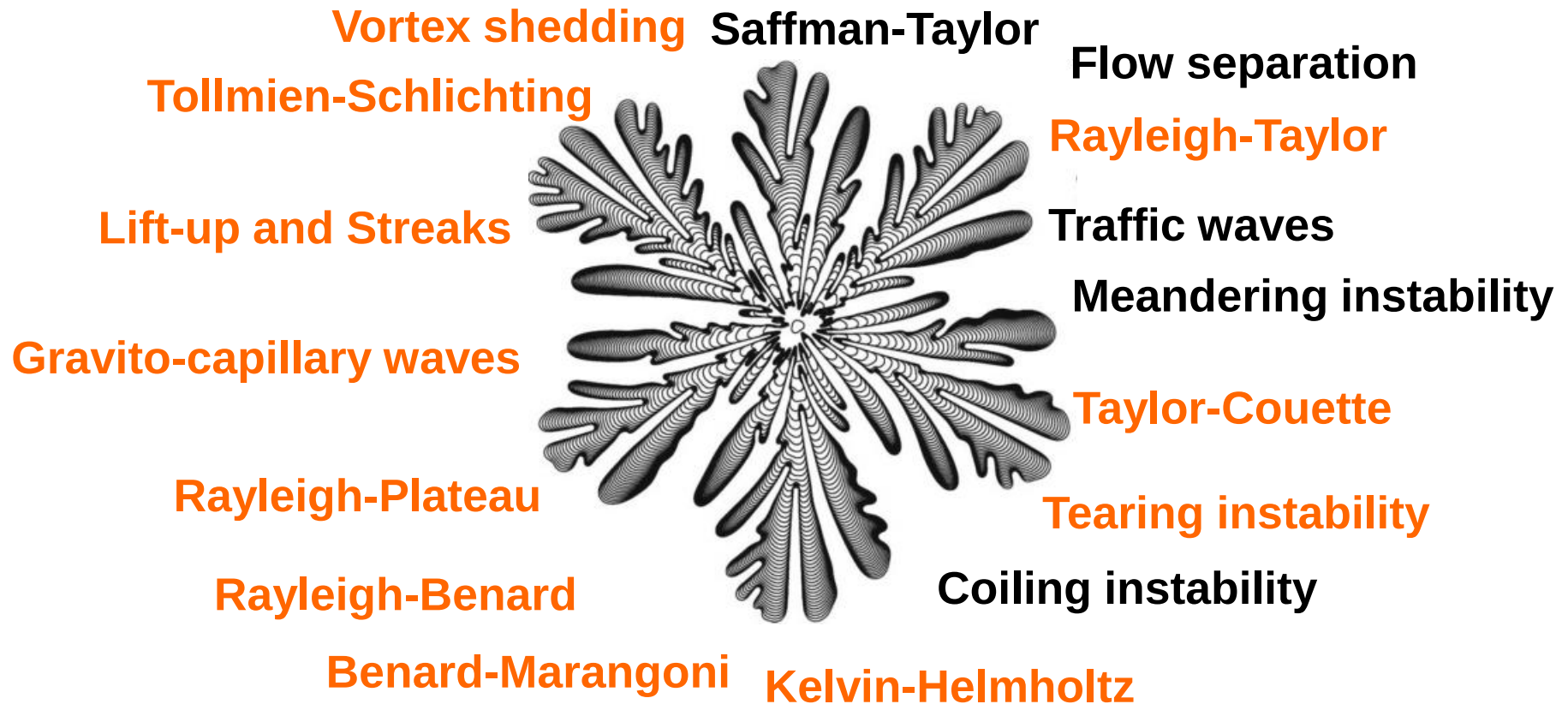


Most flows are unstable...

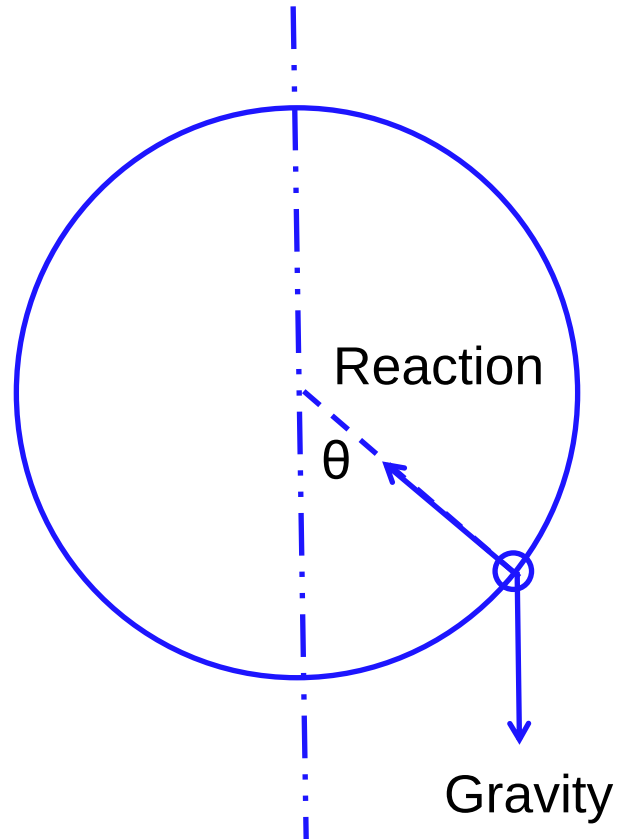


Nonlinearities: bifurcations and amplitude equations

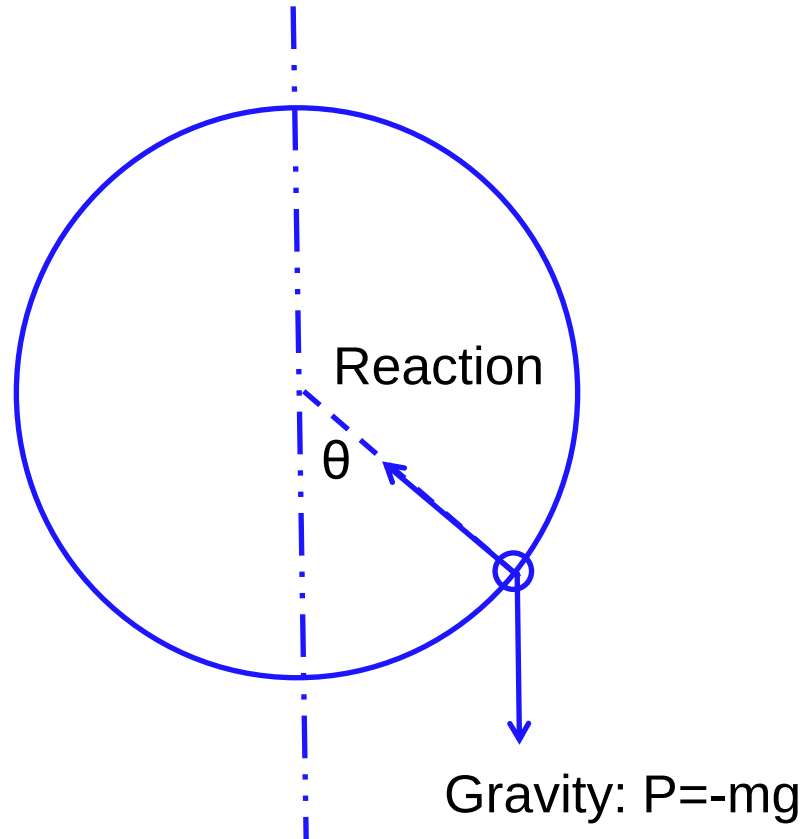
Agenda

1. The gravitational pendulum : a poor watch
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2. A simple example of bifurcation
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3. Classical bifurcations
4. Hopf bifurcation and Stuart-Landau equation

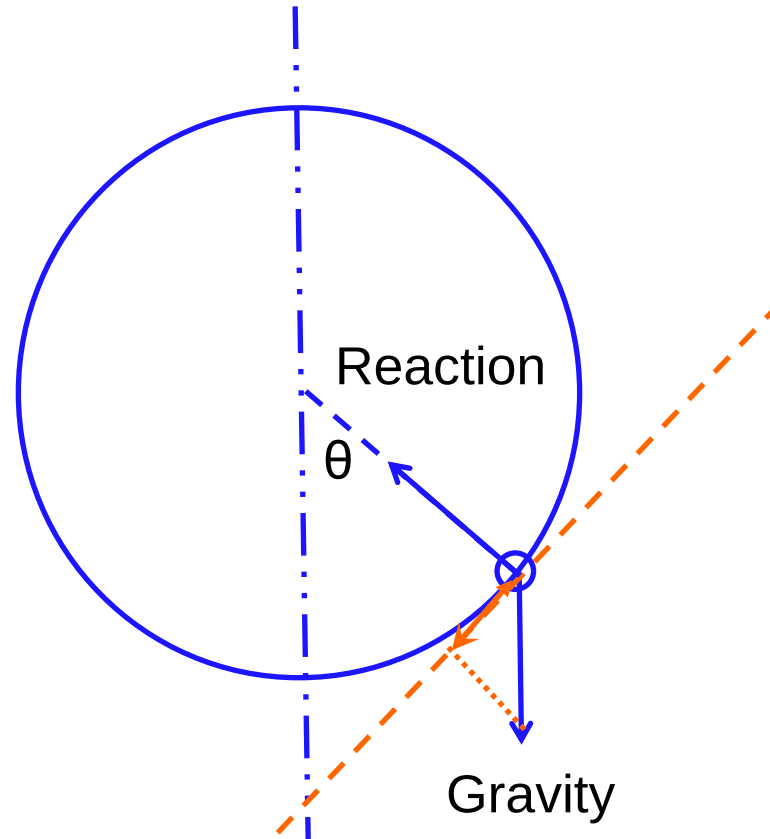
Period of a gravitational pendulum



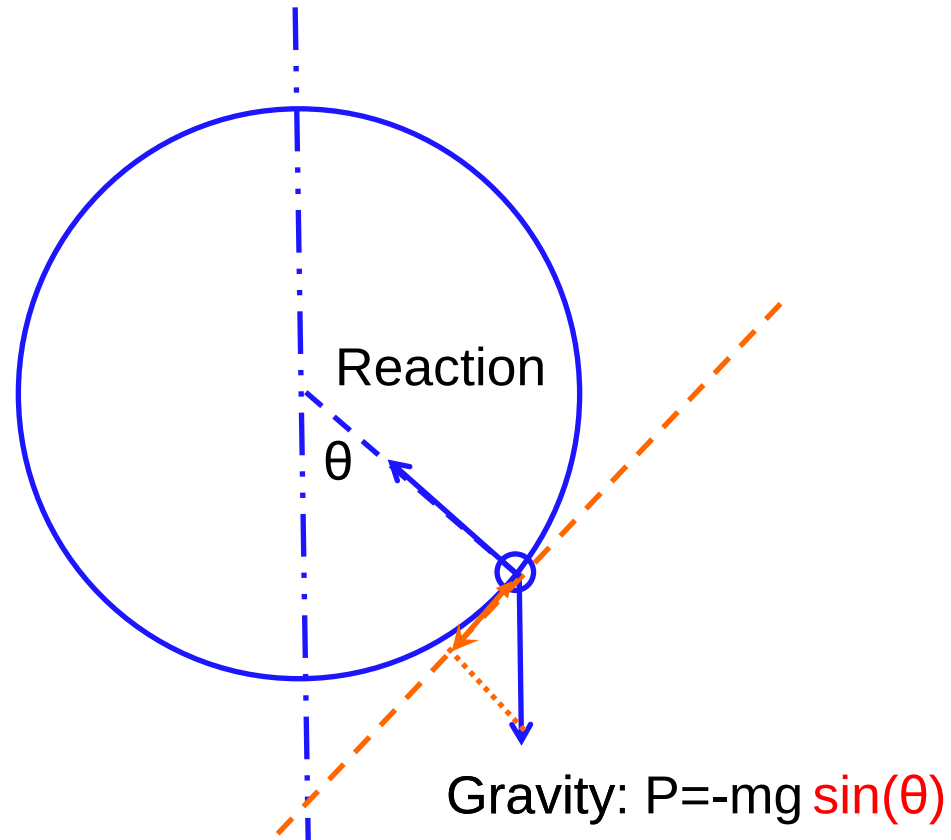
Period of a gravitational pendulum



Bifurcations and amplitude equations



Bifurcations and amplitude equations



$$mR\ddot{\theta} = -mg \sin(\theta)$$

Governing equations

$$\ddot{\theta} = -\omega_0^2 \sin(\theta)$$

$\omega_0^2 = g/R$
Pendulum frequency

Small perturbations

$$\theta = \varepsilon \theta'$$

Linearized equations

$$\ddot{\theta}' = -\omega_0^2 \theta'$$

Governing equations

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$\omega_0^2 = g/R$
Pendulum frequency

Small perturbations

$$\theta = \varepsilon \theta'$$

Linearized equations

$$\ddot{\theta}' = -\omega_0^2 \theta'$$

Period seems independant of amplitude.
Can this be true? Effect of nonlinearity on period?

Multiple time scale expansion

Weakly nonlinear approach

$$\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots$$

$$T = \epsilon^2 t \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial t}$$

$$\theta_i(t, T)$$

$$\ddot{\theta} = -\omega_0^2 \sin(\theta)$$

Order ϵ^0

$$\ddot{\theta}_0 = 0 \Rightarrow \theta_0 = 0$$

Multiple time scale expansion

Weakly nonlinear approach

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$$\ddot{\theta}_0 = 0 \Rightarrow \theta_0 = 0$$

Order ϵ^1

$$\ddot{\theta}_1 = -\omega_0^2 \theta_1 \Rightarrow \theta_1 = A(T) \cos(\omega_0 t + \varphi(T))$$

Multiple time scale expansion

Weakly nonlinear approach

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Order ϵ^2

$$\ddot{\theta}_2 = -\omega_0^2 \theta_2 \Rightarrow \theta_2 = B(T) \cos(\omega_0 t + \psi(T))$$

Multiple time scale expansion

Weakly nonlinear approach

$$\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots$$

$$T = \epsilon^2 t \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T}$$

$$\theta_i(t, T)$$

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$$\ddot{\theta}_1 = -\omega_0^2 \theta_1 \Rightarrow \theta_1 = A(T) \cos(\omega_0 t + \varphi(T))$$

Order ϵ^2

$$\ddot{\theta}_2 = -\omega_0^2 \theta_2 \Rightarrow \theta_2 = B(T) \cos(\omega_0 t + \psi(T))$$

Order ϵ^3

$$\ddot{\theta}_3 = -\omega_0^2 \theta_3 - 2\dot{\theta}_1' + \theta_1^3/6$$

Non-resonance condition

Order ε^3

$$\ddot{\theta}_3 = -\omega_0^2 \theta_3 - 2\ddot{\theta}_1 + \omega_0^2 \theta_1^3/6$$

$$\ddot{\theta}_1 = -\dot{A}(T)\omega_0 \sin(\omega_0 t + \varphi(T)) + \omega_0 A(T)\dot{\varphi}(T) \cos(\omega_0 t + \varphi(T))$$

$$\cos^3(\omega_0 t + \varphi(T)) = 3/4 \cos(\omega_0 t + \varphi(T)) + 1/4 \cos(3\omega_0 t + \varphi(T))$$

$$\theta_1^3 = A^3(T) \cos^3(\omega_0 t + \varphi(T))$$

Non-resonance condition

Order ε^3

$$\ddot{\theta}_3 = -\omega_0^2 \theta_3 - 2\ddot{\theta}_1 + \omega_0^2 \theta_1^3/6$$

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$$\theta_1^3 = A^3(T) \cos^3(\omega_0 t + \varphi(T))$$

Attention! What happens if you force a linear system at its natural frequency?

Example: $\ddot{\theta}_3 + \omega_0^2 \theta_3 = \cos(\omega_0 t)$

The particular solution is $\theta_{3f} = t \sin(\omega_0 t) / 2\omega_0$

It grows linearly in time and diverges! This should be avoided

Non-resonance condition

Order ε^3

$$\ddot{\theta}_3 = -\omega_0^2 \theta_3 - 2\ddot{\theta}_1 + \omega_0^2 \theta_1^3/6$$

$$\dot{\theta}_1 = -\dot{A}(T)\omega_0 \sin(\omega_0 t + \varphi(T)) + \omega_0 A(T)\dot{\varphi}(T) \cos(\omega_0 t + \varphi(T))$$

$$\cos^3(\omega_0 t + \varphi(T)) = 3/4 \cos(\omega_0 t + \varphi(T)) + 1/4 \cos(3\omega_0 t + \varphi(T))$$

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The particular solution is $\theta_{3f} = t \sin(\omega_0 t) / 2\omega_0$

It grows linearly in time and diverges! This should be avoided

Therefore the resonant RHS should be zero

$$\dot{A}(T)\omega_0 \sin(\omega_0 t + \varphi(T)) - \omega_0 A(T)\dot{\varphi}(T) \cos(\omega_0 t + \varphi(T)) + \omega_0^2 A^3(T) \cos(\omega_0 t + \varphi(T))/16 = 0$$

Nonlinear frequency correction

Therefore the resonant RHS should be zero

$$A'(T)\omega_0\sin(\omega_0 t+\varphi(T))-\omega_0 A(T)\dot{\varphi}(T)\cos(\omega_0 t+\varphi(T))+\omega_0^2 A^3(T)\cos(\omega_0 t+\varphi(T))/16=0$$

Nonlinear frequency correction

Therefore the resonant RHS should be zero

$$A'(T)\omega_0\sin(\omega_0 t + \phi(T)) - \omega_0 A(T)\phi'(T)\cos(\omega_0 t + \phi(T)) + \omega_0^2 A^3(T)\cos(\omega_0 t + \phi(T))/16 = 0$$

$$-\omega_0 A(T)\phi'(T) + \omega_0^2 A^3(T)/16 = 0$$

$$A'(T)\omega_0 = 0$$

Nonlinear frequency correction

Therefore the resonant RHS should be zero

$$A'(T)\omega_0\sin(\omega_0 t+\varphi(T))-\omega_0 A(T)\dot{\varphi}(T)\cos(\omega_0 t+\varphi(T))+\omega_0^2 A^3(T)\cos(\omega_0 t+\varphi(T))/16=0$$

$$-\omega_0 A(T)\dot{\varphi}(T) + \omega_0^2 A^3(T)/16=0$$

$$A'(T)\omega_0=0$$

$$\varphi(T)=\varphi_0+\omega_0 A_0^2 T/16$$

$$A(T)=A_0$$

Nonlinear frequency correction

Therefore the resonant RHS should be zero

$$A'(T)\omega_0\sin(\omega_0 t + \varphi(T)) - \omega_0 A(T)\dot{\varphi}(T)\cos(\omega_0 t + \varphi(T)) + \omega_0^2 A^3(T)\cos(\omega_0 t + \varphi(T))/16 = 0$$

$$-\omega_0 A(T)\dot{\varphi}(T) + \omega_0^2 A^3(T)/16 = 0$$

$$A'(T)\omega_0 = 0$$

$$\varphi(T) = \varphi_0 + \omega_0 A_0^2 T / 16$$

$$A(T) = A_0$$

The oscillation frequency depends on the amplitude

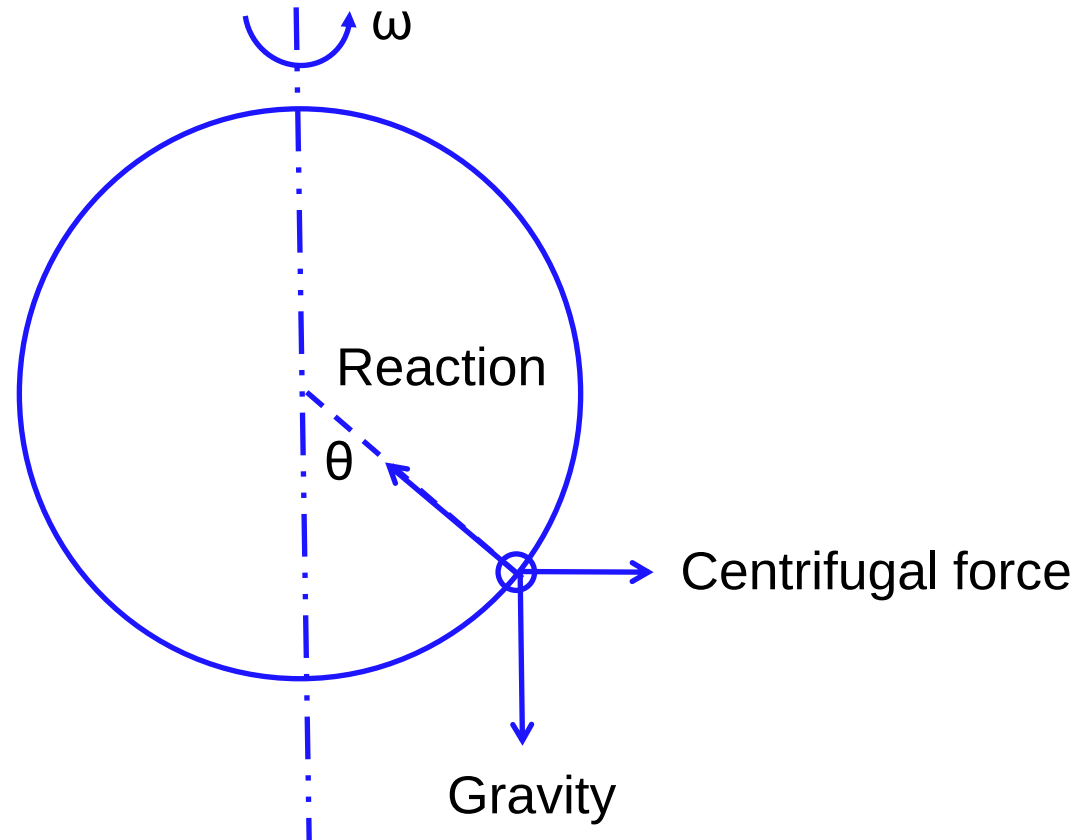
$$\omega = \omega_0(1 + A_0^2/16) \text{ Borda's Formula}$$

A gravitational pendulum is not a good oscillator for a watch!

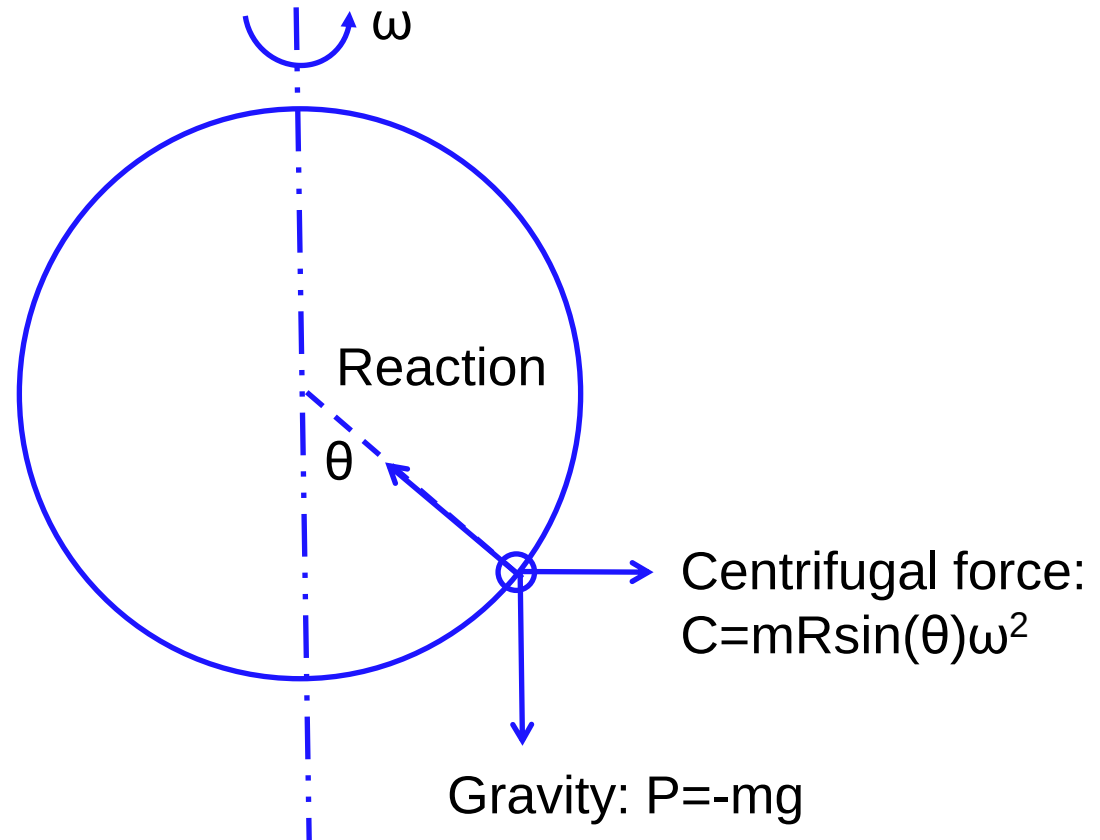
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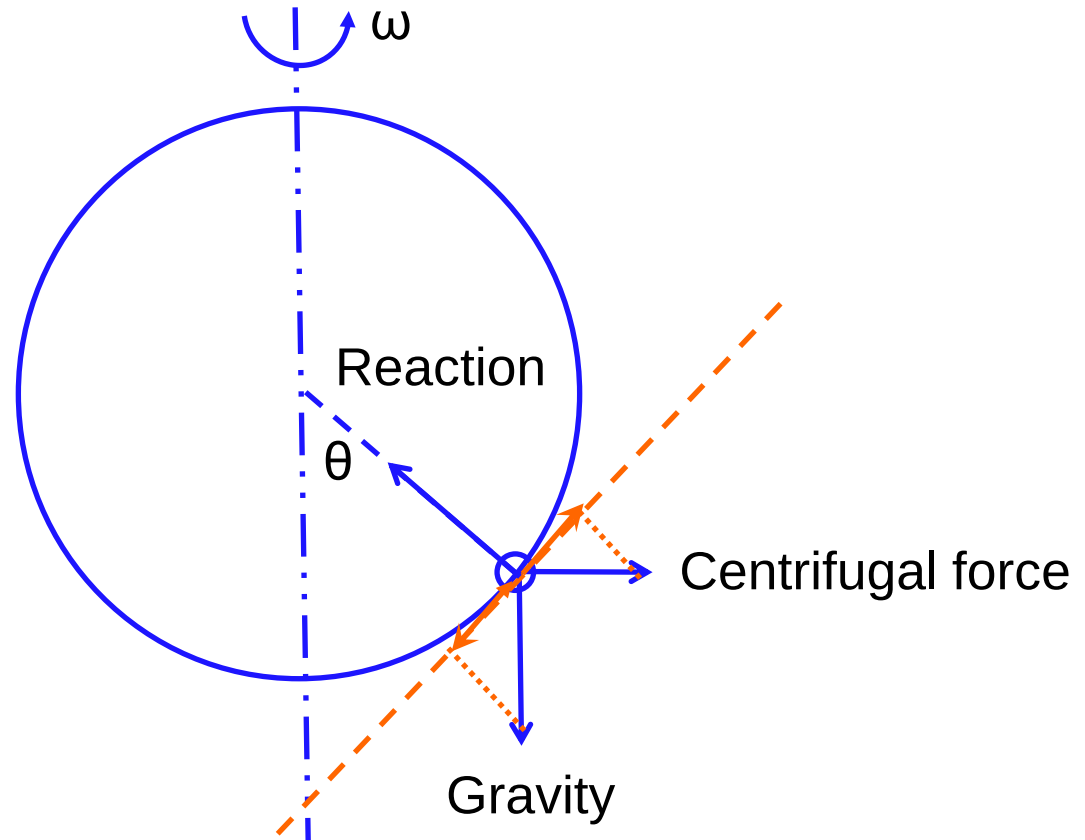
Bifurcations and amplitude equations



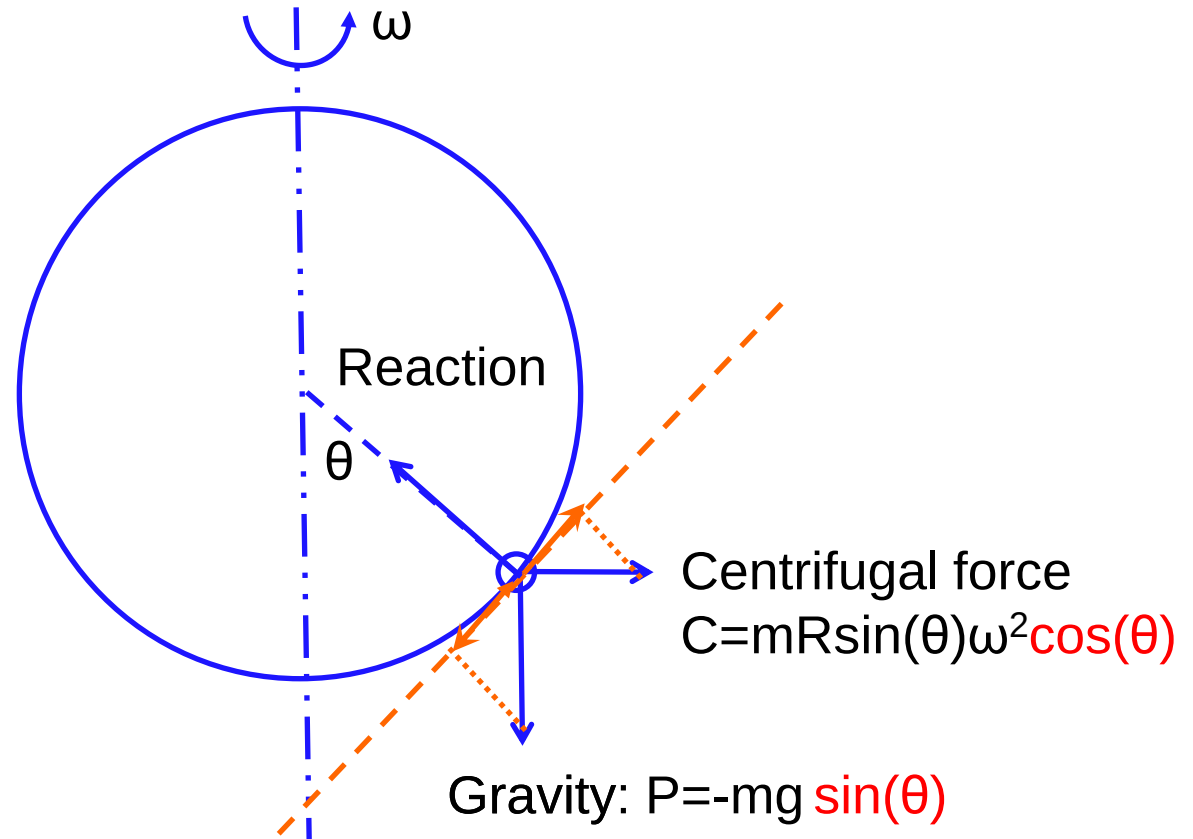
Bifurcations and amplitude equations



Bifurcations and amplitude equations



Bifurcations and amplitude equations



$$mR\ddot{\theta} = -mg\sin(\theta) + mR\sin(\theta)\omega^2\cos(\theta)$$

Governing equations

$$\ddot{\theta} = -\omega_0^2 \sin(\theta) + \omega^2 \sin(\theta) \cos(\theta)$$

$$\omega_0^2 = g/R$$

Pendulum frequency

Base flow

$$\theta = 0$$

Small perturbations

$$\theta = 0 + \varepsilon \theta'$$

Linearized equations

$$\ddot{\theta}' = -\omega_0^2 \theta' + \omega^2 \theta'$$

Linearized equations

$$\ddot{\theta}' = -\omega_0^2 \theta' + \omega^2 \theta'$$

Normal mode

$$\theta' = A \exp(st)$$

Dispersion relation

$$s^2 = \omega^2 - \omega_0^2$$

$$\omega^2 < \omega_0^2$$

$$\omega^2 > \omega_0^2$$

$$s = \pm i(\omega_0^2 - \omega^2)^{1/2}$$

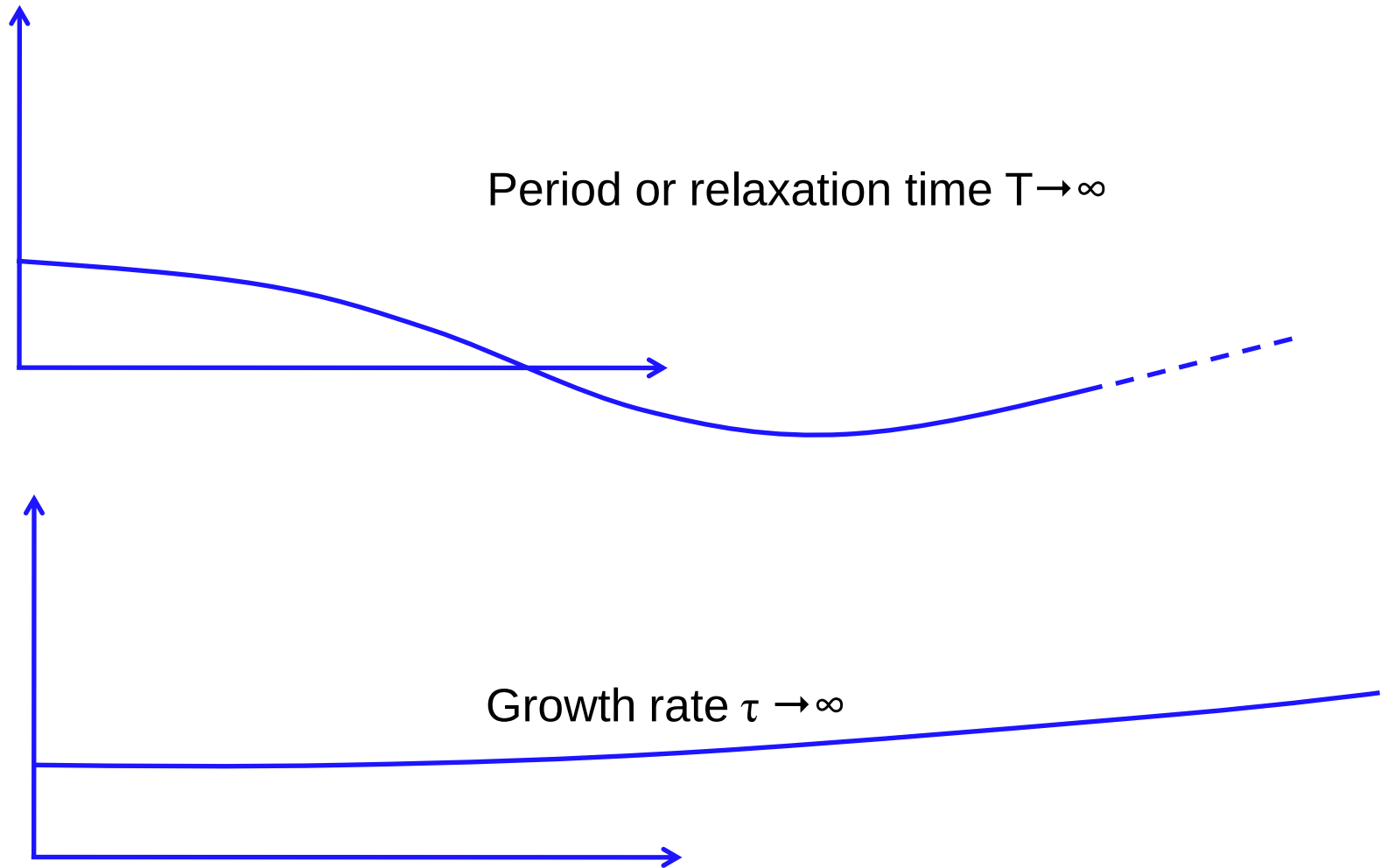
STABLE

$$s = \pm (\omega^2 - \omega_0^2)^{1/2}$$

UNSTABLE

Important concept: critical slowing

When $\omega^2 \sim \omega_0^2$, the characteristic time $\tau = 1/|s|$ diverges



What about nonlinearities?

Weakly nonlinear theory : multiscale expansion

Slow time scale $T = \epsilon^2 t$

Close to threshold $\omega^2 = \omega_0^2 + \epsilon^2 \Delta$

Asymptotic expansion $\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots$
 $\theta_i(t, T)$

$$\ddot{\theta}' = -\omega_0^2 \theta' + \omega^2 \theta'$$

Order 0

$$\theta_0 = 0$$

Base state

Order 1

$$\frac{\partial^2 \theta_1}{\partial t^2} = 0$$



$$\theta_1 = A_1(T)$$

Constant 1st order
perturbation

What about nonlinearities?

$$\ddot{\theta}' = -\omega_0^2 \theta' + \omega^2 \theta'$$

Order 0

$$\theta_0 = 0$$

Base state

Order 1

$$\frac{\partial^2 \theta_1}{\partial t^2} = 0$$

Constant (in t!) 1st order
perturbation



$$\theta_1 = A_1(T)$$

Order 2

$$\frac{\partial^2 \theta_2}{\partial t^2} = 0$$

Constant (in t!) 2nd order
perturbation



$$\theta_2 = A_2(T)$$

Order 3

$$\frac{\partial^2 \theta_3}{\partial t^2} = - \left(\frac{1}{2} \omega_0^2 A_1^3 - \Delta A_1 \right)$$



Secularity condition

= Non-resonance condition

= Compatibility condition

If not, θ_3 would grow like t^2 and ruin the
ordering in the expansion

What about nonlinearities?

$$\ddot{\theta}' = -\omega_0^2 \theta' + \omega^2 \theta'$$

Order 0

$$\theta_0 = 0$$

Order 1

$$\frac{\partial^2 \theta_1}{\partial t^2} = 0$$



$$\theta_1 = A_1(T)$$

Order 2

$$\frac{\partial^2 \theta_2}{\partial t^2} = 0$$



$$\theta_2 = A_2(T)$$

Order 3

$$\frac{\partial^2 \theta_3}{\partial t^2} = -\left(\frac{1}{2}\omega_0^2 A_1^3 - \Delta A_1\right)$$

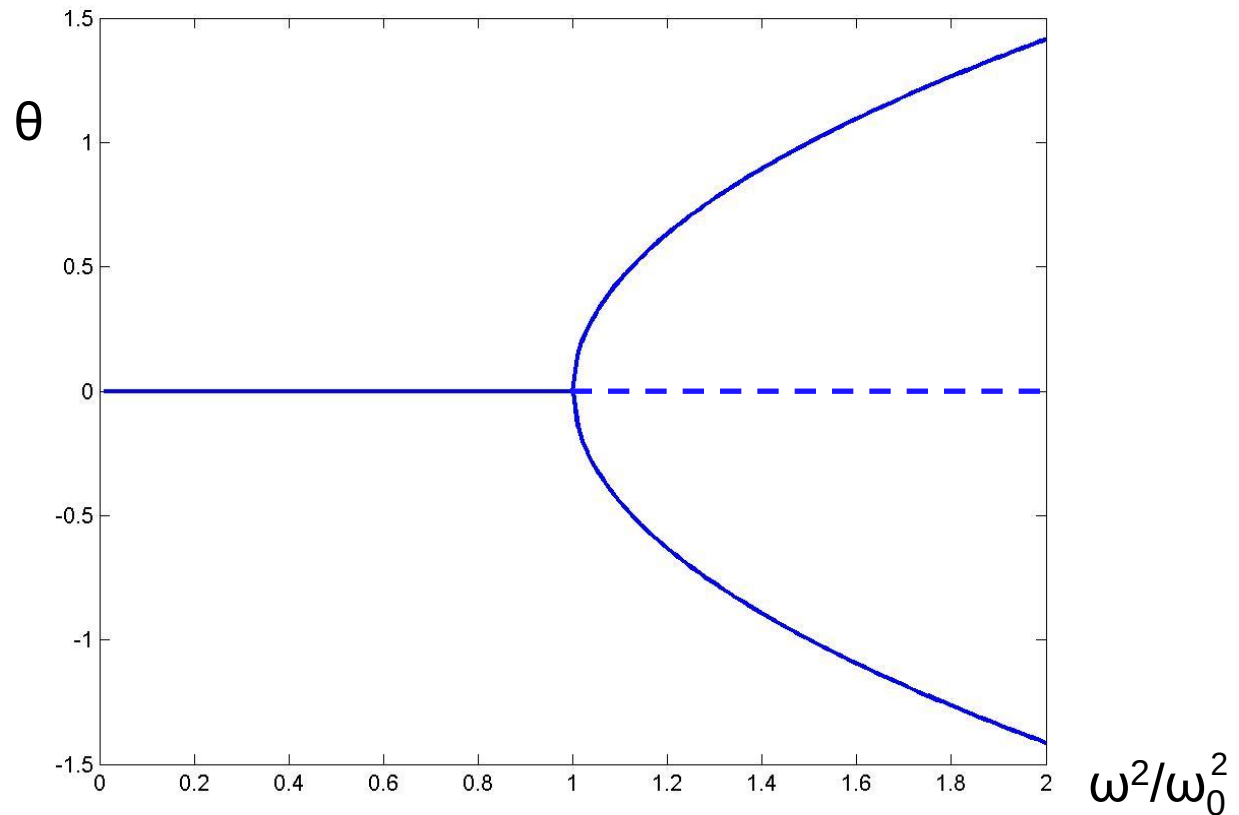


$$A_1 = \sqrt{2\frac{\Delta}{\omega_0^2}}$$

What about nonlinearities?

$$A_1 = \sqrt{2 \frac{\Delta}{\omega_0^2}}$$

$$\epsilon A_1 = \sqrt{2 \left(\frac{\omega^2}{\omega_0^2} - 1 \right)}$$



What about nonlinearities?

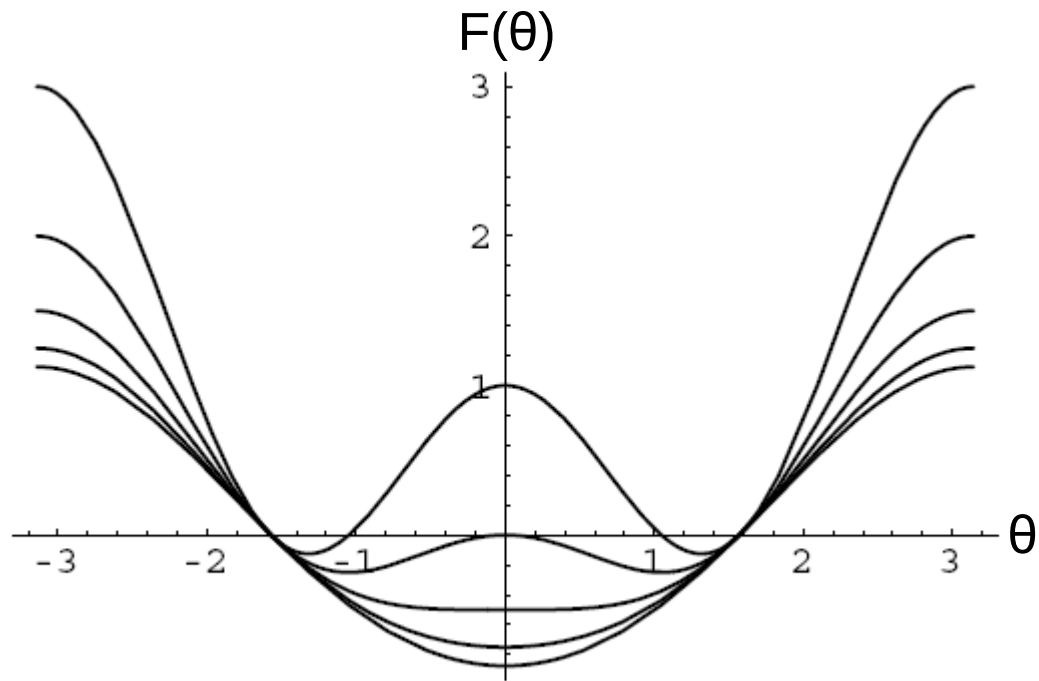
But recall the full nonlinear equation

$$\ddot{\theta} = -\omega_0^2 \sin(\theta) + \omega^2 \sin(\theta) \cos(\theta)$$

It has another 2 steady solutions

Physical interpretation (Potential)

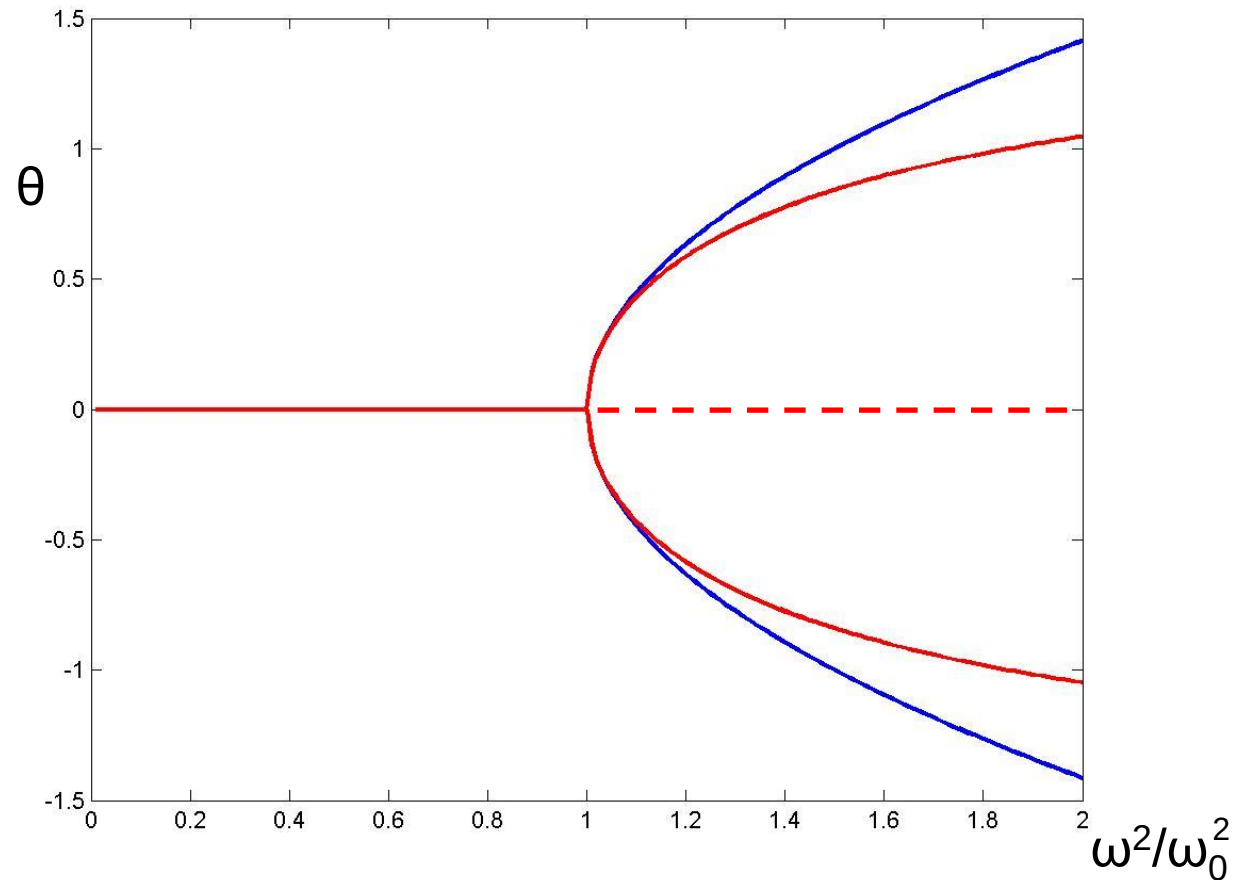
$$m\ddot{\theta} = F'(\theta)$$



What about nonlinearities?

It has another 2 steady solutions

$$\theta_s = \arccos((\omega_0/\omega)^2)$$



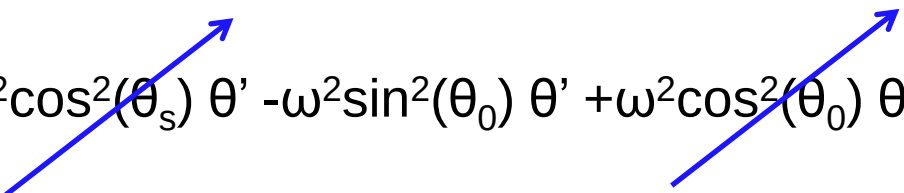
Stability of these bifurcated branches?

$$\ddot{\theta} = -\omega_0^2 \sin(\theta) + \omega^2 \sin(\theta) \cos(\theta)$$

$$\omega_0^2 = \omega^2 \cos(\theta_s)$$

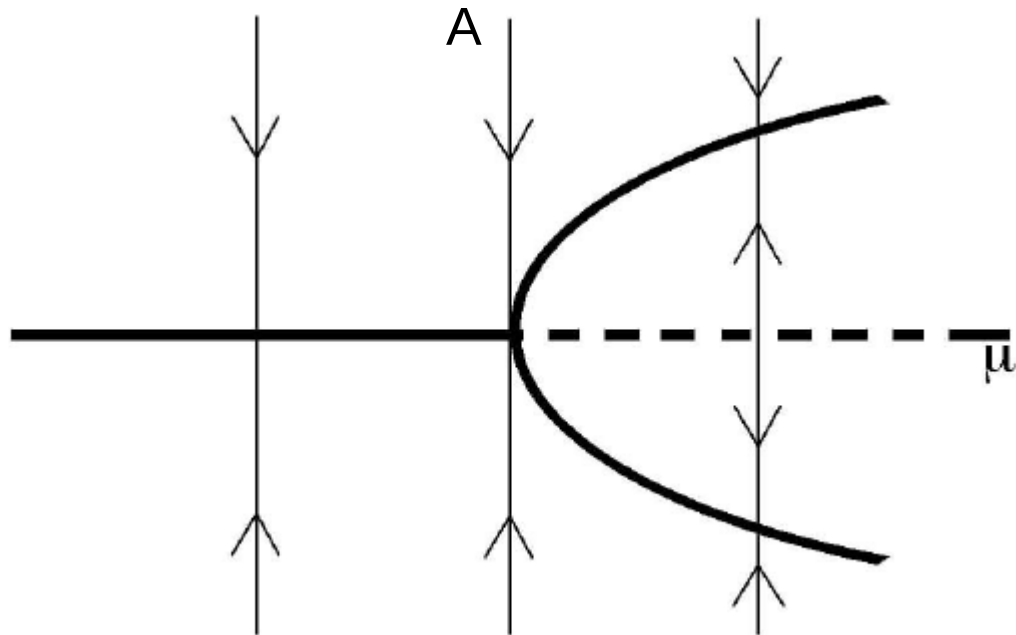
$$\theta = \theta_s + \theta'$$

$$\ddot{\theta}' = -\omega_0^2 \cos(\theta_s) \theta' - \omega^2 \sin^2(\theta_s) \theta' + \omega^2 \cos^2(\theta_s) \theta'$$

$$\ddot{\theta}' = -\omega^2 \cos^2(\theta_s) \theta' - \omega^2 \sin^2(\theta_0) \theta' + \omega^2 \cos^2(\theta_0) \theta'$$


$$\ddot{\theta}' = -\omega^2 \sin^2(\theta_0) \theta'$$

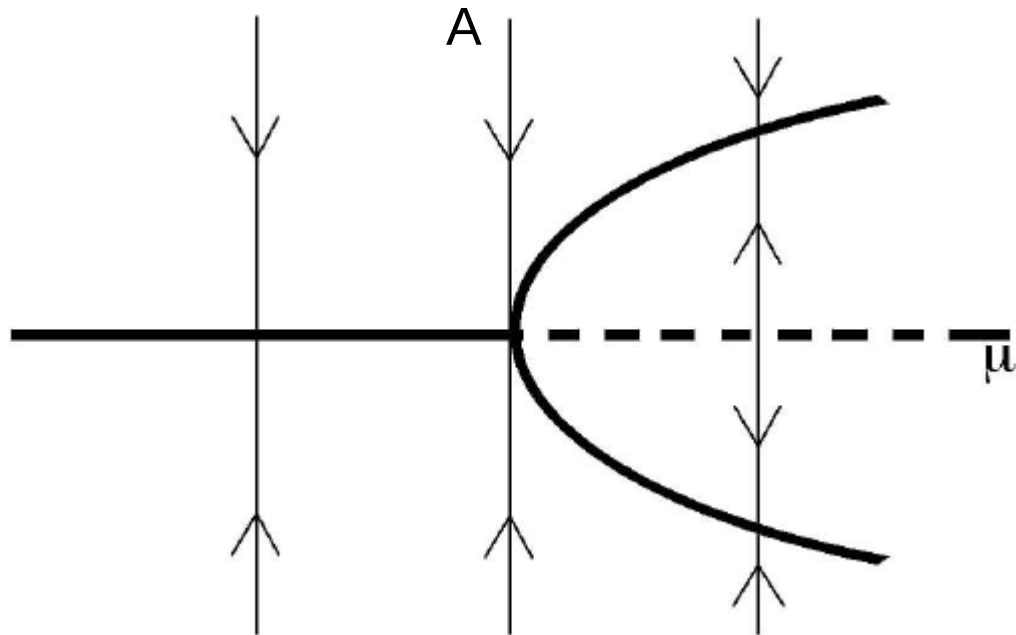
STABLE!



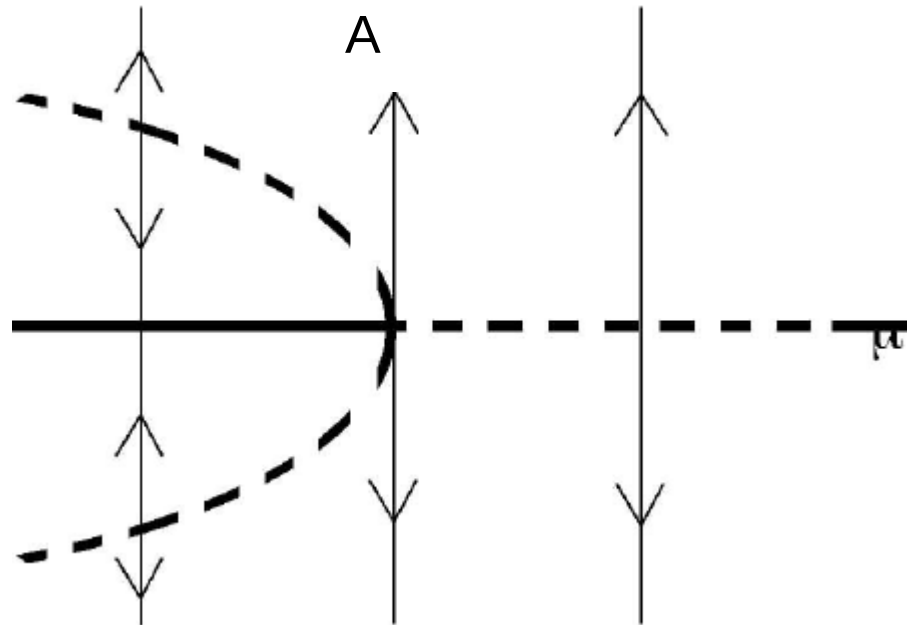
Supercritical fork bifurcation

Agenda

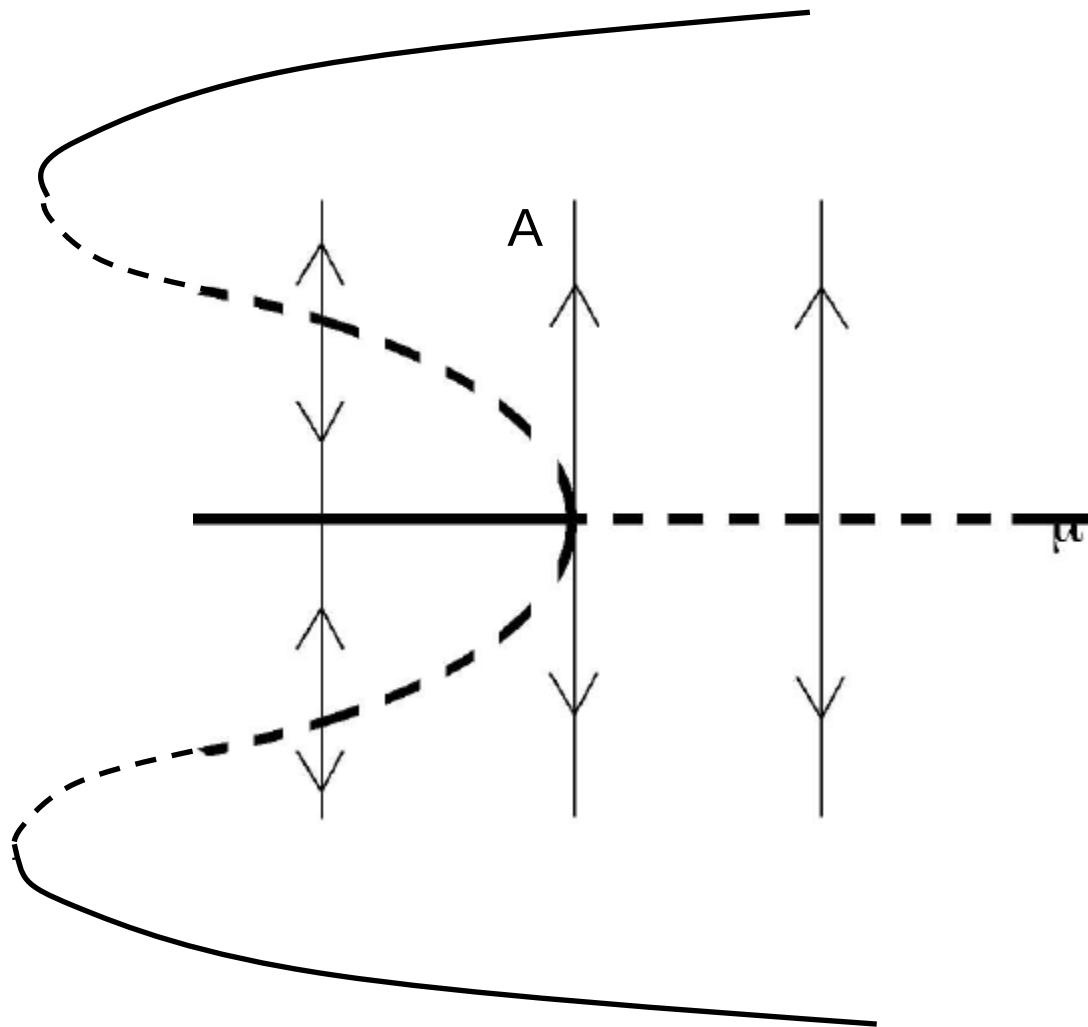
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Supercritical fork bifurcation

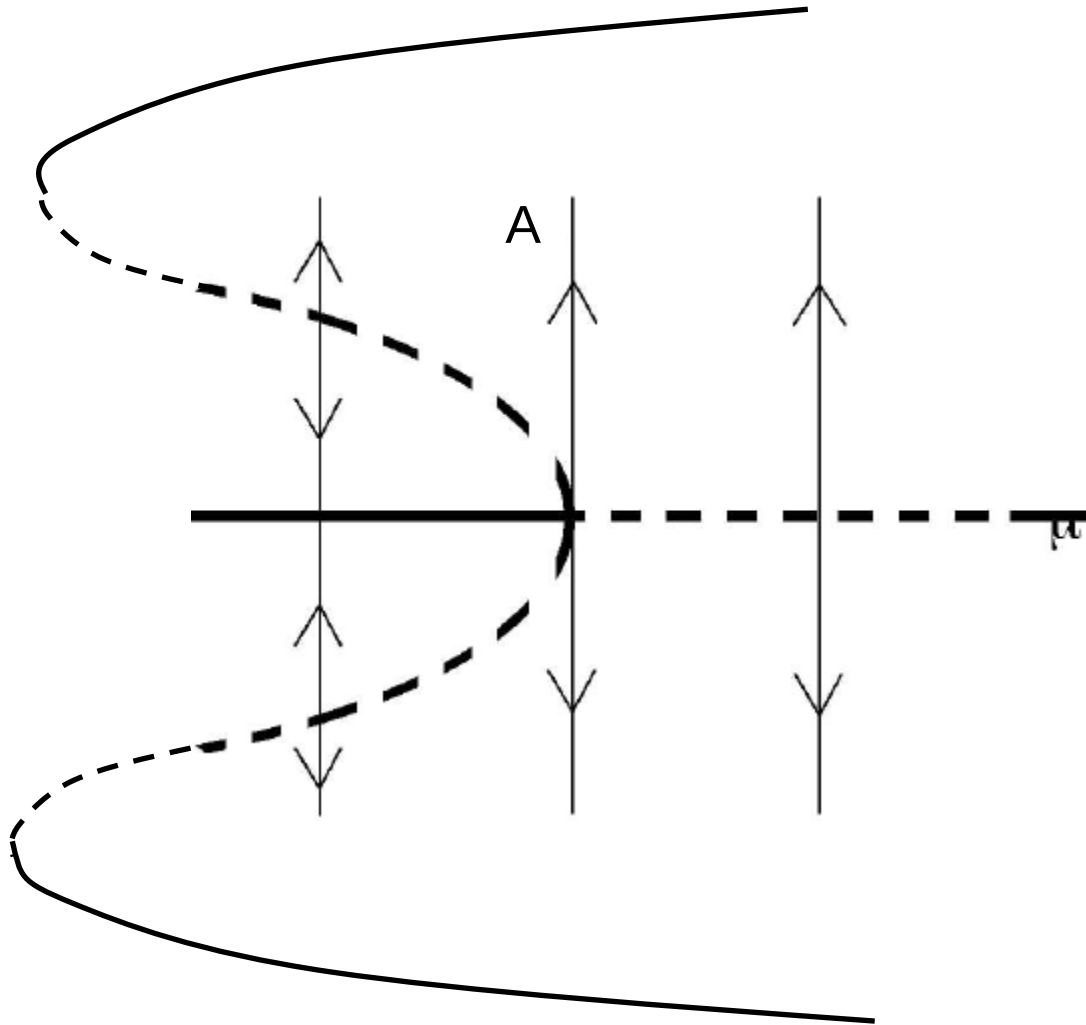


Subcritical fork bifurcation

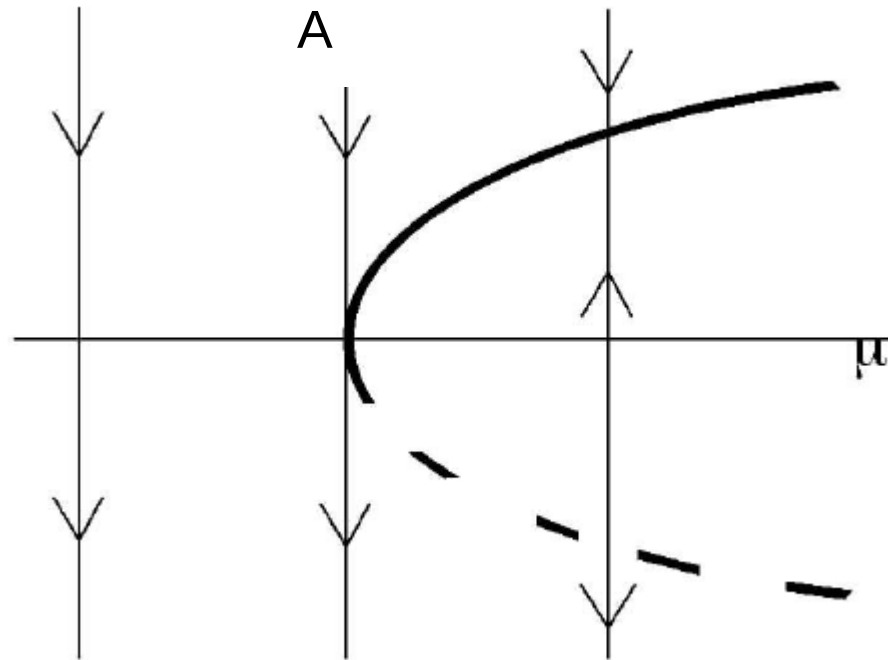


Subcritical bifurcation

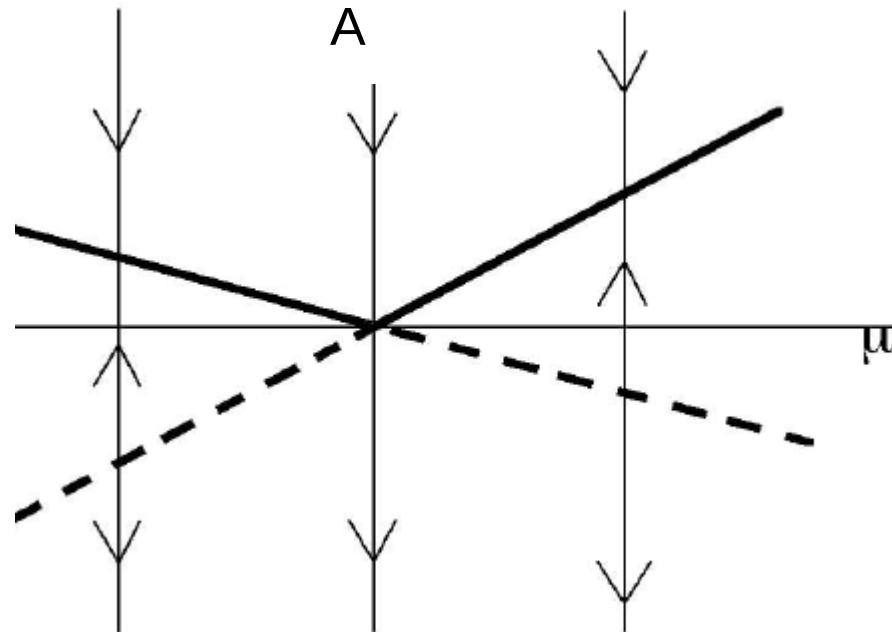
What about nonlinearities?



Hysteresis cycle!

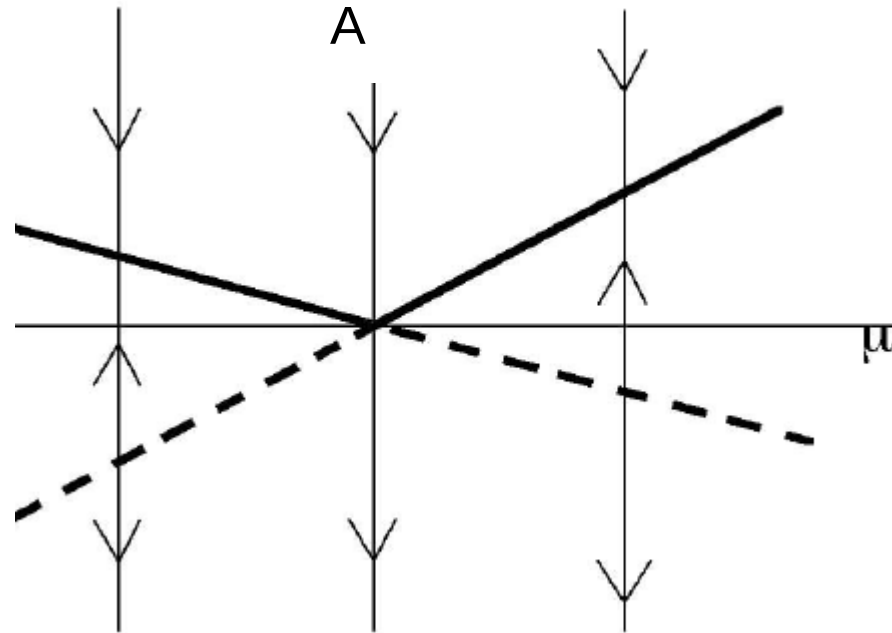


Saddle Node bifurcation

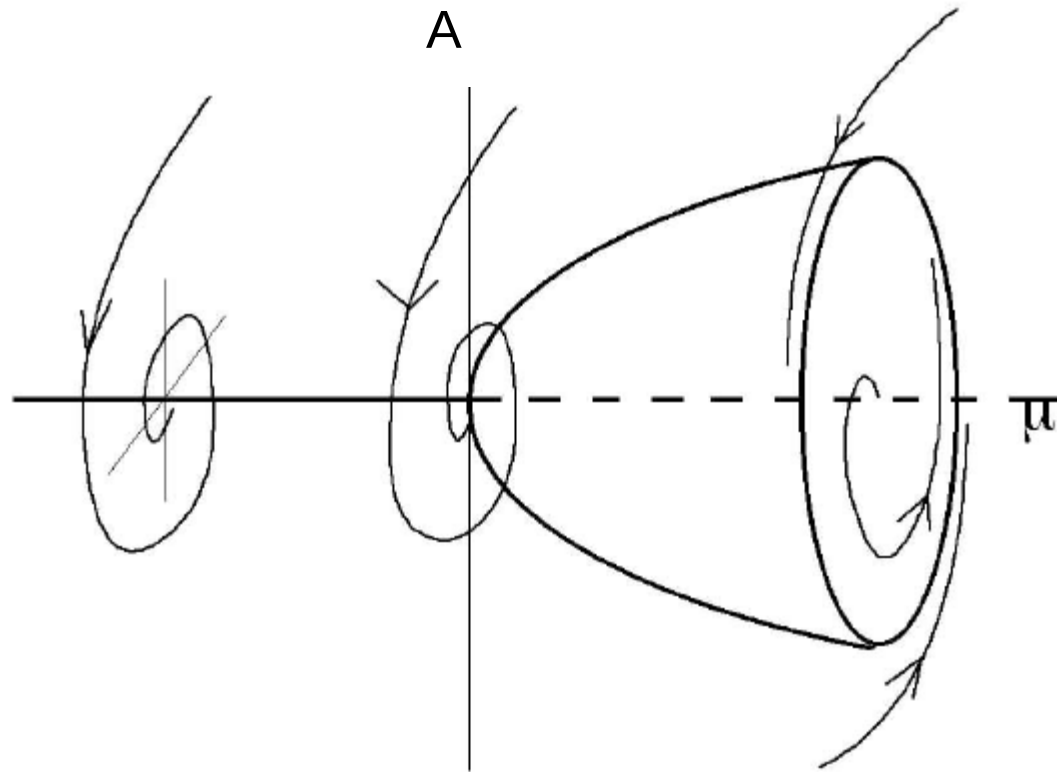


Transcritical bifurcation

What about nonlinearities?



Transcritical bifurcation



Hopf bifurcation

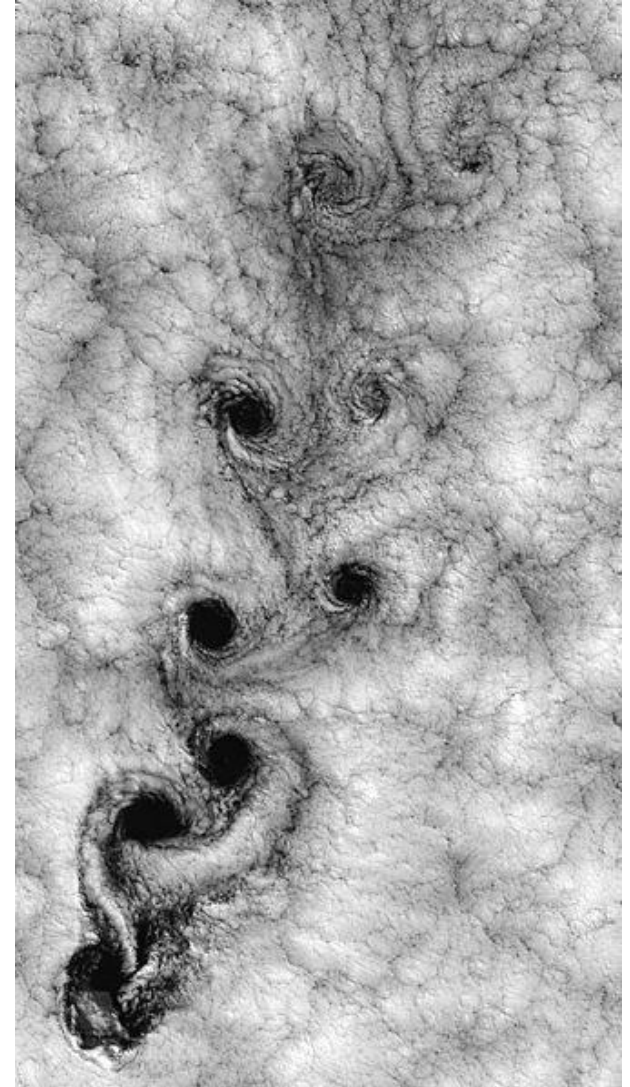
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Natural oscillators



<http://envsci.rutgers.edu/~lintner/teaching.html>



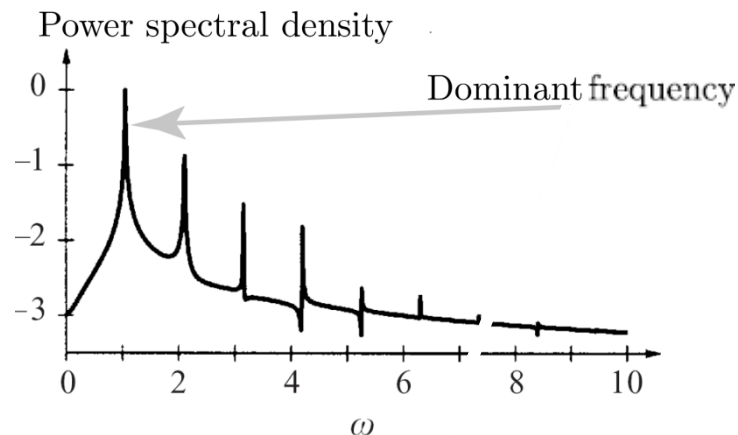
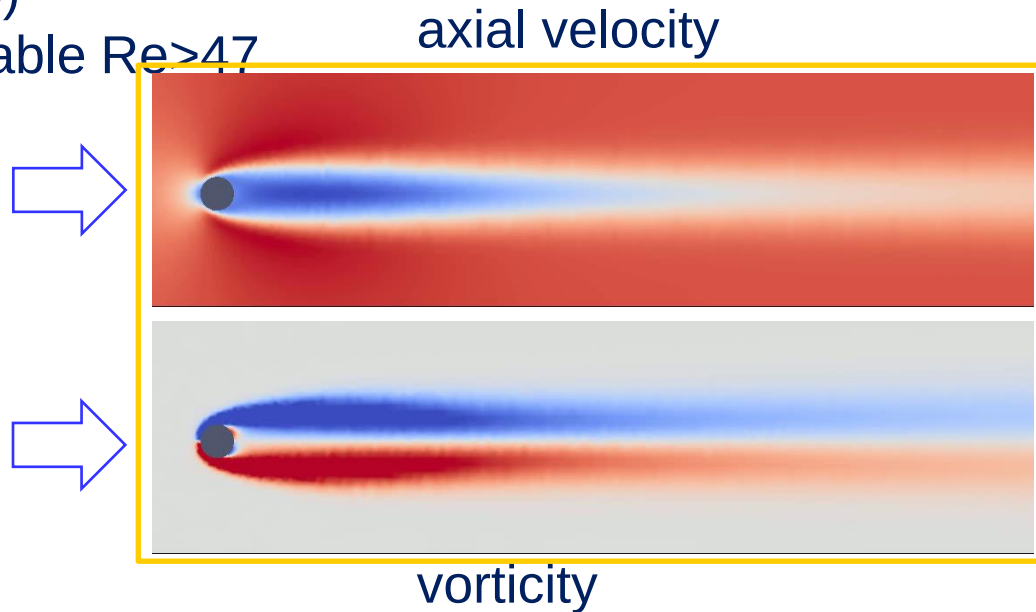
en.wikipedia.org/wiki/File:Vortex-street-1.jpg

Cylinder wake

Cylinder wake

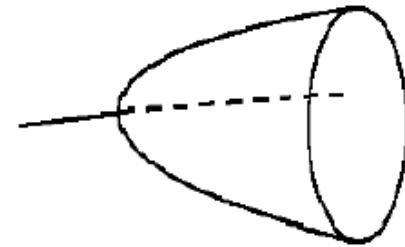
Oscillator, intrinsic dynamics, absolutely unstable (Triantafyllou 86, Monkewitz 88)

Globally unstable $Re > 47$

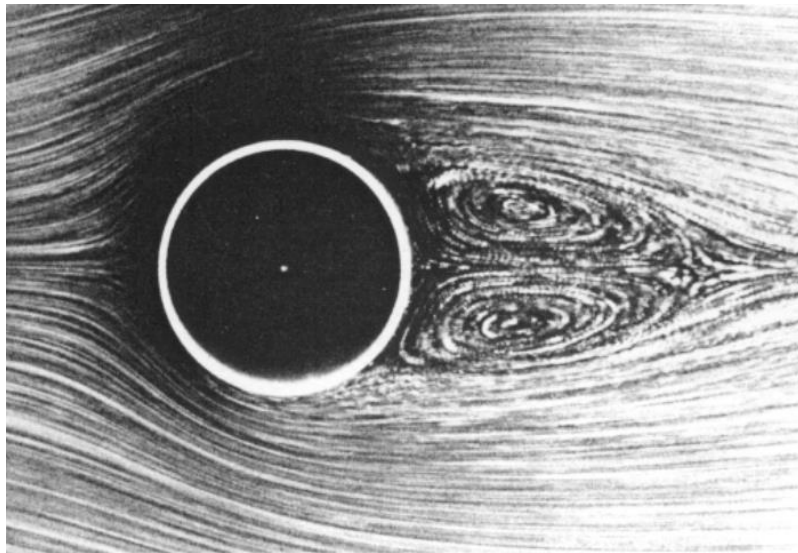


Canonical example of Hopf bifurcation Bénard-von Karman street

Supercritical Hopf Bifurcation



$$Re = 26 < Re_c$$



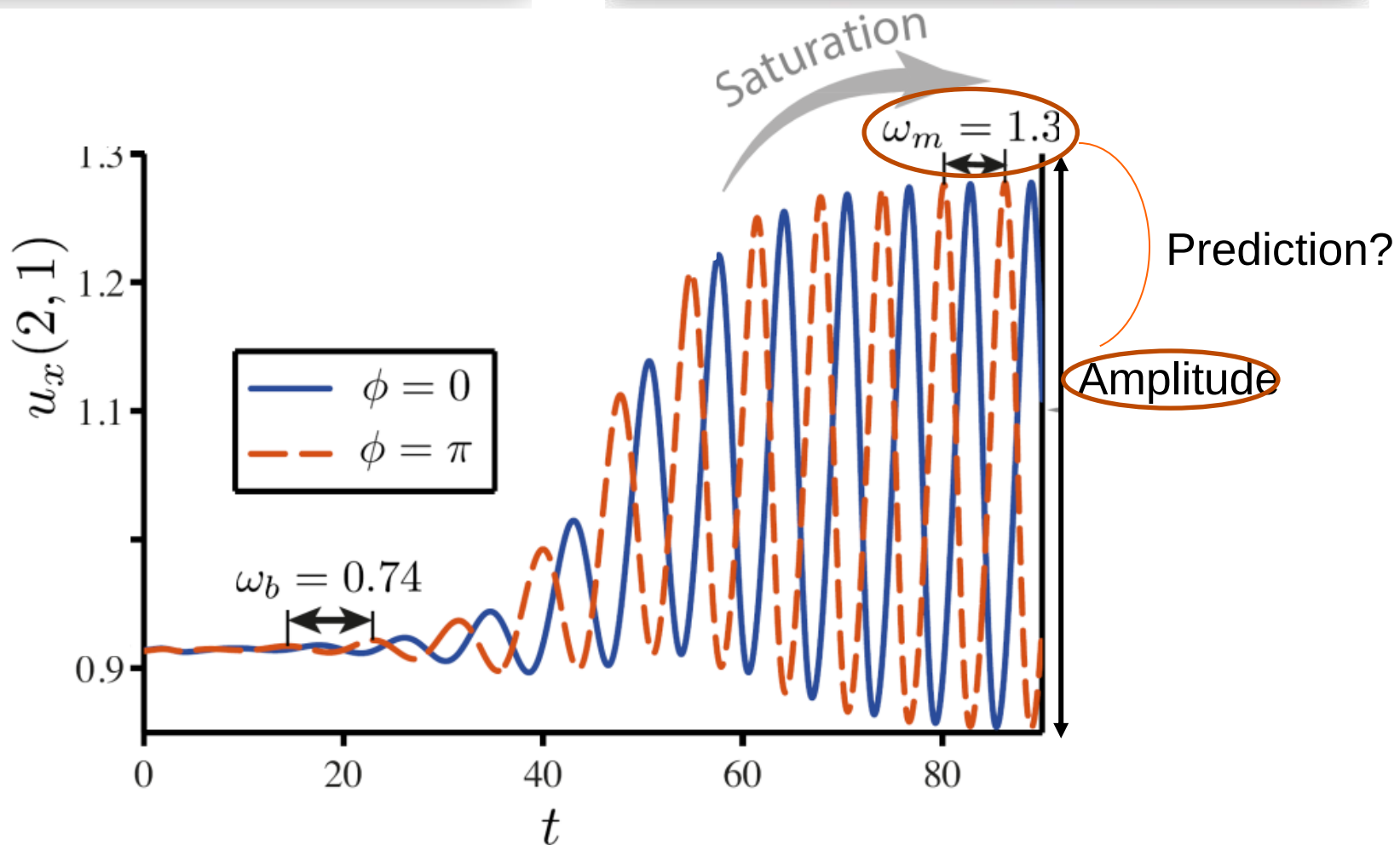
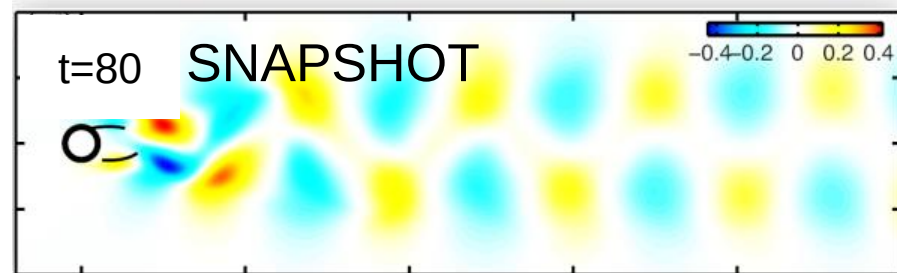
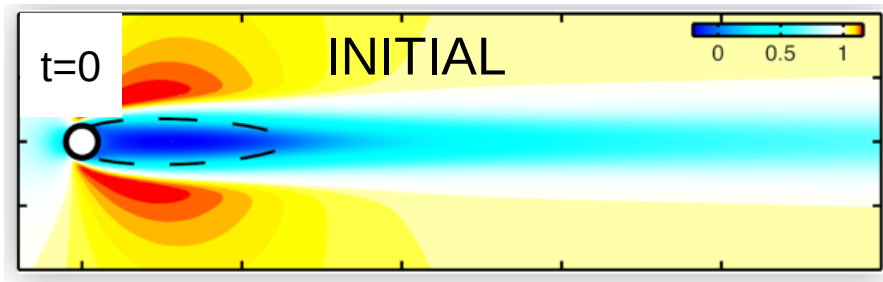
$$Re = 140 > Re_c$$



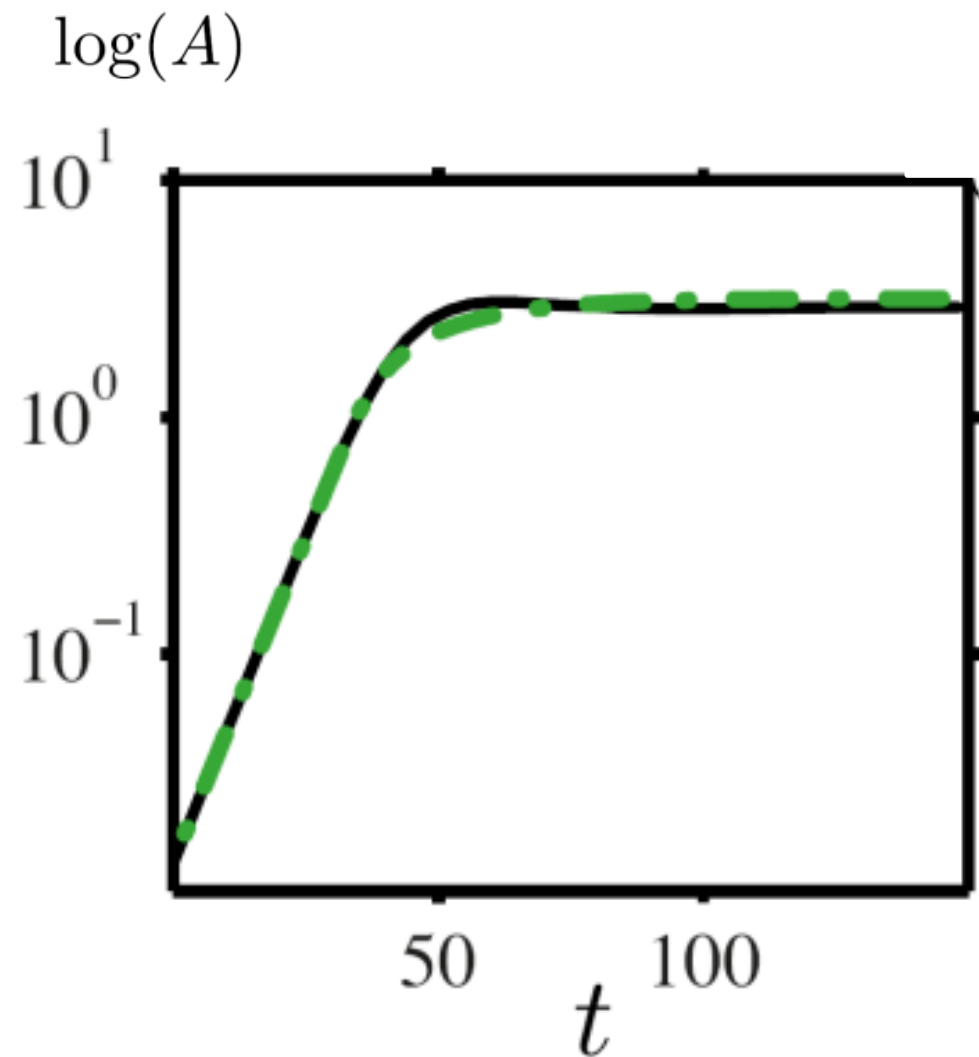
$$Re_c \approx 47$$

Threshold

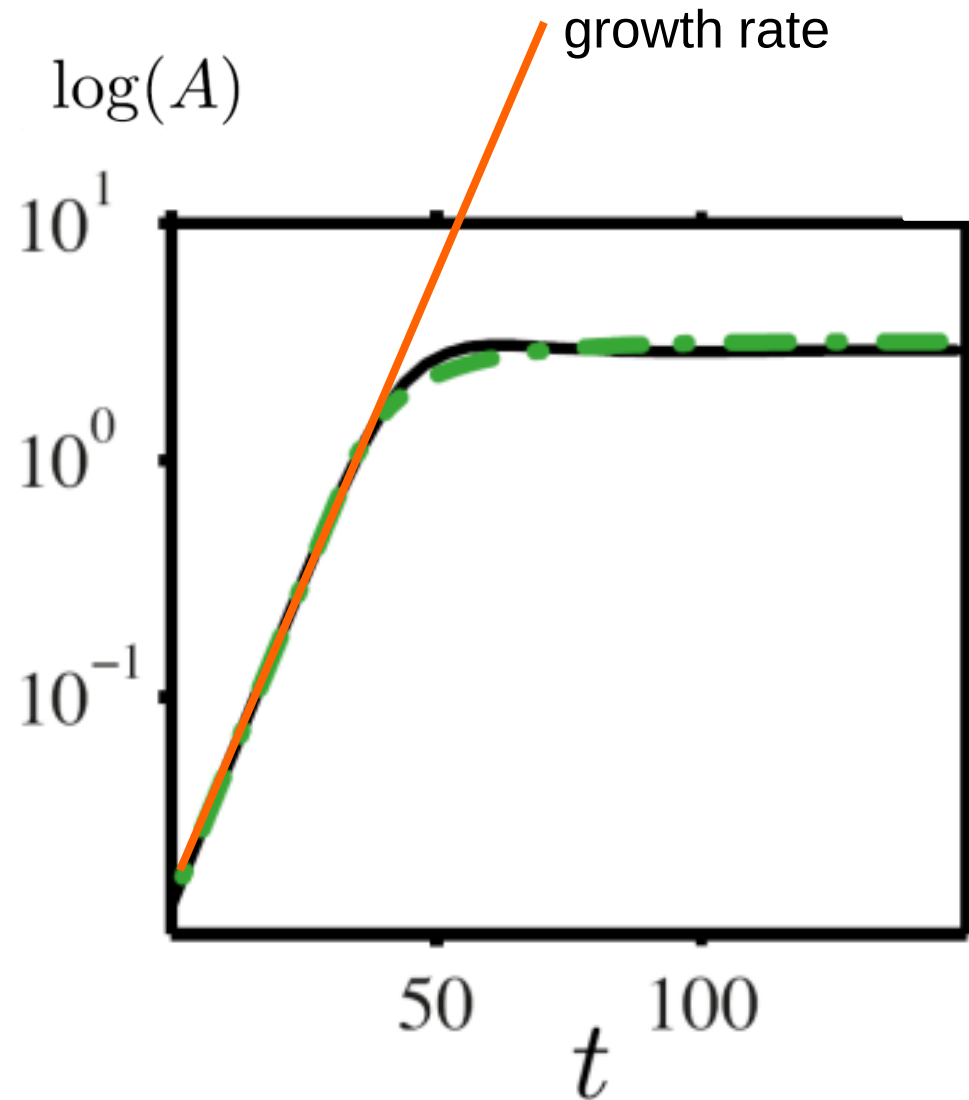
Nonlinear Saturation



Saturation



Saturation...preceded by exponential growth



Linear stability analysis

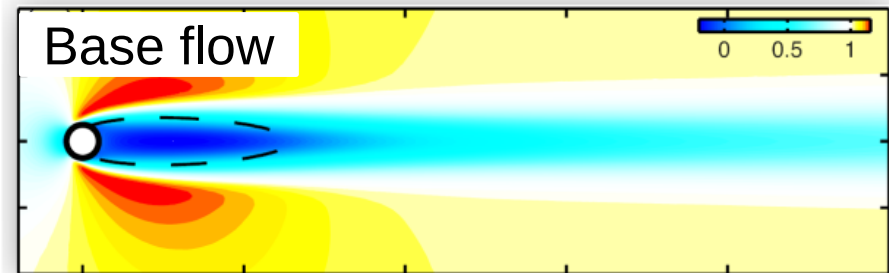
Perturbation expansion

$$(\mathbf{u}, p) = (\mathbf{U}, P) + (\mathbf{u}', p')$$

Stationary base flow Perturbations

Base flow equations

$$\begin{aligned}\nabla \mathbf{U} \cdot \mathbf{U} &= -\nabla P + Re^{-1} \nabla^2 \mathbf{U}, \\ \nabla \cdot \mathbf{U} &= 0\end{aligned}$$



Linearized perturbation equations

$$\begin{aligned}\partial_t \mathbf{u}' + \nabla \mathbf{U} \cdot \mathbf{u}' + \nabla \mathbf{u}' \cdot \mathbf{U} + \cancel{\nabla \mathbf{u}' \cdot \mathbf{u}'} &= -\nabla p' + Re^{-1} \nabla^2 \mathbf{u}', \\ \nabla \cdot \mathbf{u}' &= 0\end{aligned}$$

Global stability analysis

$$(\mathbf{u}', p')(x, y, t) = (\hat{\mathbf{u}}, \hat{p})(x, y) \exp[\sigma t]$$

$\uparrow$
 Global mode

$\sigma = \lambda + i\omega$

$\uparrow \quad \uparrow$
 Growth-rate frequency

$St = \frac{\omega}{2\pi}$

Singular generalized eigenvalue problem

$$\sigma \hat{\mathbf{u}} + \nabla \hat{\mathbf{u}} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \hat{\mathbf{u}} = -\nabla \hat{p} + Re^{-1} \nabla^2 \hat{\mathbf{u}},$$

$$\nabla \cdot \hat{\mathbf{u}} = 0,$$

Global stability analysis solvers

For a given value of Re , numerically solve

- Non linear equations,

$$\nabla U \cdot U = -\nabla P + Re^{-1} \nabla^2 U,$$

(Newton method)

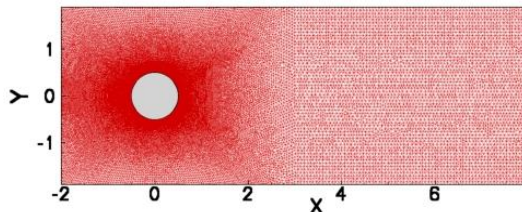
- Eigenvalue problem

$$\sigma \hat{u} + \nabla \hat{u} \cdot U + \nabla U \cdot \hat{u} = -\nabla \hat{p} + Re^{-1} \nabla^2 \hat{u},$$

(Krylov-Arnoldi method)

Spatial discretization = finite element methods

(FreeFem++ freeware)



Taylor-Hood finite elements (P2,P2,P1)

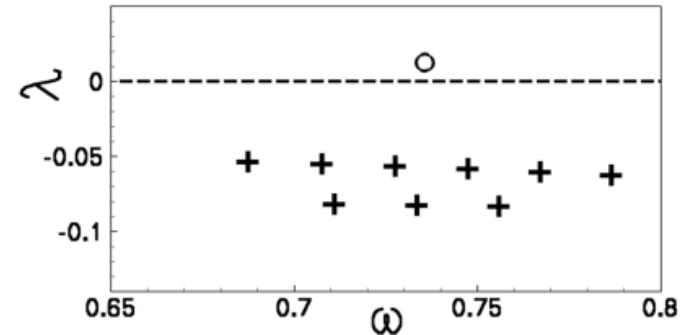
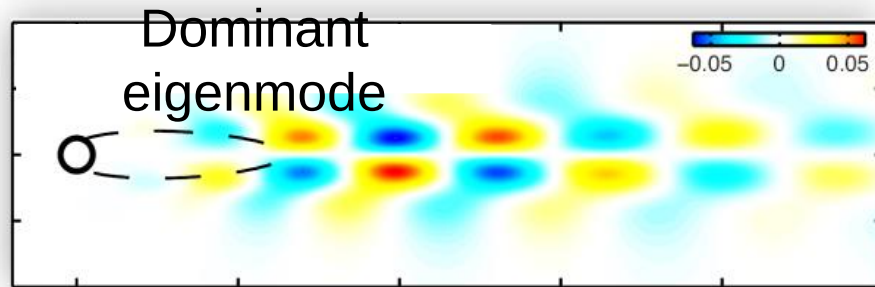
→ number of degrees of freedom ~ $O(10^6)$

Dominant eigenvalue

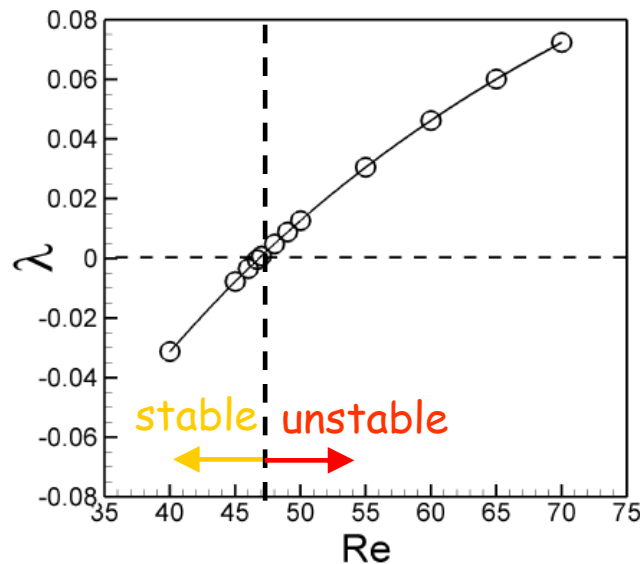
$$(\hat{u}, \hat{p}) \exp[\sigma t]$$

$$\sigma = \lambda + i\omega$$

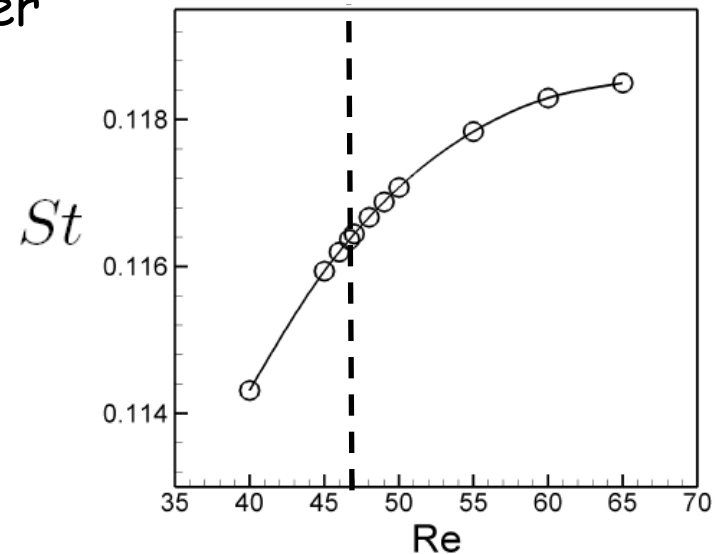
Spectrum at $Re = 50$



Evolution as a function of the Reynolds number



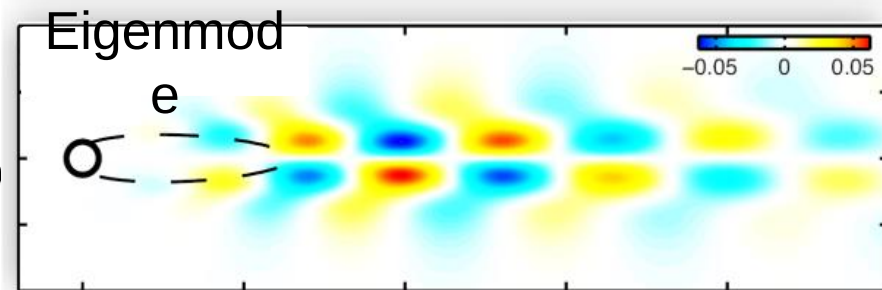
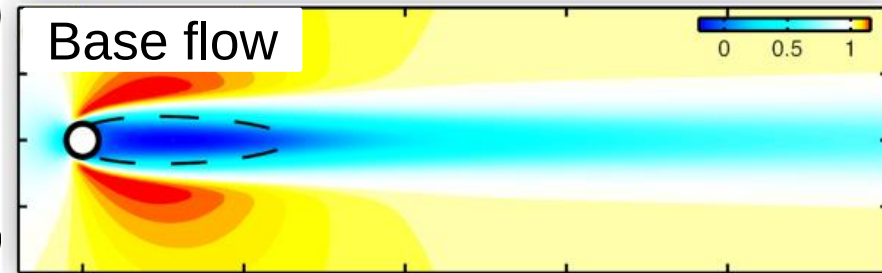
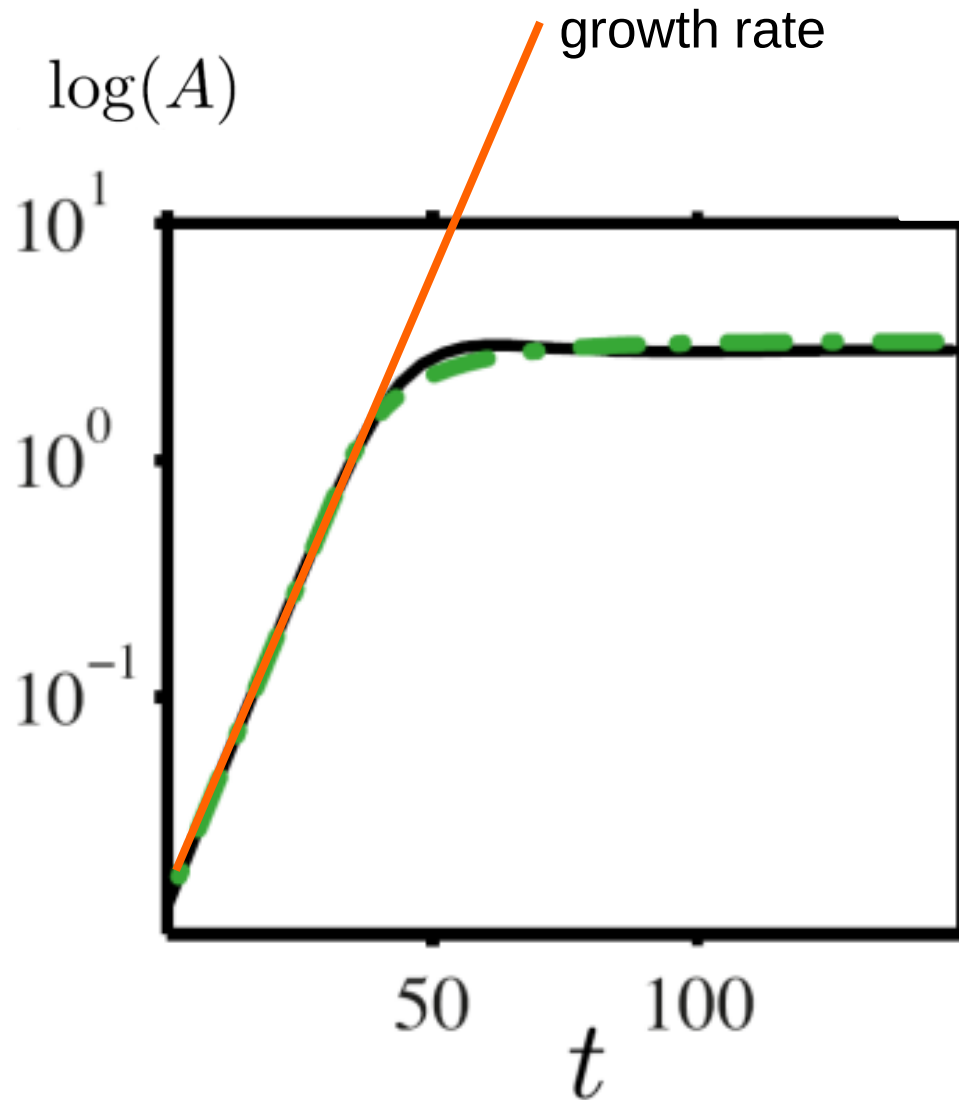
$Re_c \sim 47$



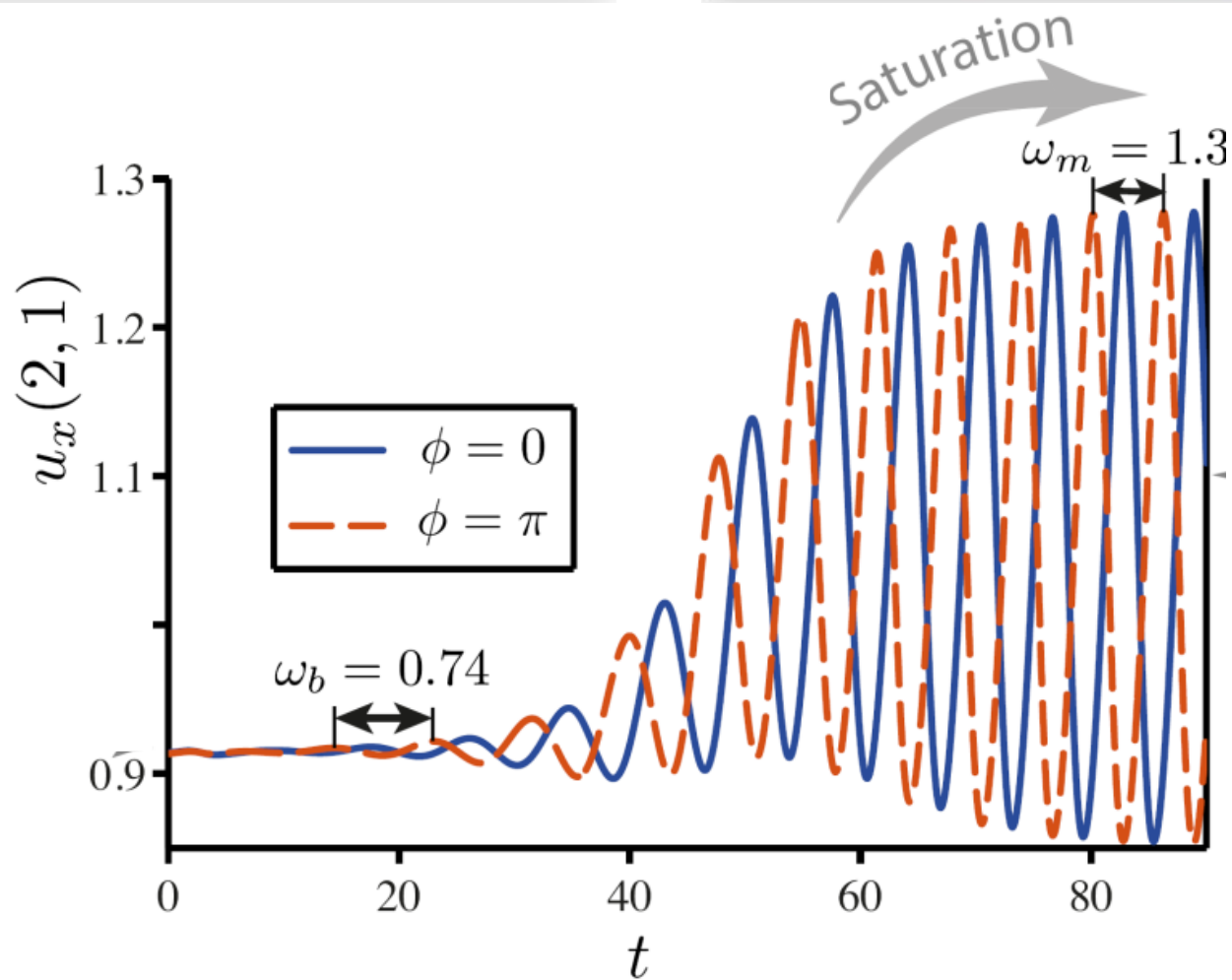
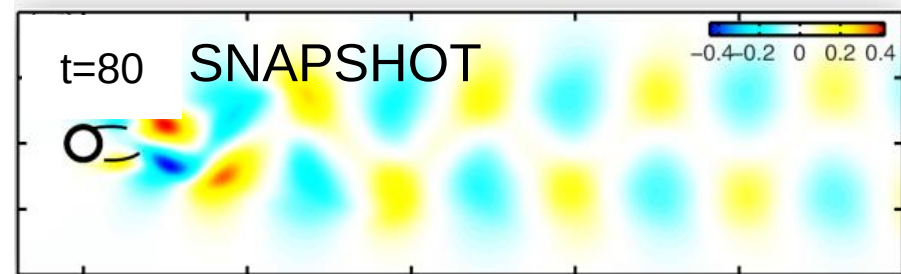
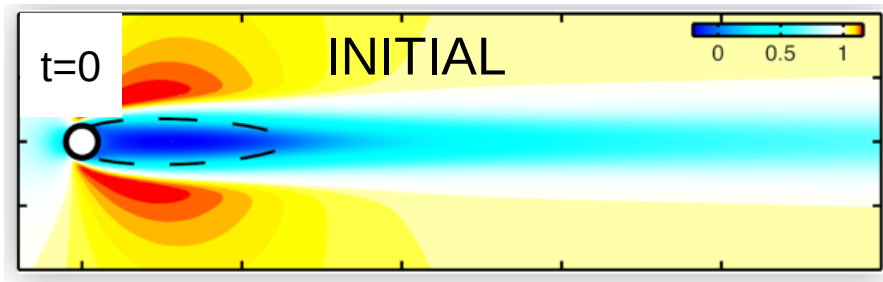
$St_c \sim 0.11$

Jackson (1987), Zebib (1987), Ding & Kawahara (1999), Barkley (2006), Giannetti & Luchini (2003, 2007), Sipp & Lebedev (2007), Marquet, Sipp & Jacquin (2009)...

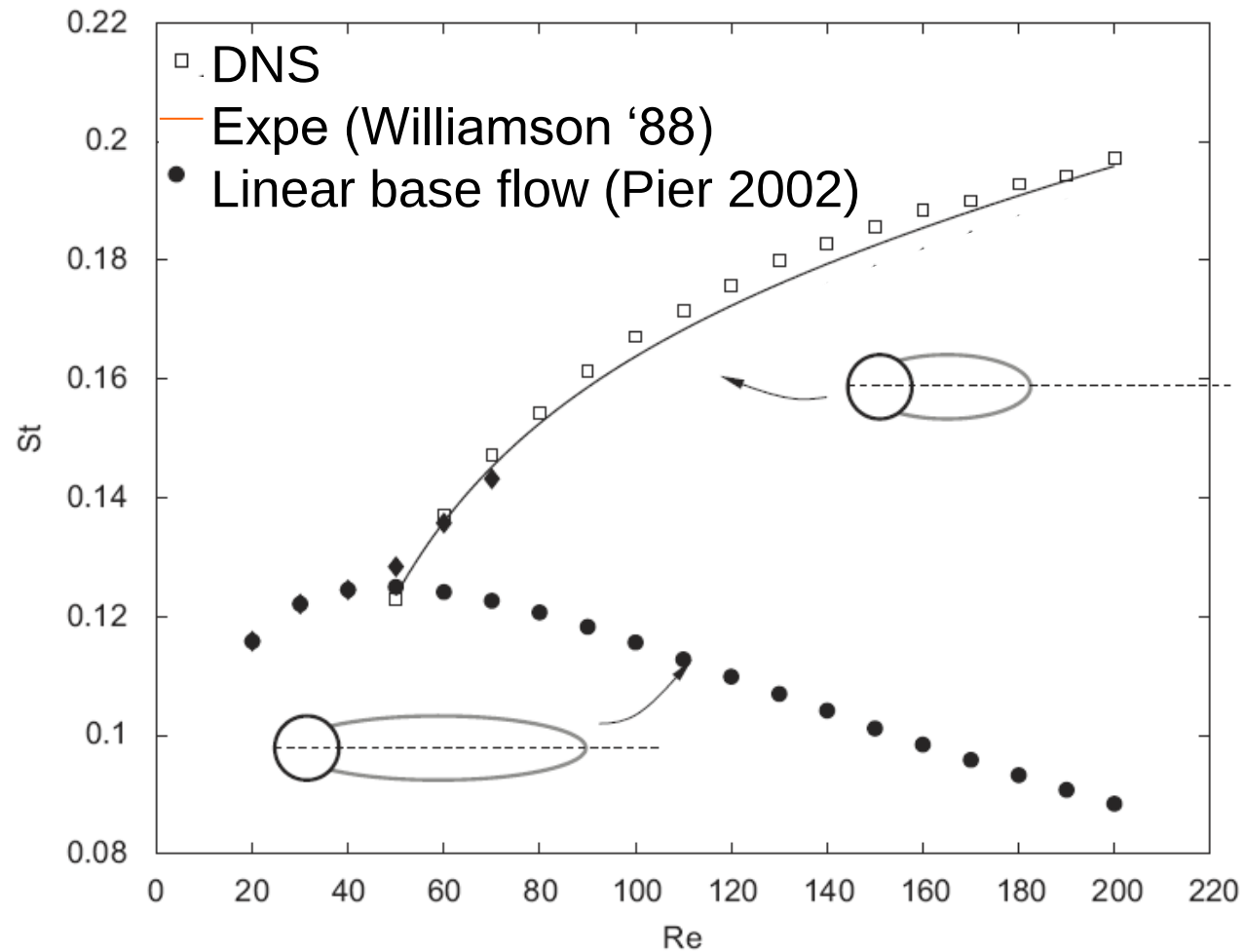
Saturation...preceded by exponential growth



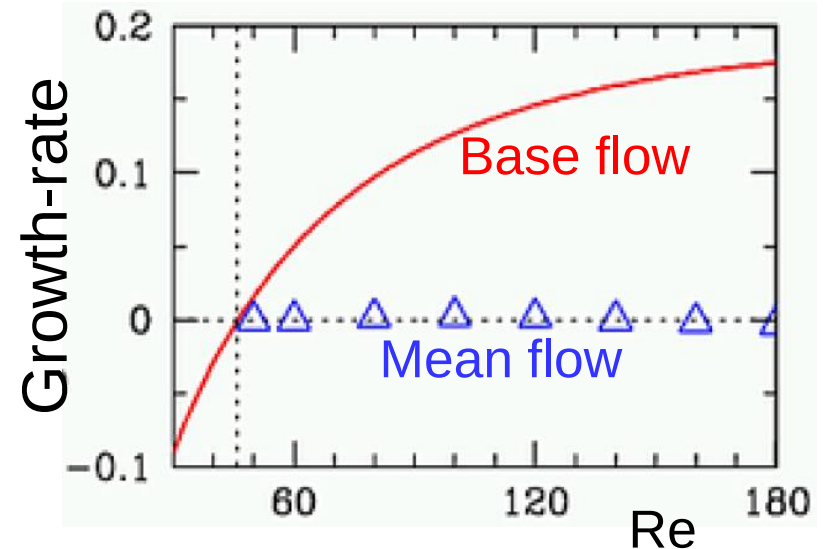
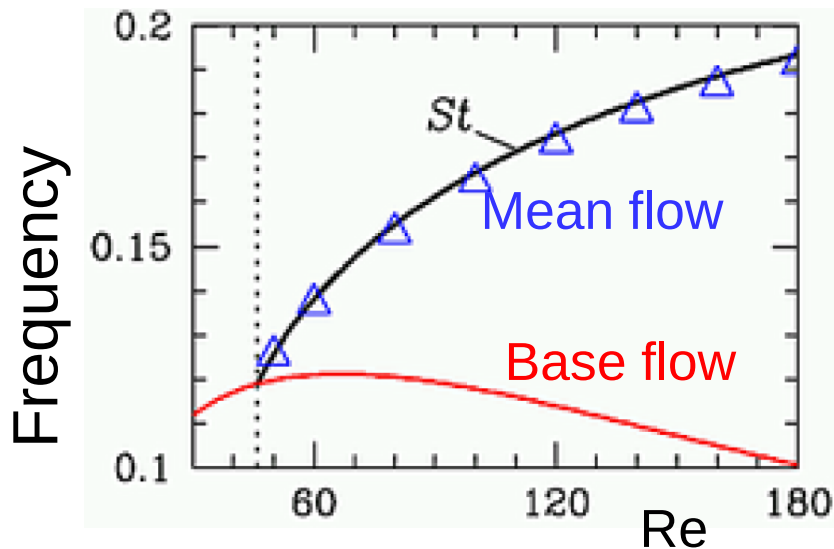
Frequency correction



Frequency correction



The mean flow is
neutrally (marginally) stable

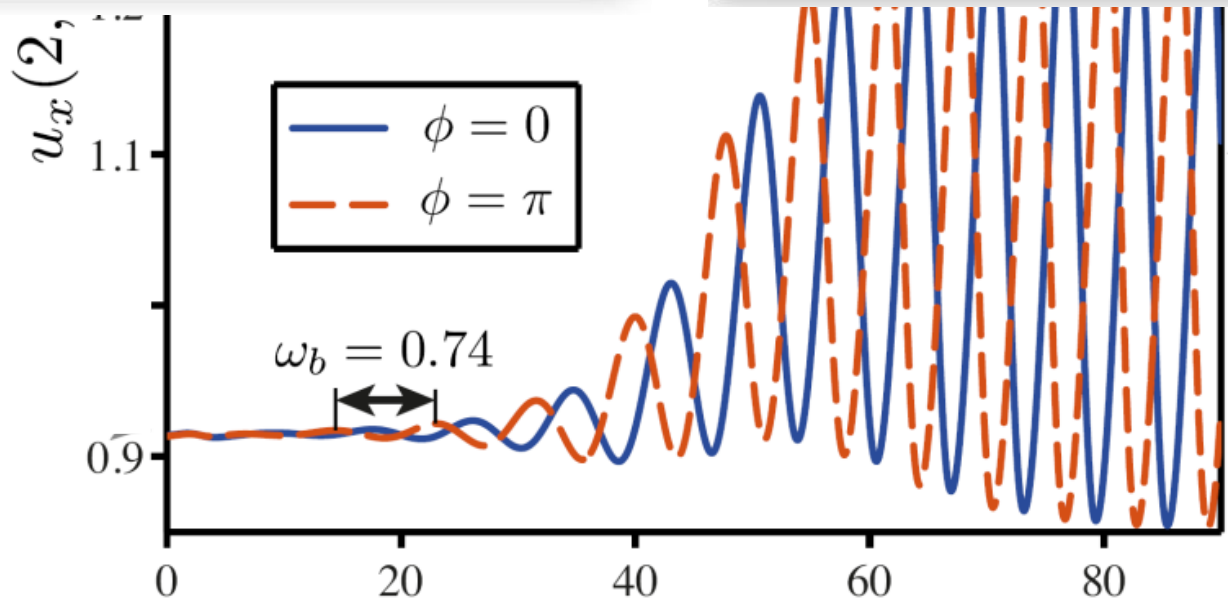
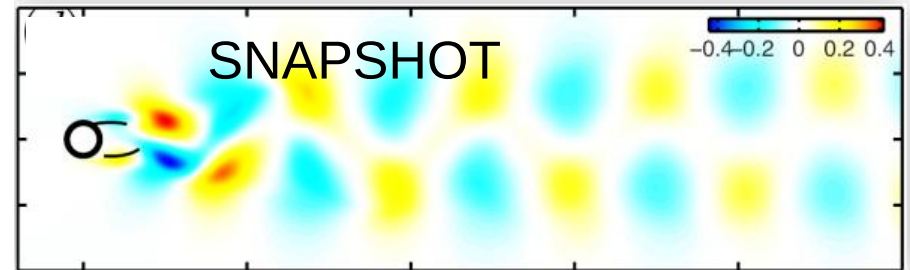
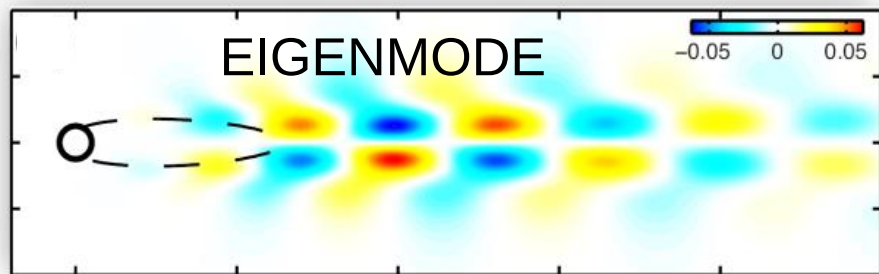
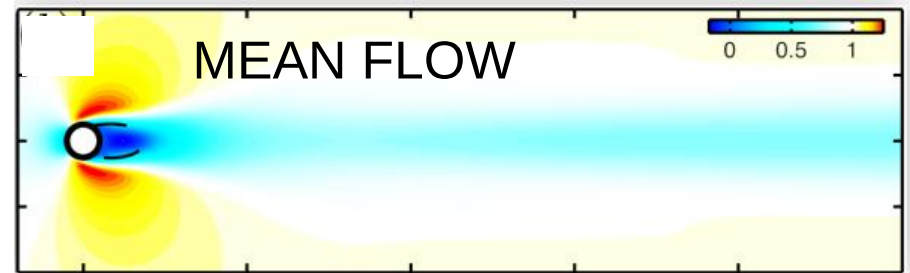
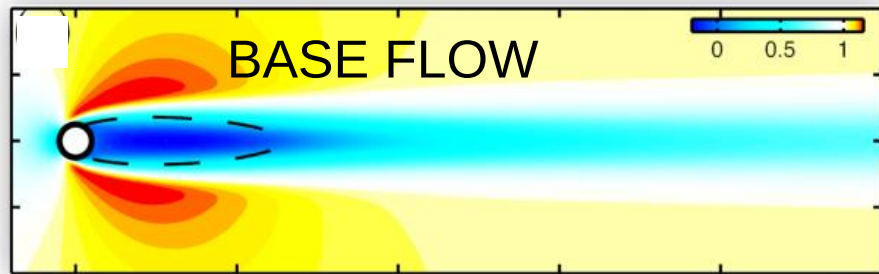


Barkley (2006), Malkus (1956)

Two limitations:

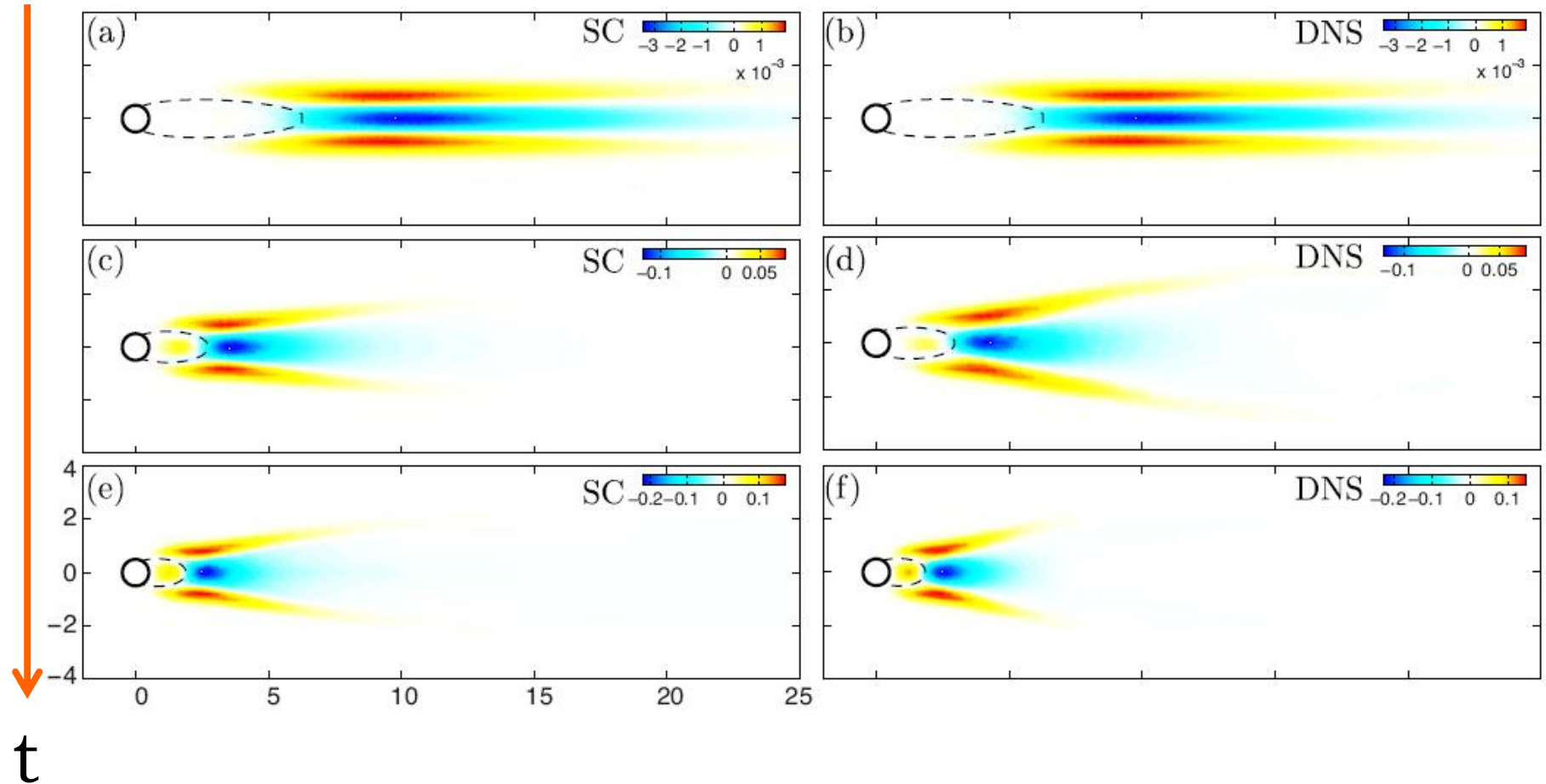
A posteriori prediction: need mean flow
No information on amplitude

Mean flow distortion



Stuart 1958, Maurel, Pagneux and Wesfreid 1995 , Barkley 2006

Transient mean flow correction



Stuart-Landau amplitude equation

$$\frac{dA}{dT} = \lambda \delta A - \mu A |A|^2 ,$$

δ : distance from threshold

Experimental

Sreenivasan, Strykowski & Olinger (1986)

Provansal, Mathis & Boyer (1987)

Numeric

Dusek, Le Gal & Fraunié (1994)

Analytic

Sipp & Lebedev (2007)

Bifurcation theory

Stuart (1960) , Sipp & Lebedev (2007)

Departure from threshold:

$$\frac{1}{Re} - \frac{1}{Re_*} = O(\epsilon^2) \equiv \epsilon^2 \delta$$

slow time scale $T = \epsilon^2 t$

Expansion:

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + \dots$$

q_0 base flow,

q_1 leading order perturbation $q_1 = A(T) \hat{q}_{1A} e^{i\omega_* t} + \text{c.c.}$

$A(T)$ unknown

q_2 second order perturbation, no secular terms with frequency ω_*

$$q_2 = \delta \hat{q}_{2\delta} + |A|^2 \hat{q}_{2|A|^2} + (A^2 \hat{q}_{2A^2} e^{2i\omega_* t} + \text{c.c.})$$

BF diffusion

Base flow
modifications

harmonics

Bifurcation theory

q_2 second order perturbation, no secular terms with frequency ω_*

$$q_2 = \delta \hat{q}_{2\delta} + |A|^2 \hat{q}_{2|A|^2} + (A^2 \hat{q}_{2A^2} e^{2i\omega_* t} + \text{c.c.})$$

BF diffusion

*Base flow
modifications*

harmonics

$$(\partial_t \mathcal{L} + \mathcal{M}) q_2 = F_2^1 + |A|^2 F_2^{|A|^2} + (A^2 e^{2i\omega_0 t} F_2^{A^2} + \text{c.c.})$$

$$F_2^1 = \begin{pmatrix} -\Delta u_0 \\ 0 \end{pmatrix},$$

$$F_2^{|A|^2} = \begin{pmatrix} -\nabla u_1^A \cdot \overline{u_1^A} - \nabla \overline{u_1^A} \cdot u_1^A \\ 0 \end{pmatrix},$$

$$F_2^{A^2} = \begin{pmatrix} -\nabla u_1^A \cdot \nabla u_1^A \\ 0 \end{pmatrix}.$$

Resonance at third order

q_3

Third order secular (**resonant**) forcing terms

$$\mathcal{B}\partial_t \mathbf{q}_3 + \mathbf{L}_* \mathbf{q}_3 = (\hat{\mathbf{F}}_{3r} e^{i\omega_* t} + \mathbf{F}_{3nr} + \text{c.c.}, 0)^T .$$

$$\hat{\mathbf{F}}_{3r} = -\frac{dA}{dT} \hat{\mathbf{u}}_{1A} + \delta A \hat{\mathbf{F}}_{3A} + A|A|^2 \hat{\mathbf{F}}_{3A|A|^2} ,$$

$$\mathbf{F}_3^A = \begin{pmatrix} -\nabla u_1^A \cdot \nabla u_2^1 - u_2^1 \cdot \nabla u_1^A - \Delta u_1^A \\ 0 \end{pmatrix} ,$$

$$\mathbf{F}_3^{A|A|^2} = \begin{pmatrix} -\nabla u_1^A \cdot \nabla u_2^{|A|^2} - u_2^{|A|^2} \cdot \nabla u_1^A \\ 0 \end{pmatrix} ,$$

$$\mathbf{F}_3^{\bar{A}A^2} = \begin{pmatrix} -\nabla \bar{u}_1^A \cdot \nabla u_2^{A^2} - u_2^{A^2} \cdot \nabla \bar{u}_1^A \\ 0 \end{pmatrix} ,$$

Resonance at third order

q_3

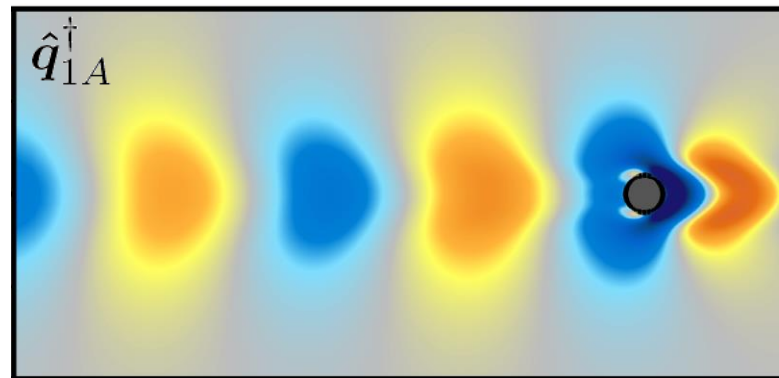
Third order secular (resonant) forcing terms

$$\mathcal{B}\partial_t \mathbf{q}_3 + \mathbf{L}_* \mathbf{q}_3 = (\hat{\mathbf{F}}_{3r} e^{i\omega_* t} + \mathbf{F}_{3nr} + \text{c.c.}, 0)^T .$$

$$\hat{\mathbf{F}}_{3r} = -\frac{dA}{dT} \hat{\mathbf{u}}_{1A} + \delta A \hat{\mathbf{F}}_{3A} + A|A|^2 \hat{\mathbf{F}}_{3A|A|^2} ,$$

⇒ The Fredholm alternative $\hat{\mathbf{F}}_{3r}$ orthogonal to the adjoint of $\hat{\mathbf{q}}_{1A}$

$$\nabla \cdot \hat{\mathbf{u}}^\dagger = 0, \quad \partial_t \hat{\mathbf{u}}^\dagger + \nabla U^T \cdot \hat{\mathbf{u}}^\dagger - \nabla \hat{\mathbf{u}}^\dagger \cdot \mathbf{U} + \nabla \hat{p}^\dagger - \text{Re}^{-1} \nabla^2 \hat{\mathbf{u}}^\dagger = \mathbf{0} ,$$



Giannetti & Luchini (2003), Sipp & Lebedev (2007), Marquet, Sipp & Jacquin (2009)...

Compatibility condition yields closure and the normal form

y $A\varepsilon$ leading order determined by resonant terms at ε^3

$$\frac{dA}{dT} = \lambda \delta A - \mu A |A|^2 ,$$

$$\lambda = \int_{\Sigma} \hat{q}_{1A}^{\dagger} \cdot \hat{F}_{3A} dx dy ,$$

$$\mu = \int_{\Sigma} \hat{q}_{1A}^{\dagger} \cdot \hat{F}_{3A|A|^2} dx dy .$$

$$\int_{\Sigma} \hat{q}_{1A}^{\dagger} \cdot \hat{q}_{1A} dx dy = 1$$

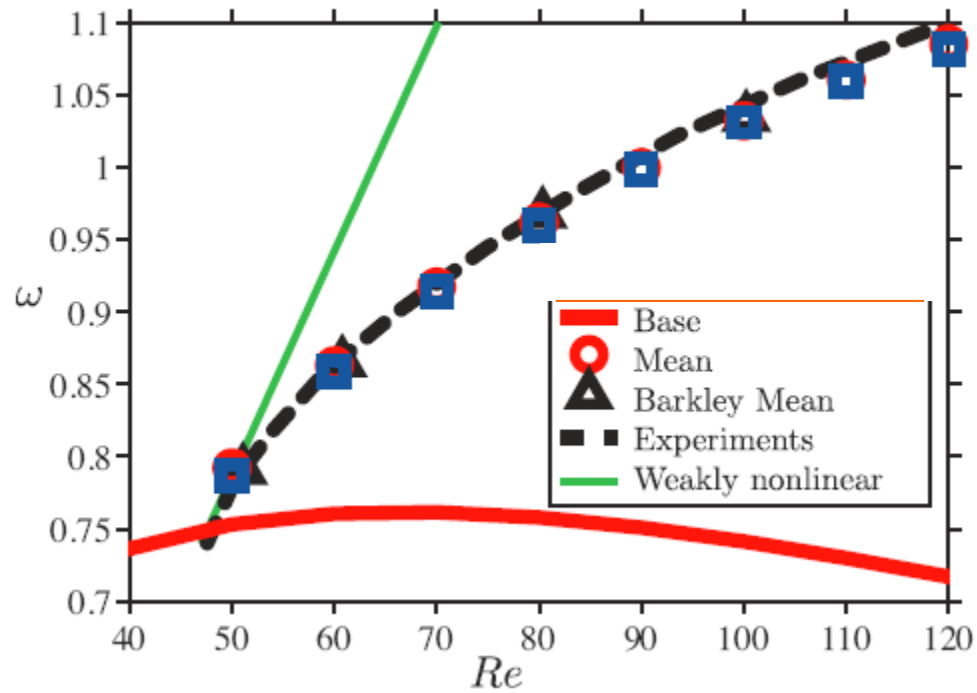
Normal form

$$\frac{dA}{dT} = \lambda \delta A - \mu A |A|^2,$$

$$\mu_r > 0 \Rightarrow \text{predicts saturation} \quad |A|^2 = \frac{\lambda_r \delta}{\mu_r}$$

$$\Rightarrow \text{nonlinear frequency correction} \quad \delta\omega = \lambda_i \delta - \mu_i |A|^2$$

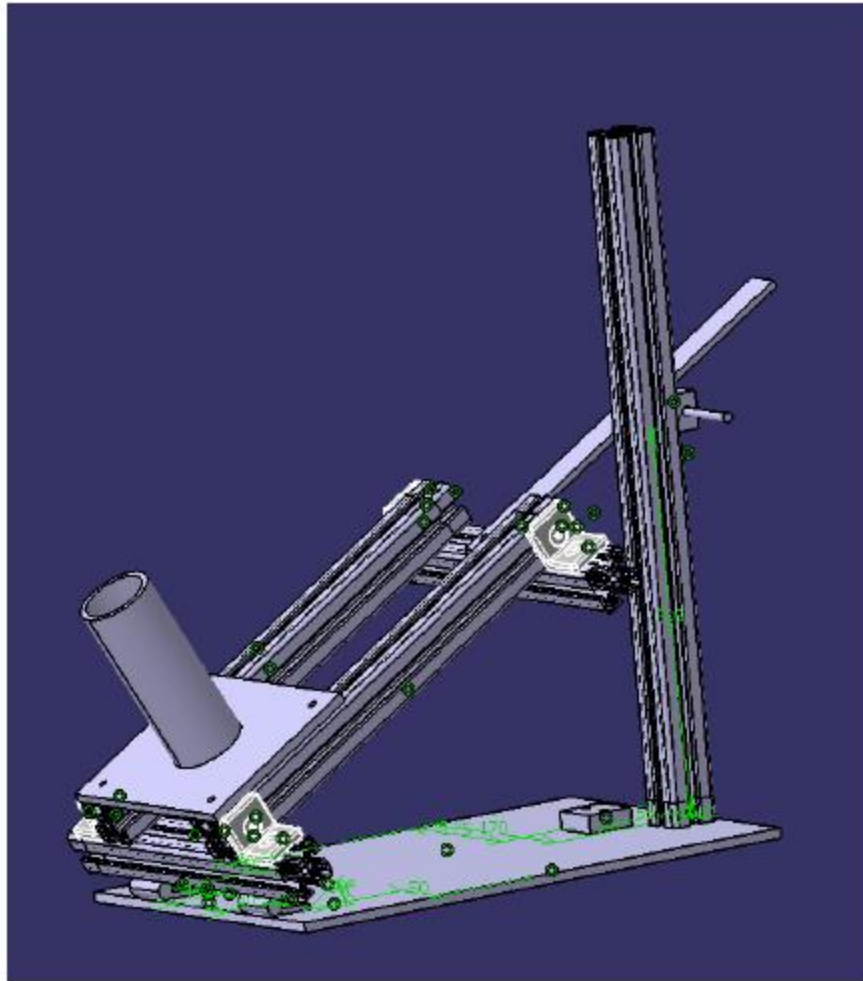
Correct near threshold



Agenda

1. The gravitational pendulum : a poor watch
First use of Multiple scale weakly nonlinear approach
2. A simple example of bifurcation
First use of Multiple scale weakly nonlinear approach
3. Classical bifurcations
4. Hopf bifurcation and Stuart-Landau equation
5. Exercise : a nonlinearly damped fluid oscillator

Impulse response

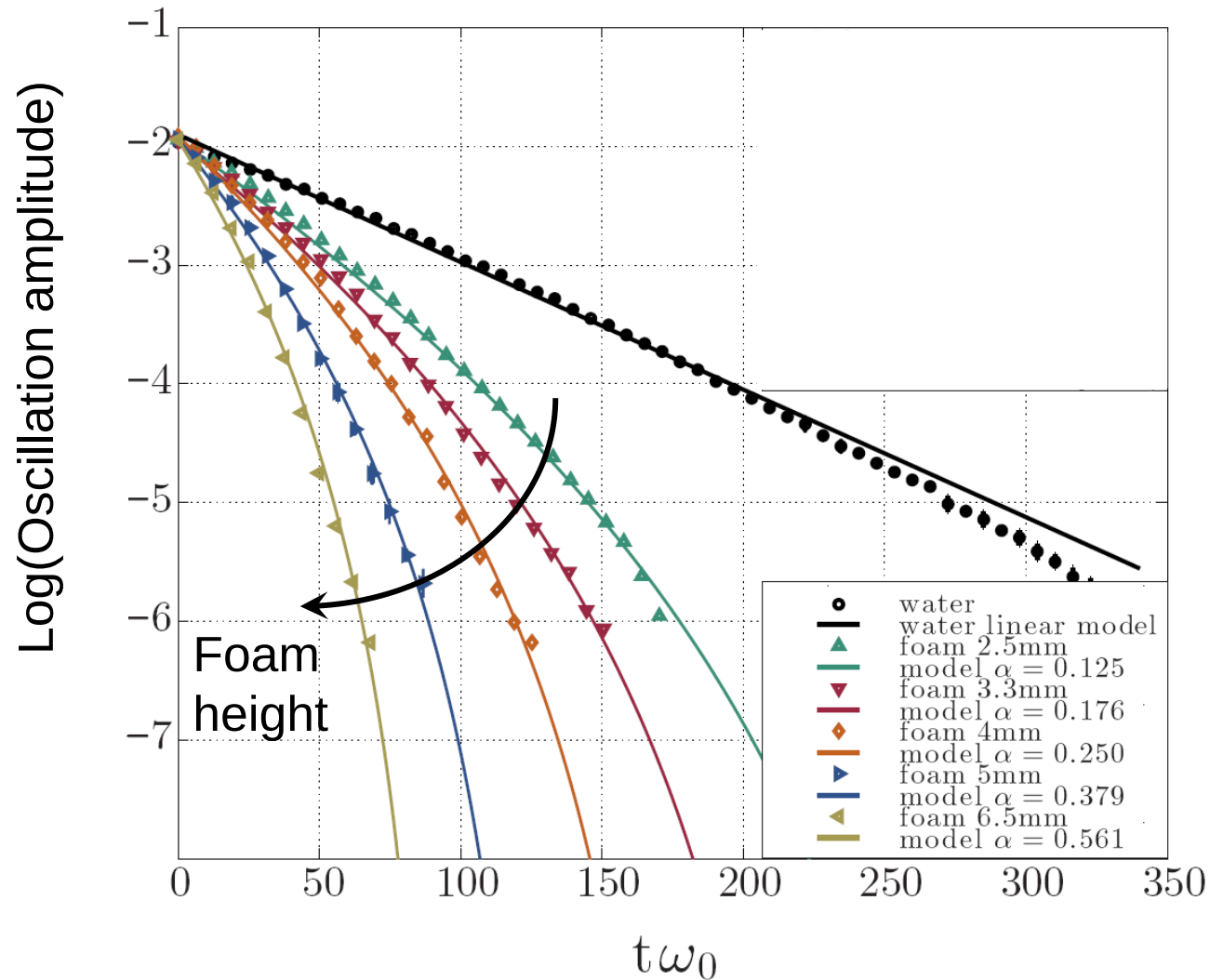


Damping by foam



see also Sauret et al. 2015

Damping by foam



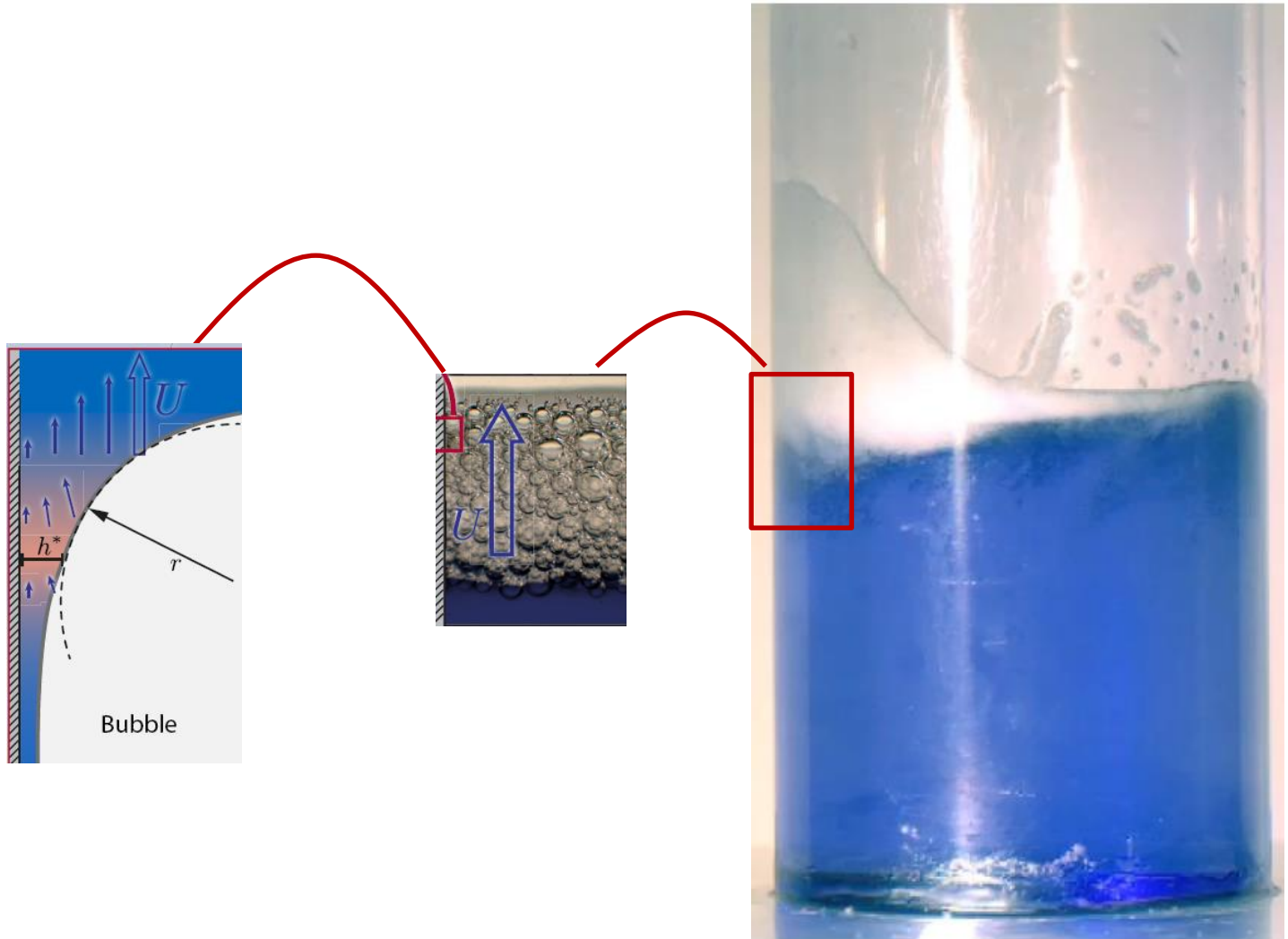
There must be something nonlinear!

Damping by foam to exacerbate nonlinear effects



see also Sauret et al. 2015

Damping by foam



A nonlinearly damped oscillator

$$\pi R^2(\rho\lambda + \rho_f h)\ddot{X} + F_g = F_w + F_b + F_s + F_f$$

- Wall friction $F_w \sim \dot{X}$
- Bulk dissipation $F_b \sim \dot{X}$
- Free surface dissipation $F_s \sim \dot{X}$
- Contact line friction $F_f \sim \dot{X}|\dot{X}|^{-1/3}$

$$\ddot{X} + \omega_0^2 X = -2\sigma\omega_0\dot{X} - \alpha\dot{X}|\dot{X}|^{-1/3}$$