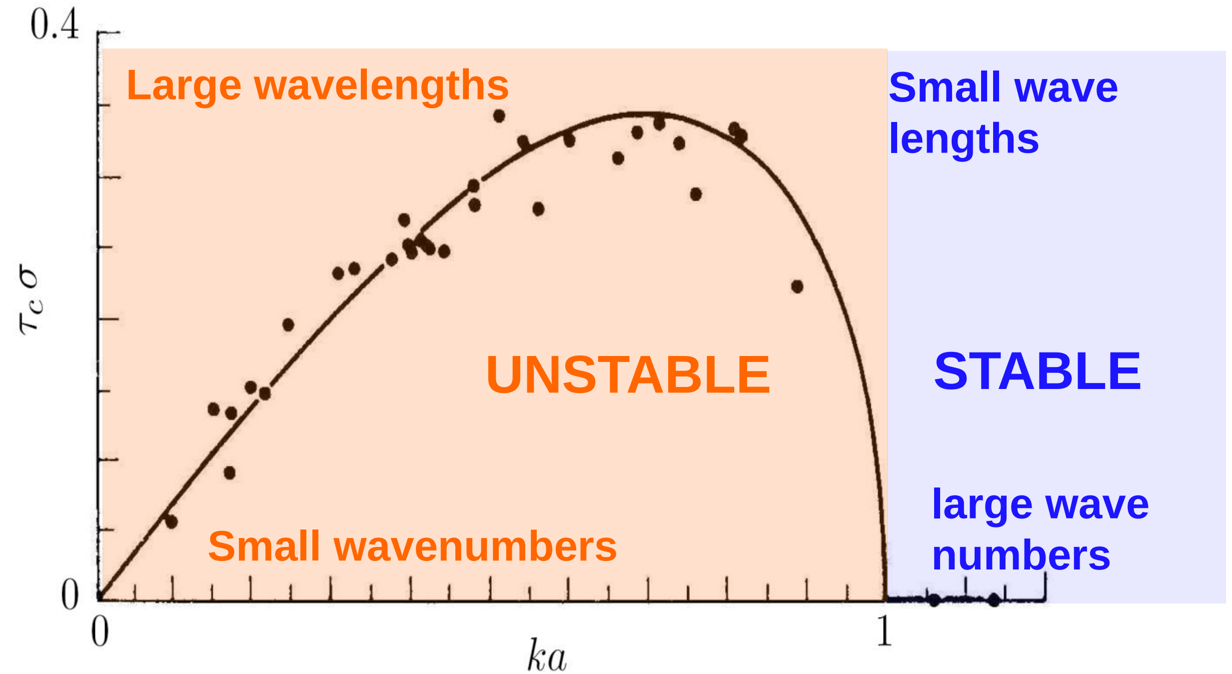
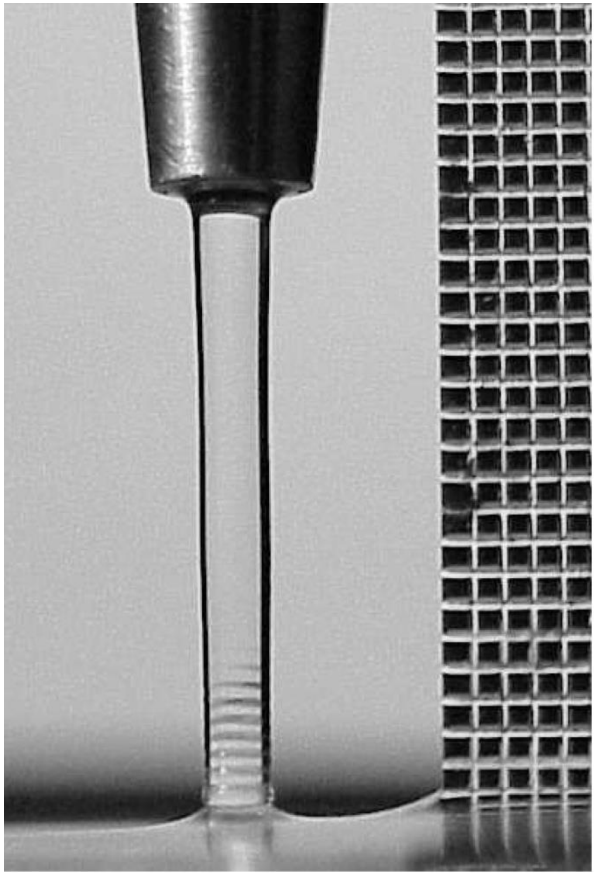


FIG. 2.10 – *Taux de croissance* $\tau_c\sigma$, *avec* $\tau_c = \sqrt{\rho a^3/\gamma}$, *de l'instabilité d'un filet fluide non visqueux, et points expérimentaux. D'après (Drazin & Reid 2004).*



Surface tension is destabilizing as a consequence of the radial curvature

Surface tension is stabilizing as a consequence of the axial curvature

Instability analysis:

1. **Equations and boundary conditions**
2. **Base state**
3. **Linearized equations**
4. **Normal mode expansion**
5. **Dispersion relation**
6. **Analysis of the dispersion relation**