Most flows are unstable...

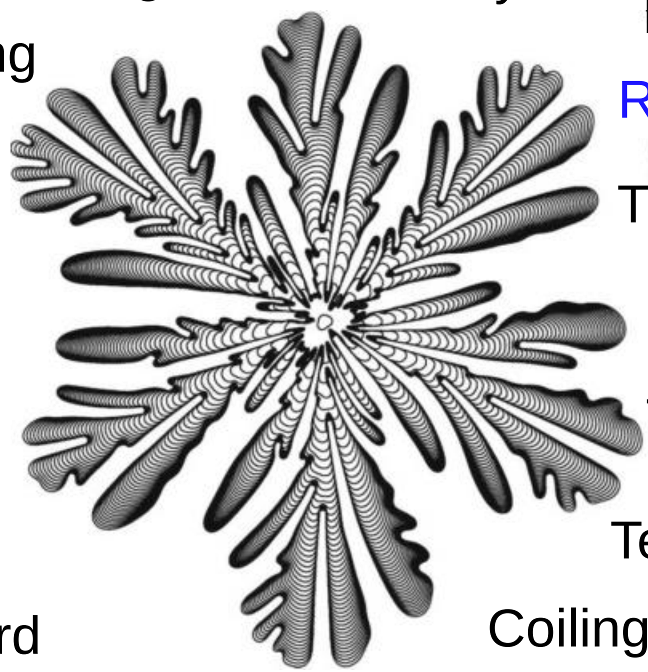
Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor



Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

Tearing instability

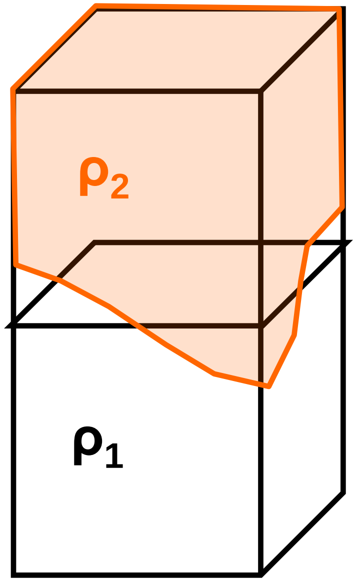
Coiling instability

Rayleigh-Benard

Benard-Marangoni

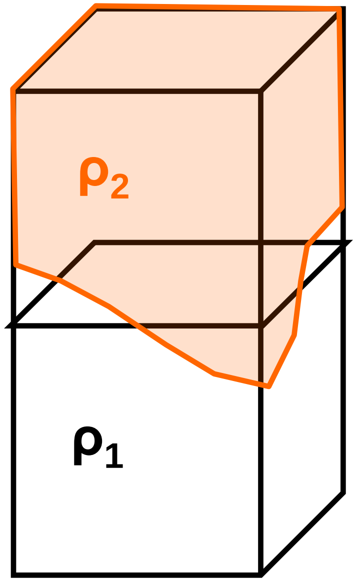
Kelvin-Helmholtz

Dispersion relation

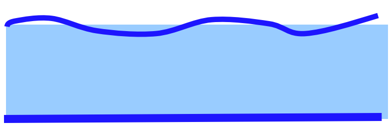


$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

Dispersion relation for water waves



$$\omega^2 = \frac{-kg(\cancel{\rho_2} - \rho_1) + \gamma k^3}{\rho_1 + \cancel{\rho_2}}$$



$$\boxed{\omega^2 = \tanh(kH)\left(\frac{\gamma k^3}{\rho} + gk\right)}$$

Dispersion relation

$$\boxed{\omega^2 = \tanh(kH)\left(\frac{\gamma k^3}{\rho} + gk\right)}$$

Capillary wavenumber: $k_c = \sqrt{\rho g/\gamma}$

Length scale: $\tilde{k} = k/k_c$

Time scale $\tilde{\omega} = \omega/\sqrt{gk_c}$

One single non-dimensional parameter $\tilde{H} = Hk_c$

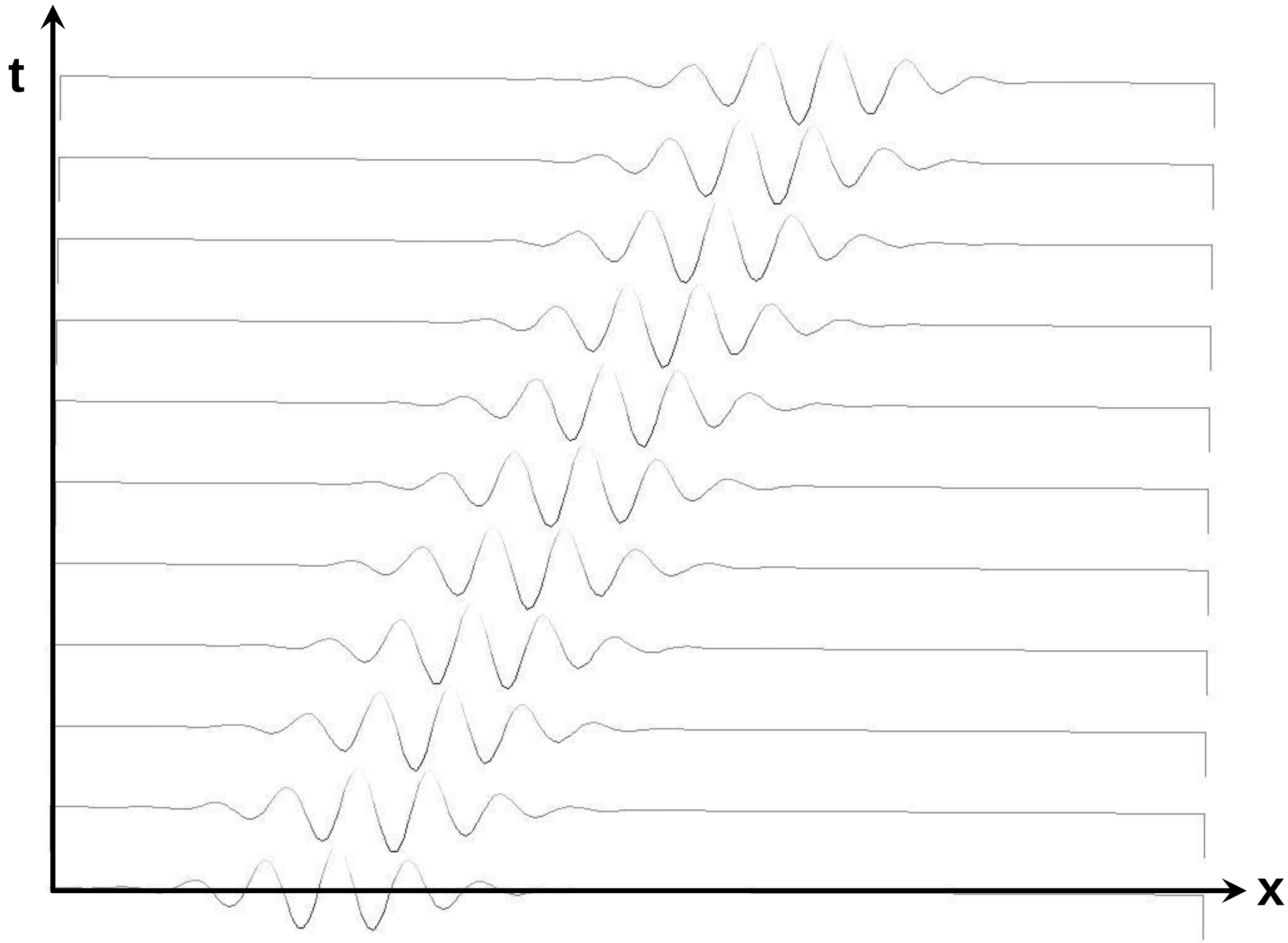
$$\boxed{\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H})\left(\tilde{k}^3 + \tilde{k}\right)}$$

Dispersion relation

$$\boxed{\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H})\left(\tilde{k}^3 + \tilde{k}\right)}$$

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\pm\tilde{k}\,\sqrt{\tilde{H}}$	$\pm\tilde{k}^2\sqrt{\tilde{H}}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\pm\sqrt{\tilde{k}}$	$\pm\tilde{k}\sqrt{\tilde{k}}$

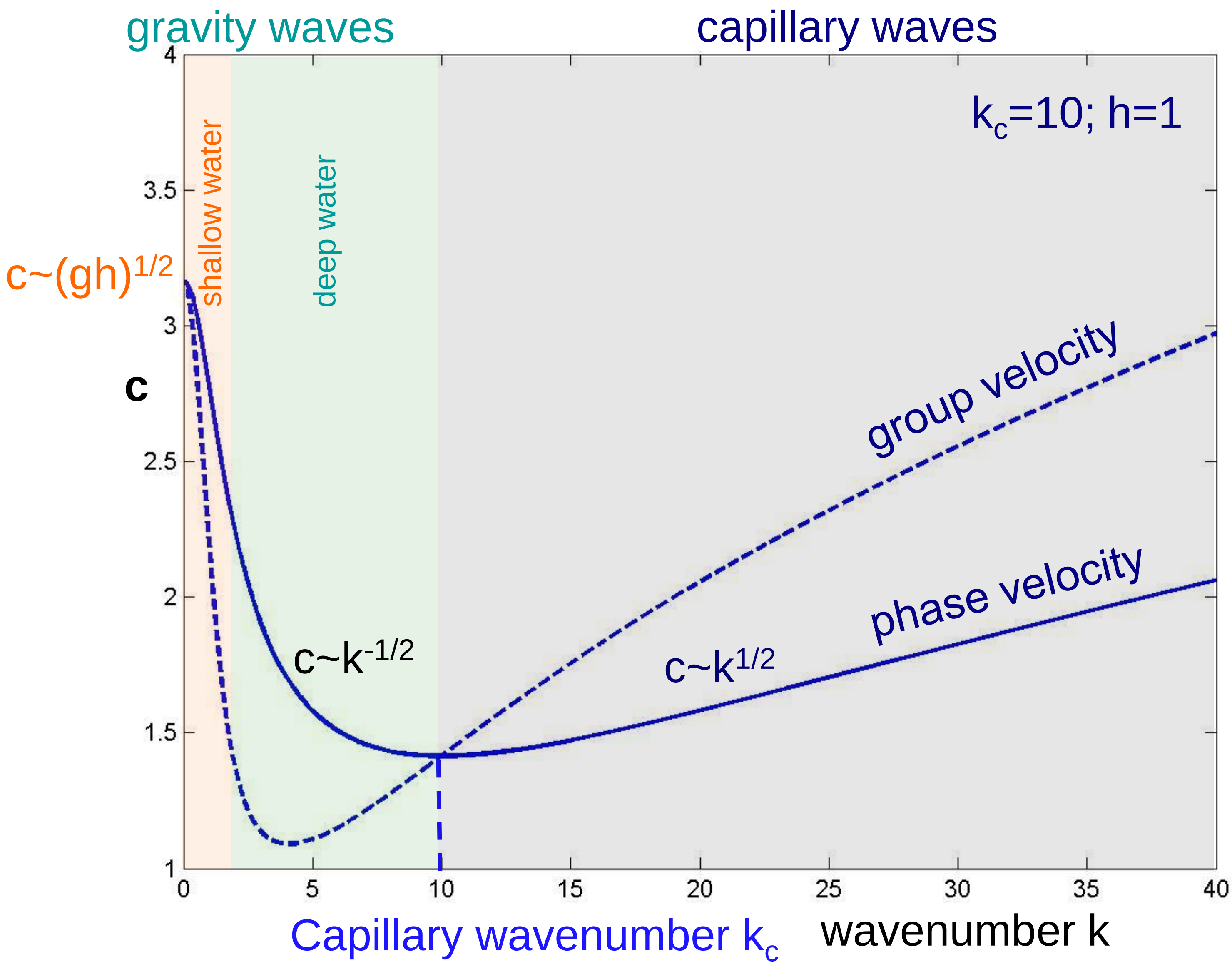
Difference between group velocity v and phase velocity c



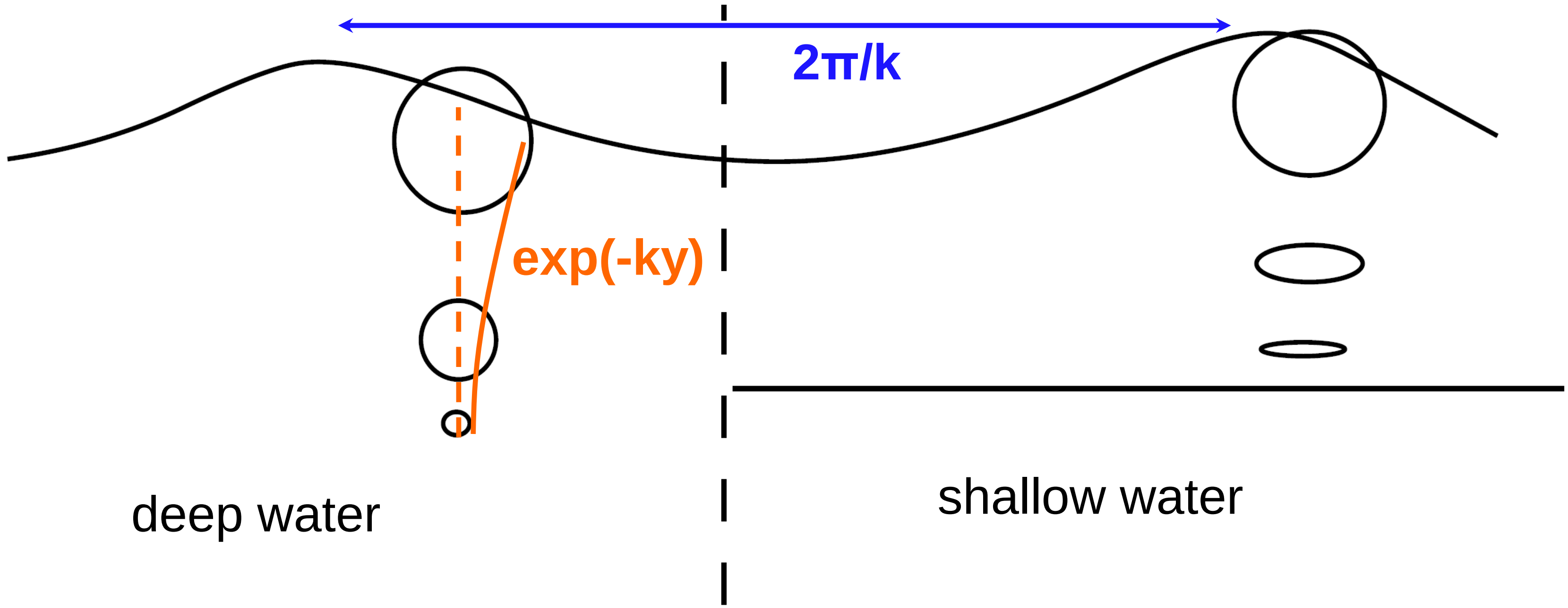
Dispersion relation

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\omega_{shallow/gravity} \sim \pm k\sqrt{gH}$ $c_{shallow/gravity} \sim \pm\sqrt{gH}$ $v_{shallow/gravity} \sim \pm\sqrt{gH}$	$\omega_{shallow/capillary} \sim \pm k^2\sqrt{\gamma H/\rho}$ $c_{shallow/capillary} \sim \pm k\sqrt{\gamma H/\rho}$ $v_{shallow/capillary} \sim \pm 2k\sqrt{\gamma H/\rho}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\omega_{deep/gravity} \sim \pm\sqrt{gk}$ $c_{deep/gravity} \sim \pm\sqrt{\frac{g}{k}}$ $v_{deep/gravity} \sim \pm\frac{1}{2}\sqrt{\frac{g}{k}}$	$\omega_{deep/capillary} \sim \pm k^{3/2}\sqrt{\gamma/\rho}$ $c_{deep/capillary} \sim \pm k^{1/2}\sqrt{\gamma/\rho}$ $v_{deep/capillary} \sim \pm 3/2k^{1/2}\sqrt{\gamma/\rho}$

Dispersion relation



Trajectories below waves

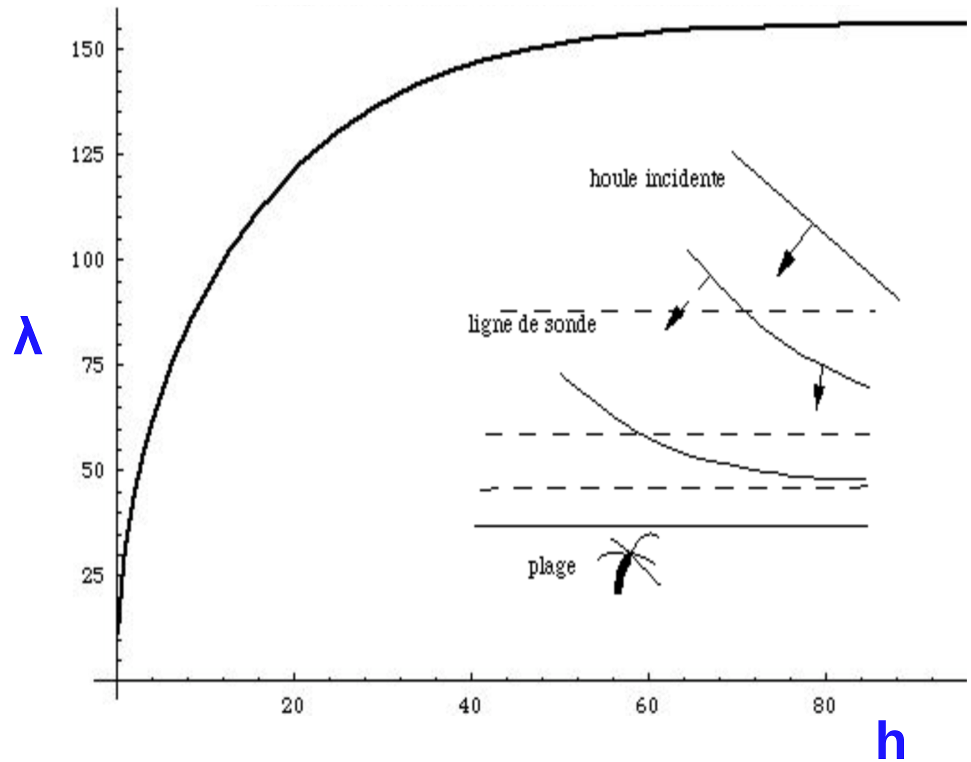


Why are the waves parallel to the shore?

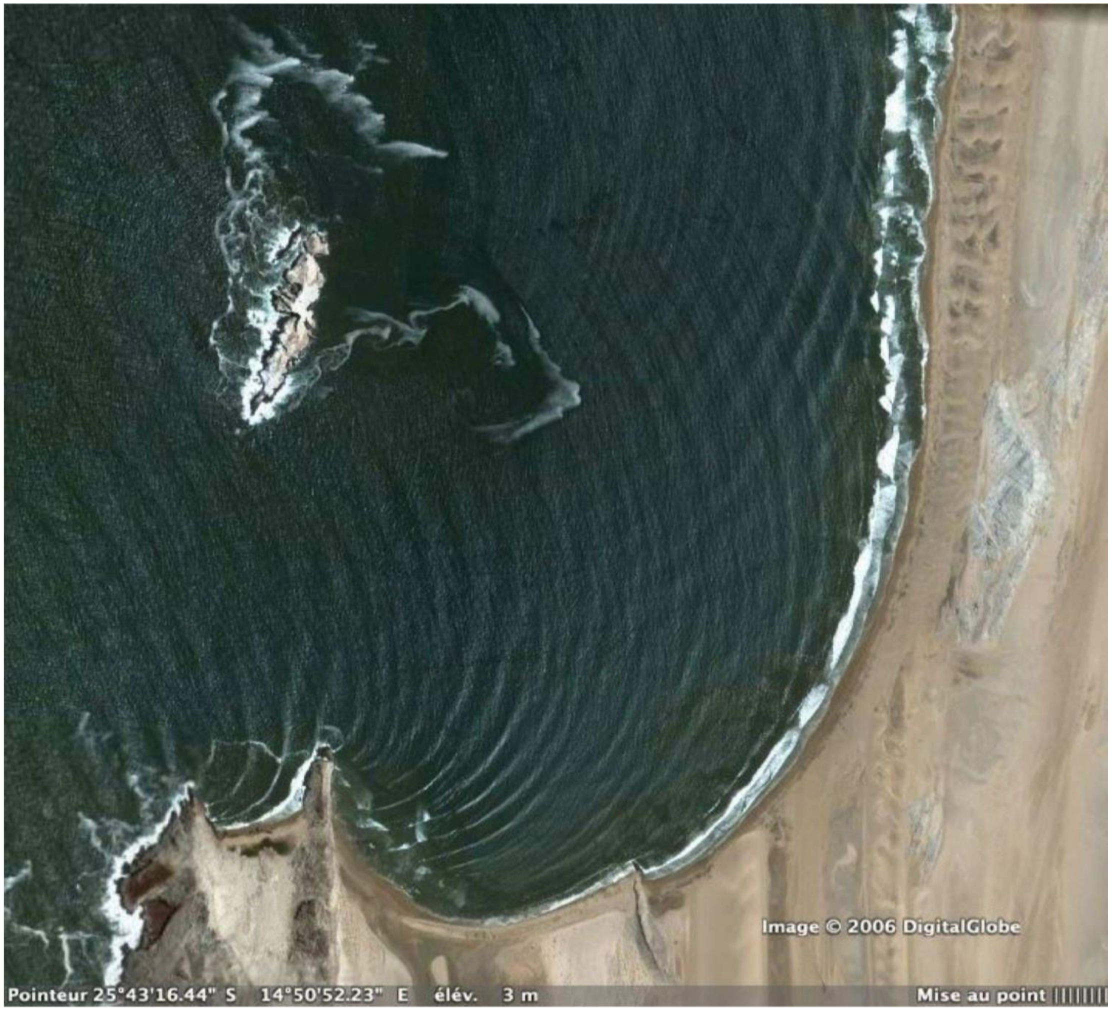
$c \sim (gh)^{1/2}$

$\lambda \sim T(gh)^{1/2}$

T=10s; ω=0.62

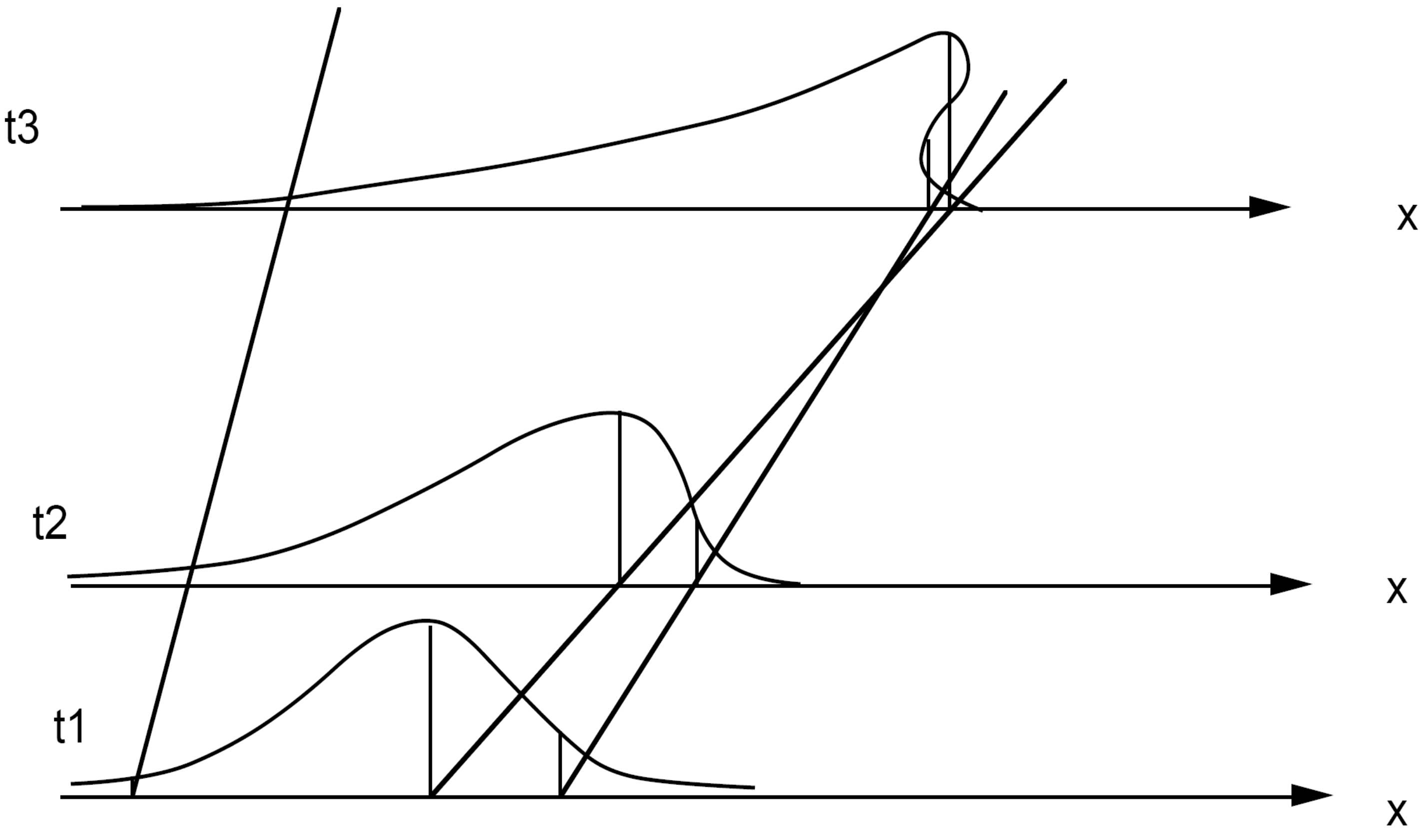


Refraction and diffraction of waves



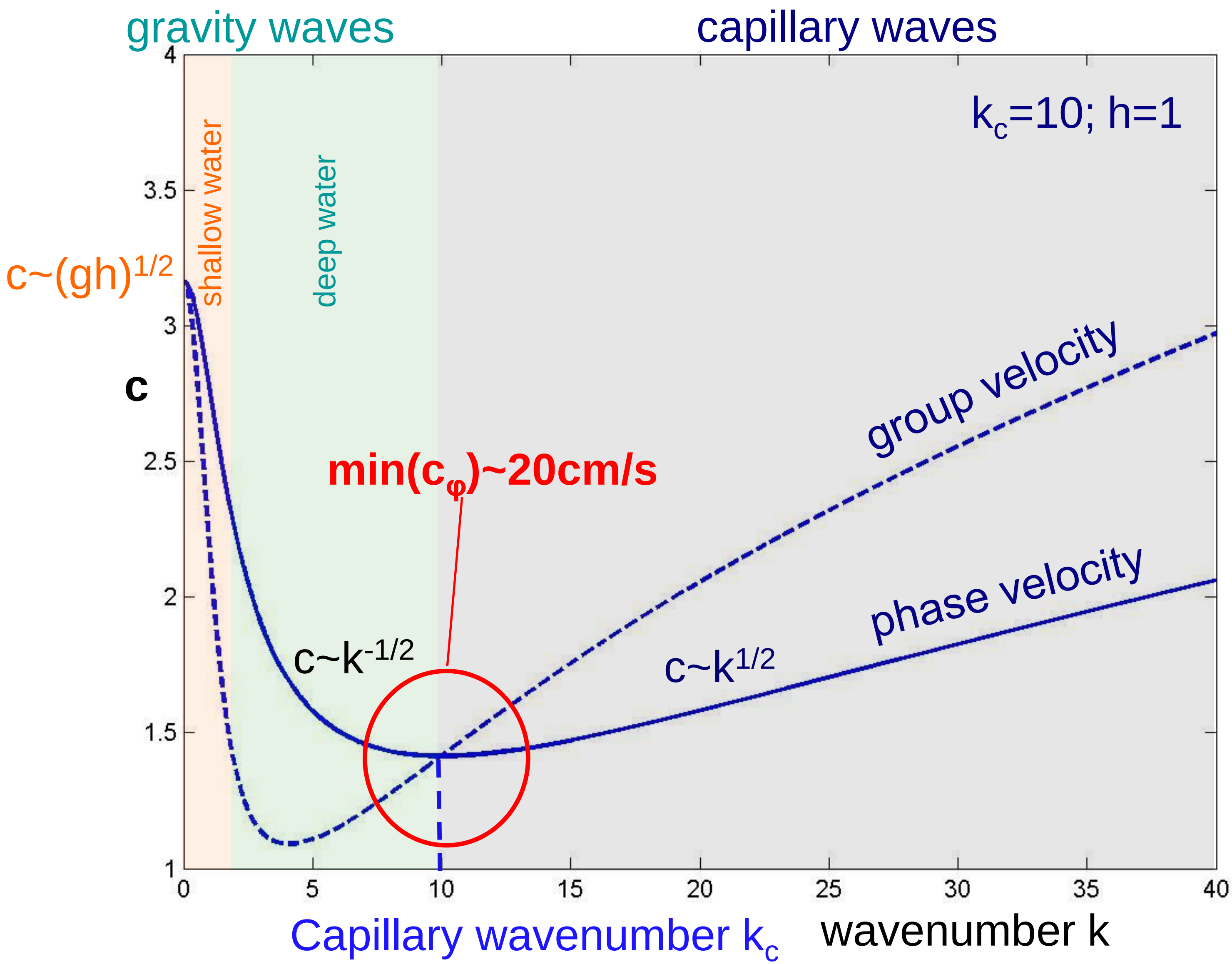
Satellite view of the shore of Namibia

Nonlinear waves/wave breaking



The velocity increases with amplitude

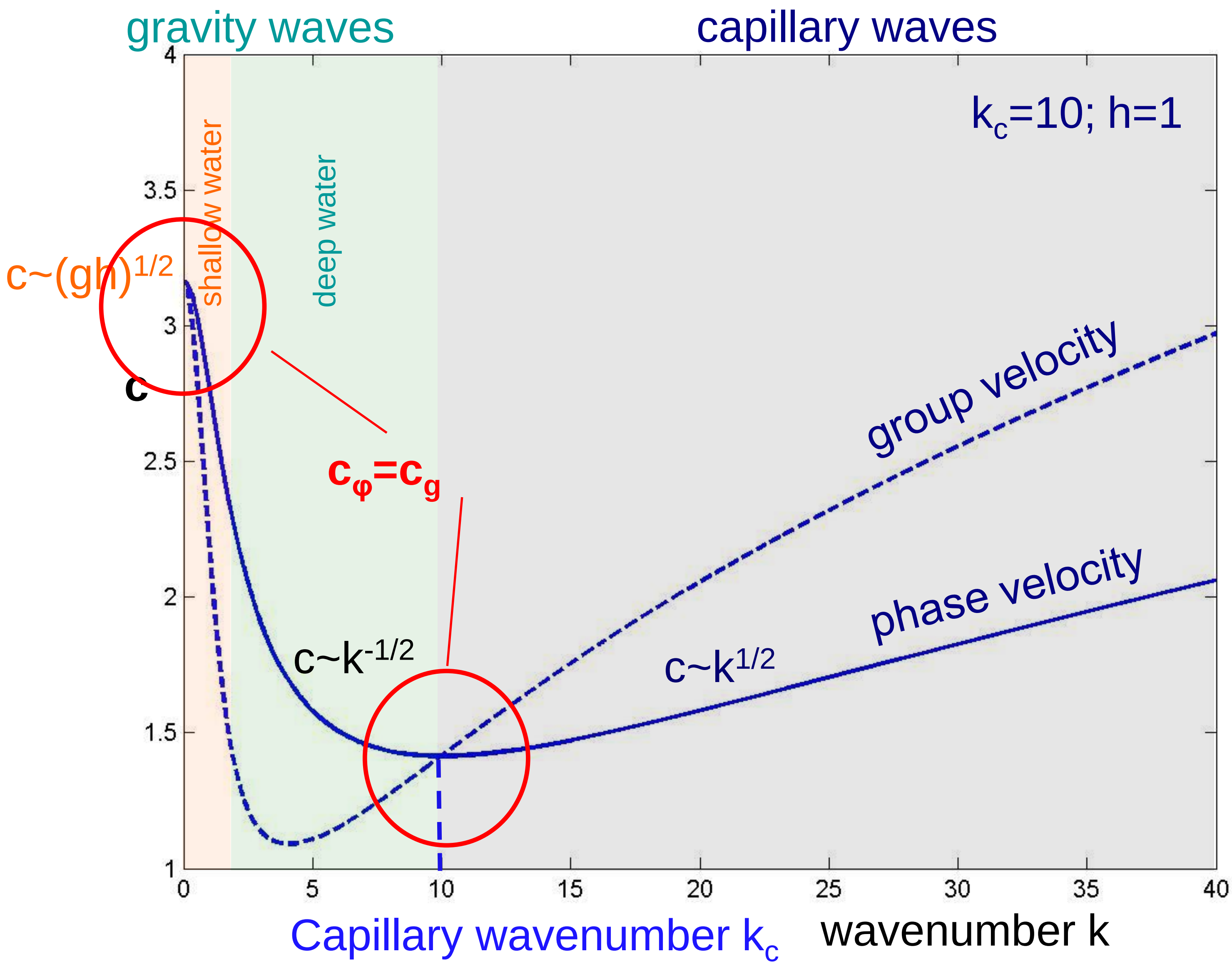
Dispersion relation



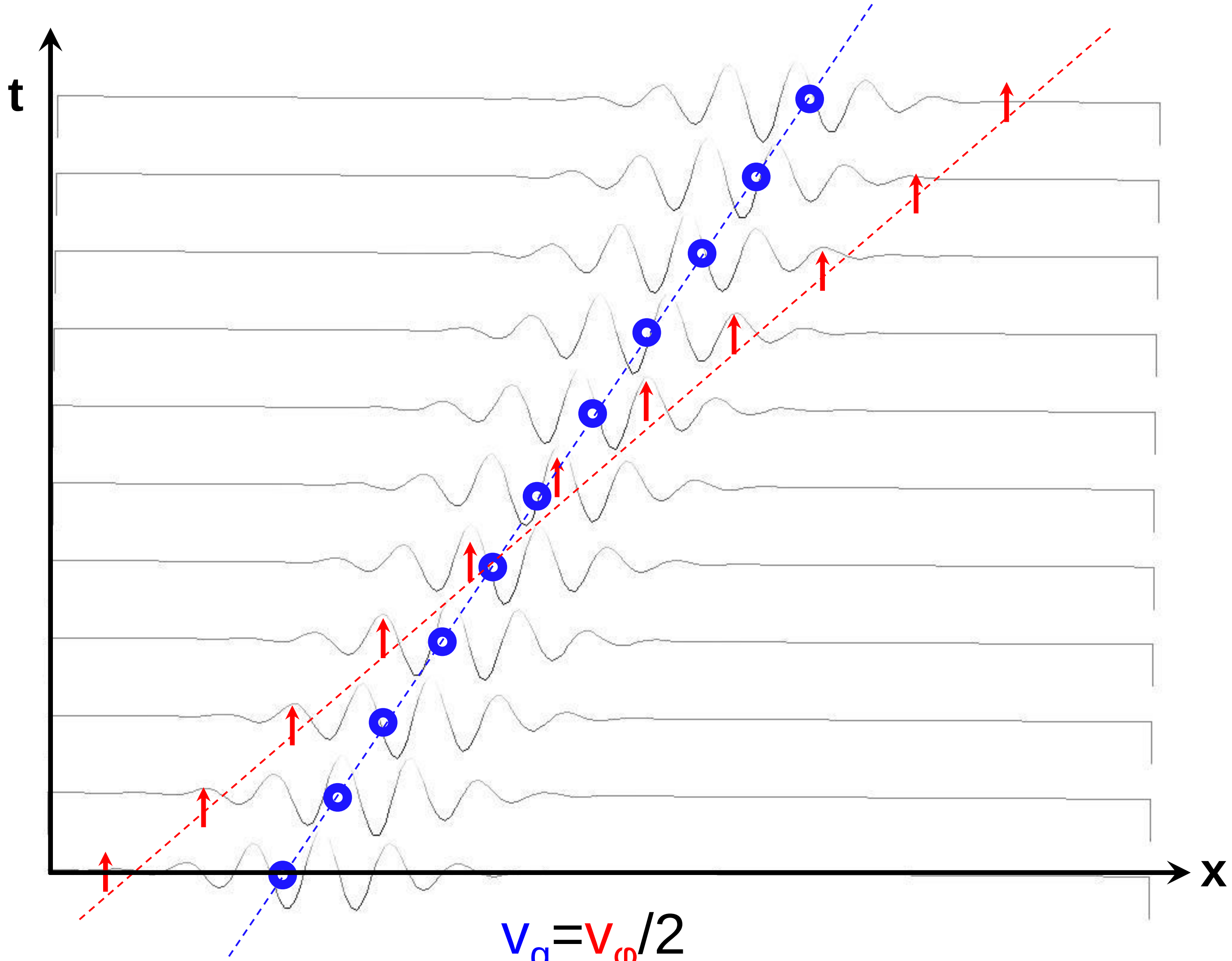
Conditions for wave pattern formation?

$$V_{duck} \lesseqgtr c_{min} \quad ?$$

Dispersion relation

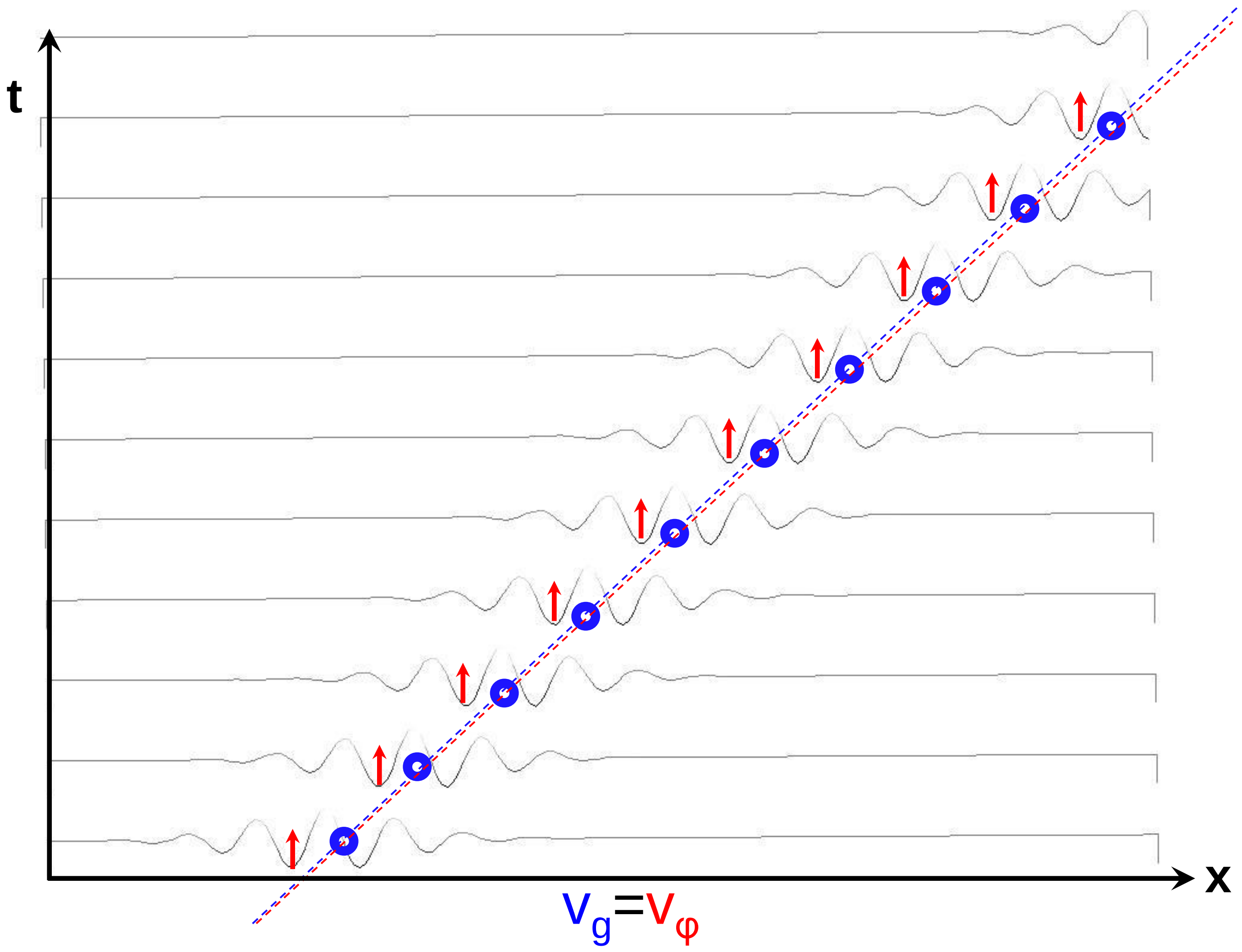


Spatio-temporel diagram

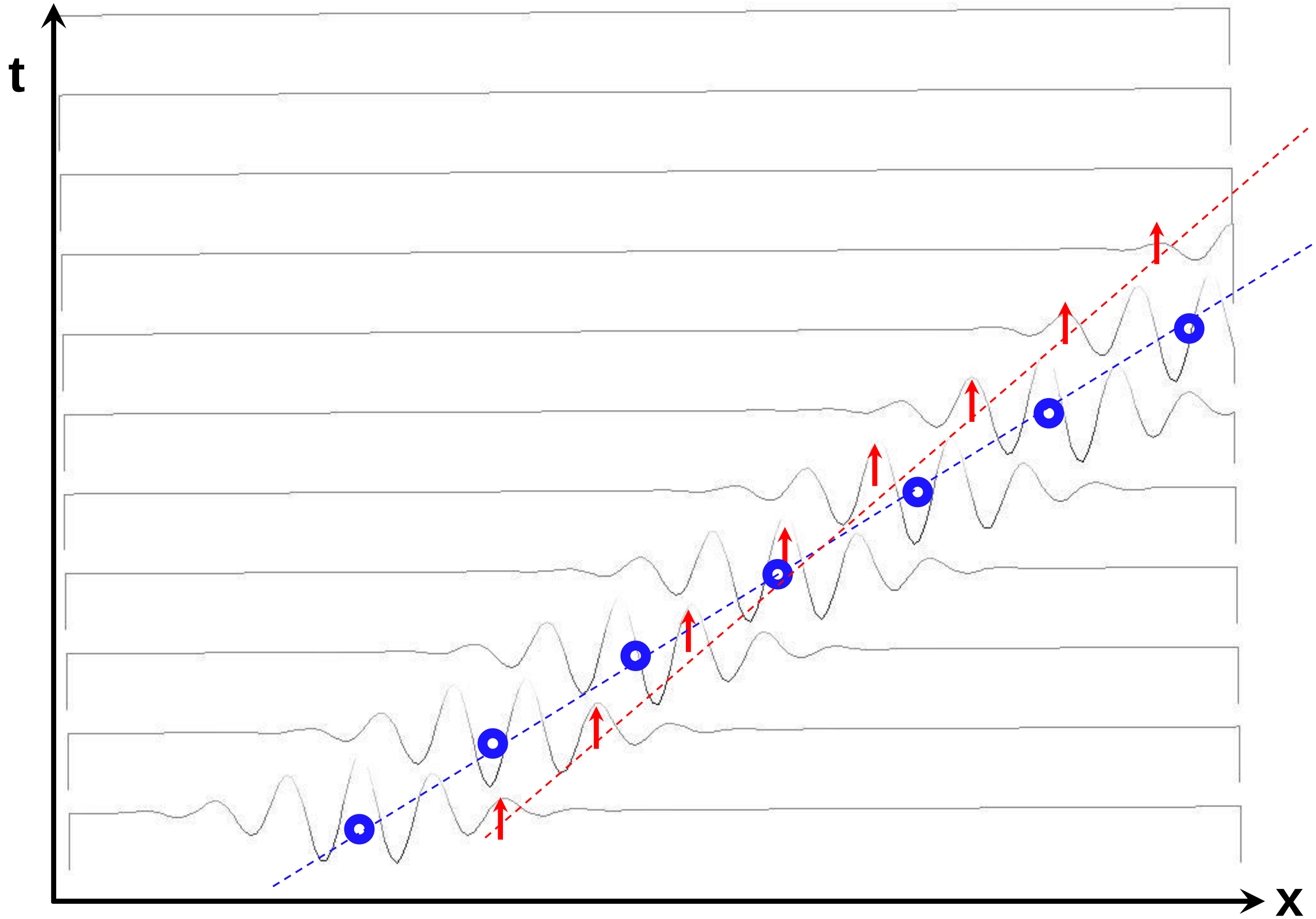


$v_g = v_\varphi / 2$

Spatio-temporal diagram



Spatio-temporal diagram



$v_g = 3/2 v_\varphi$

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{u}\,(k,\omega)\, e^{i(kx-\omega t)}\, dk\, d\omega$$

$$\hat{u}(k,\omega) = (2\pi)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u\,(x,t)\, e^{-i(kx-\omega t)}\, dx\, dt$$

Direct Fourier Transform

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{u}\,(k,\omega)\, e^{i(kx-\omega t)}\, dk\, d\omega$$

Use dispersion relation ω(k)!

Spectral analysis

Fourier transform:
$$\boxed{u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k + c.c.}$$

Carrier/enveloppe

Carrier/enveloppe : $$\boxed{u(x,t) = \frac{1}{2}A(x,t)\,\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + c.c.}$$

Spectral analysis

Fourier transform:

$$\boxed{u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + c.c.}$$

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Spectral analysis

Fourier transform:

$$\boxed{u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + c.c.}$$

Carrier/enveloppe :

$$\boxed{u(x,t) = \frac{1}{2}A(x,t)\,\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.}$$

Enveloppe :

$$\boxed{A(x,t) = \int_0^{\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.}$$

Spectral analysis at time=0

Fourier transform: $$\boxed{u(x,0) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(kx)}\,\mathrm{d}k + \text{c.c.}}$$

$\hat{u}(k)$ is given by Fourier transform at time t=0

Enveloppe : $$\boxed{A(x,0) = \int_0^{\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(k-k_0)x}\,\mathrm{d}k + \text{c.c.}}$$

Spectral analysis

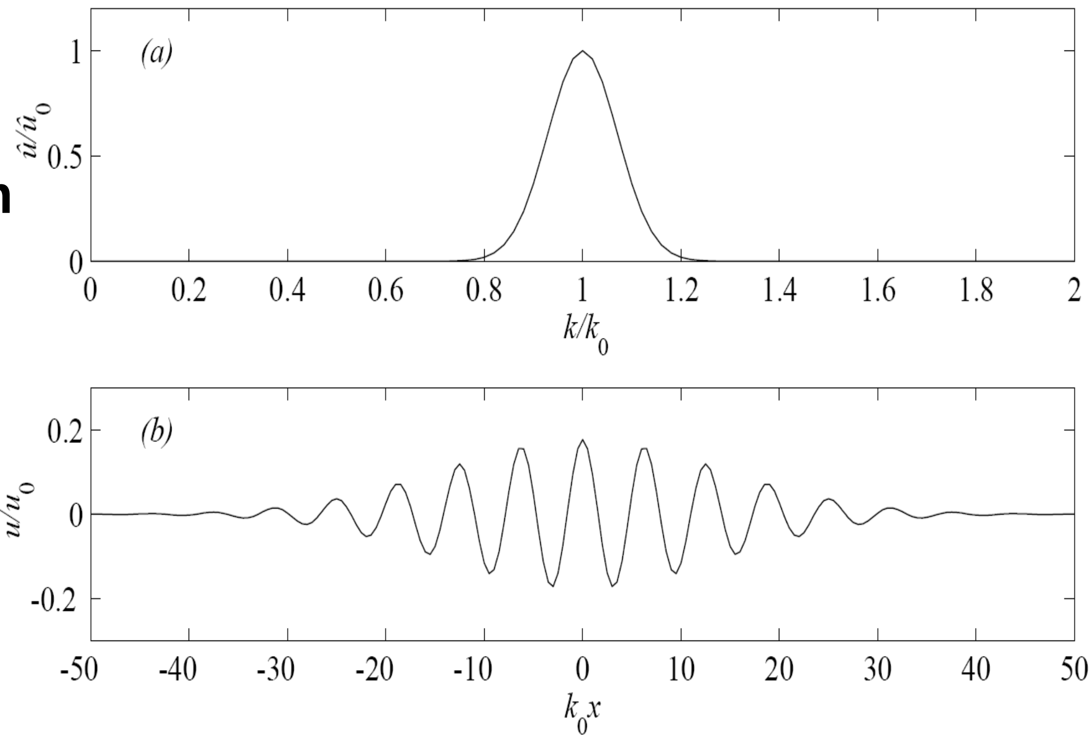
Gaussian spectrum: $$\hat{u}(k) = u_0 \, \mathrm{e}^{-\sigma^2 (k-k_0)^2}$$

Initial enveloppe : $$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma} \, \mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

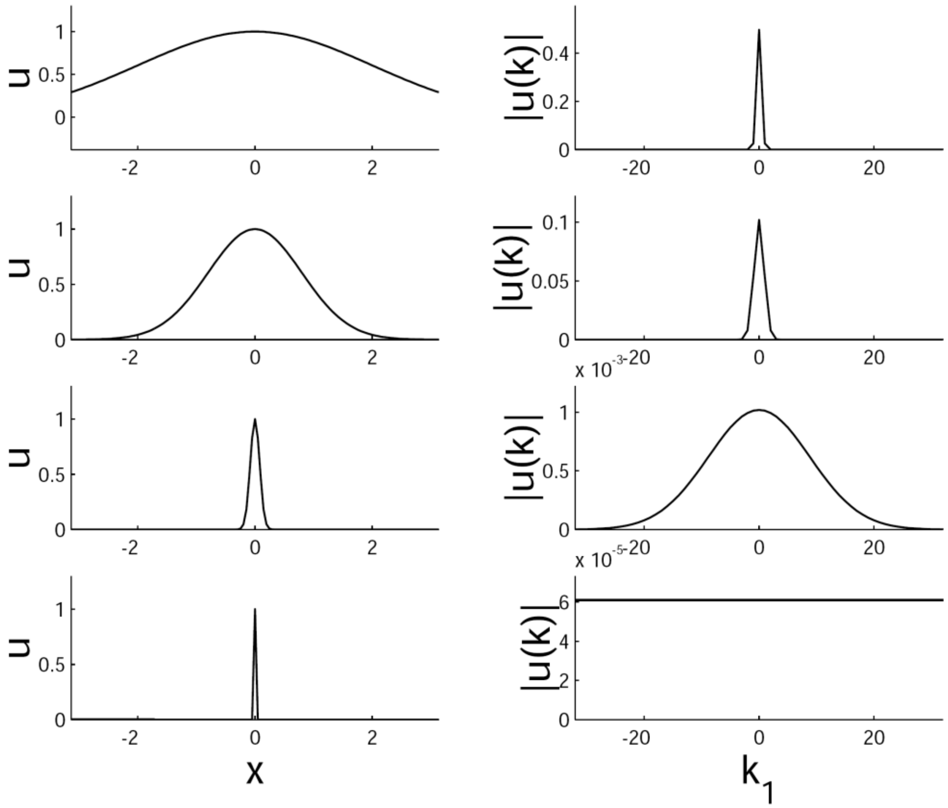
Gaussian spectrum

spectrum

wave

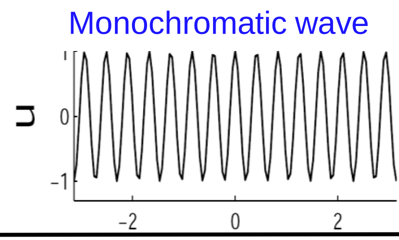
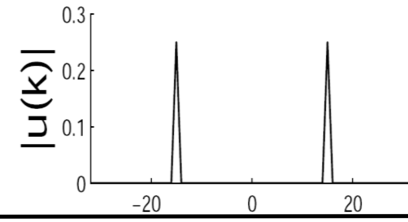
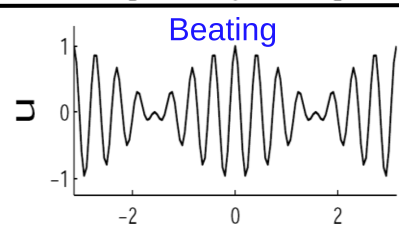
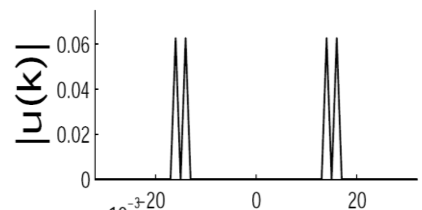
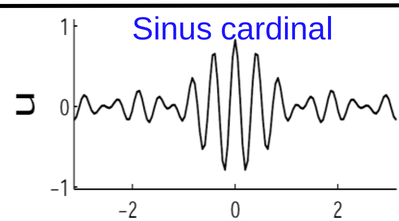
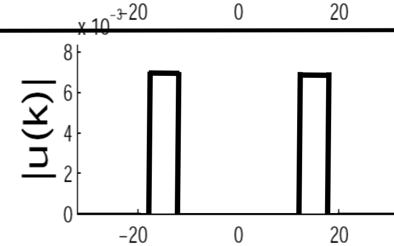
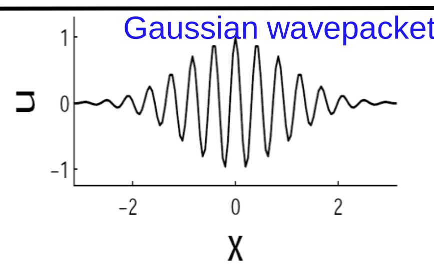
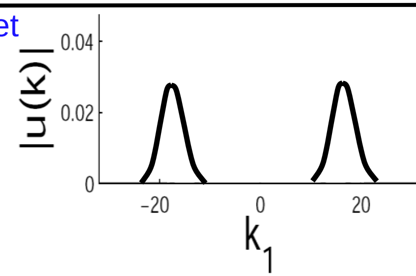


Gaussian wavepackets



$u_0(x)$	$\widehat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\,\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}}\ \exp\left(-\frac{\sigma^2\,k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Waves and spectra

$2\ \cos(k_0\, x)$			$\delta(k_1+k_0)+\delta(k_1-k_0)$
$\cos[(k_0-\kappa)x]+\cos[(k_0+\kappa)x]$ $=2\ \cos(k_0\, x)\ \cos(\kappa\, x)$			$\delta(k_1+k_0+\kappa)+\delta(k_1-k_0-\kappa)$ $+\delta(k_1+k_0-\kappa)+\delta(k_1-k_0+\kappa)$
$\int_{k_0-\kappa}^{k_0+\kappa}\cos(k_1\, x)\ dk_1$ $=4\,\kappa\ \cos(k_0\, x)\ \frac{\sin(\kappa\, x)}{\kappa\ x}$			$=1$ pour $\lvert k_1\pm k_0\rvert\le\kappa$ $=0$ sinon
$2\,\kappa\sqrt{2\pi}\ \cos(k_0\, x)\exp\left(-\frac{\kappa^2\, x^2}{2}\right)$			$\exp\left[-\frac{(k_1+k_0)^2}{2\,\kappa^2}\right]+\exp\left[-\frac{(k_1-k_0)^2}{2\,\kappa^2}\right]$

Spectral analysis

Initial enveloppe :
$$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:
$$\hat{u}(k) = u_0\,\mathrm{e}^{-\sigma^2(k-k_0)^2}$$

Spectral analysis

Initial enveloppe :

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Gaussian spectrum:

$$\hat{u}(k) = u_0\,\mathrm{e}^{-\sigma^2(k-k_0)^2}$$

Evolution of enveloppe :

$$\boxed{A(x,t) = \int_0^{\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.}$$

Spectral analysis

Initial enveloppe :

$$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:

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Evolution of enveloppe :

$$\boxed{A(x,t) = \int_0^\infty \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.}$$

Definition group velocity

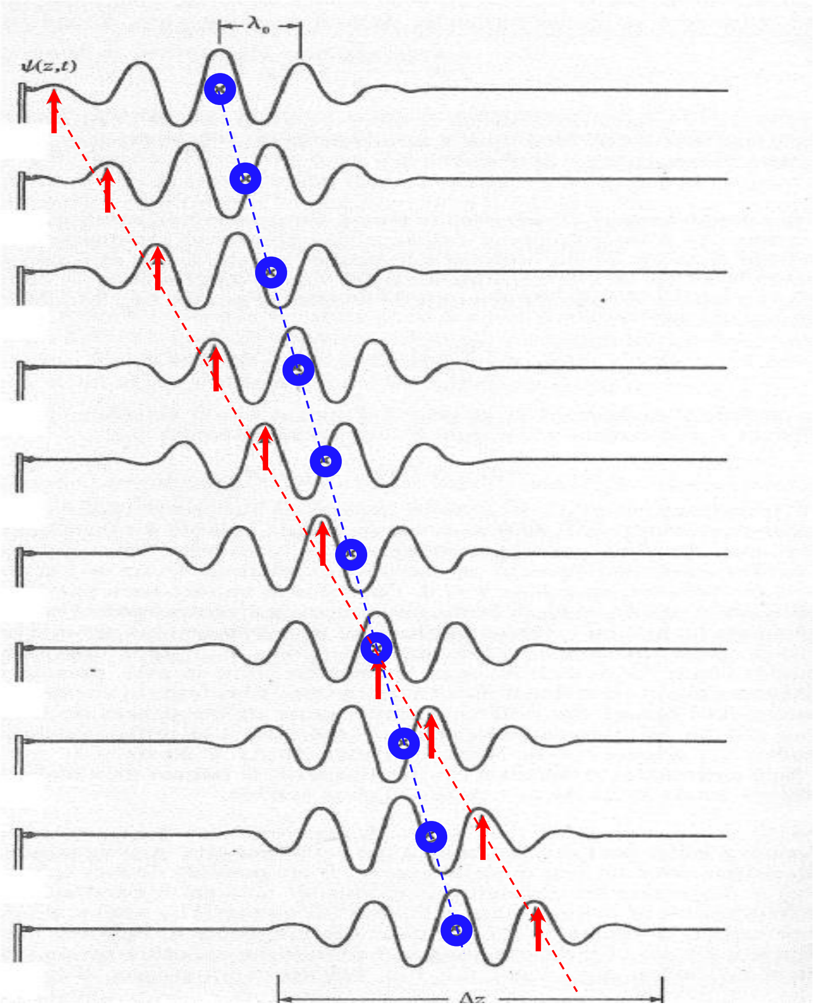
$$\boxed{\omega - \omega_0 = c_g(k-k_0), \qquad c_g = \frac{\partial\omega}{\partial k}(k_0)}$$

Spectral analysis

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0),$ $\qquad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$\boxed{A(x,t) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{(x-c_g t)^2}{4\sigma^2}}}$$

Group velocity



Wavepacket

Spectral analysis

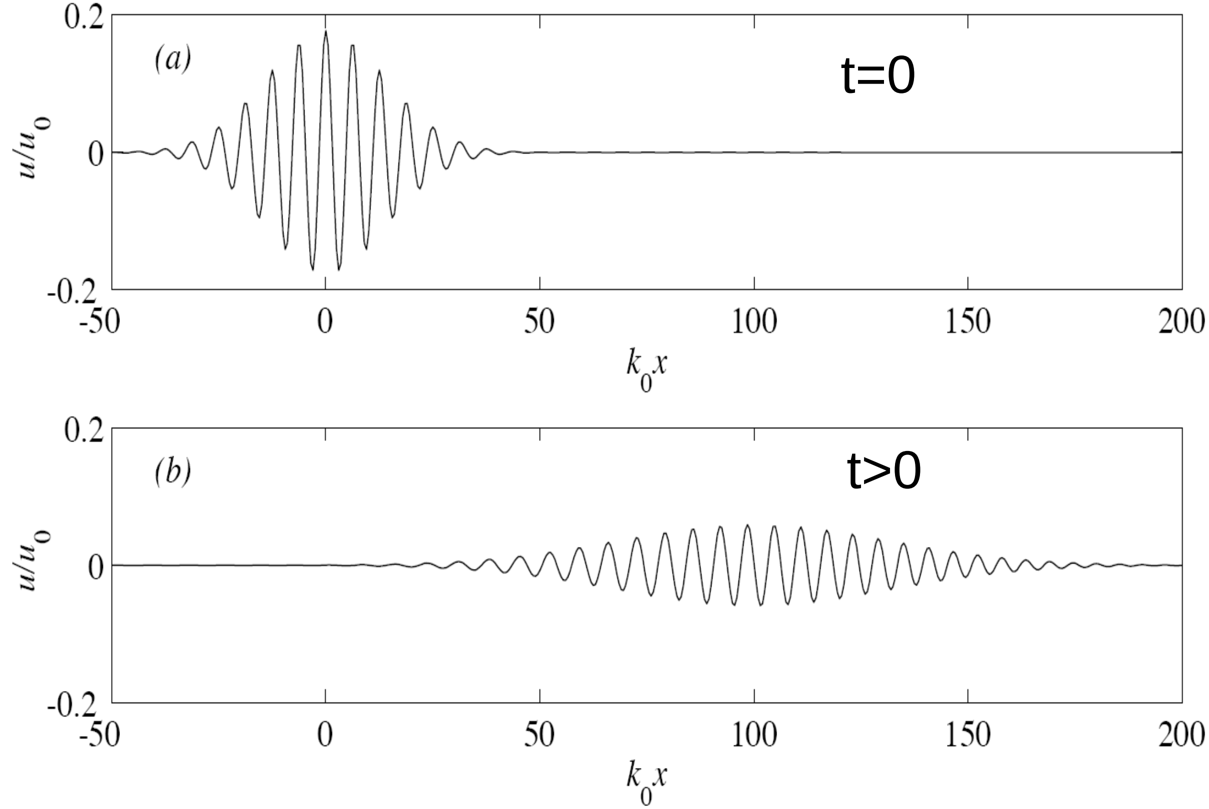
Higher order development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$

$$c_g = \frac{\partial \omega}{\partial k}(k_0), \qquad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

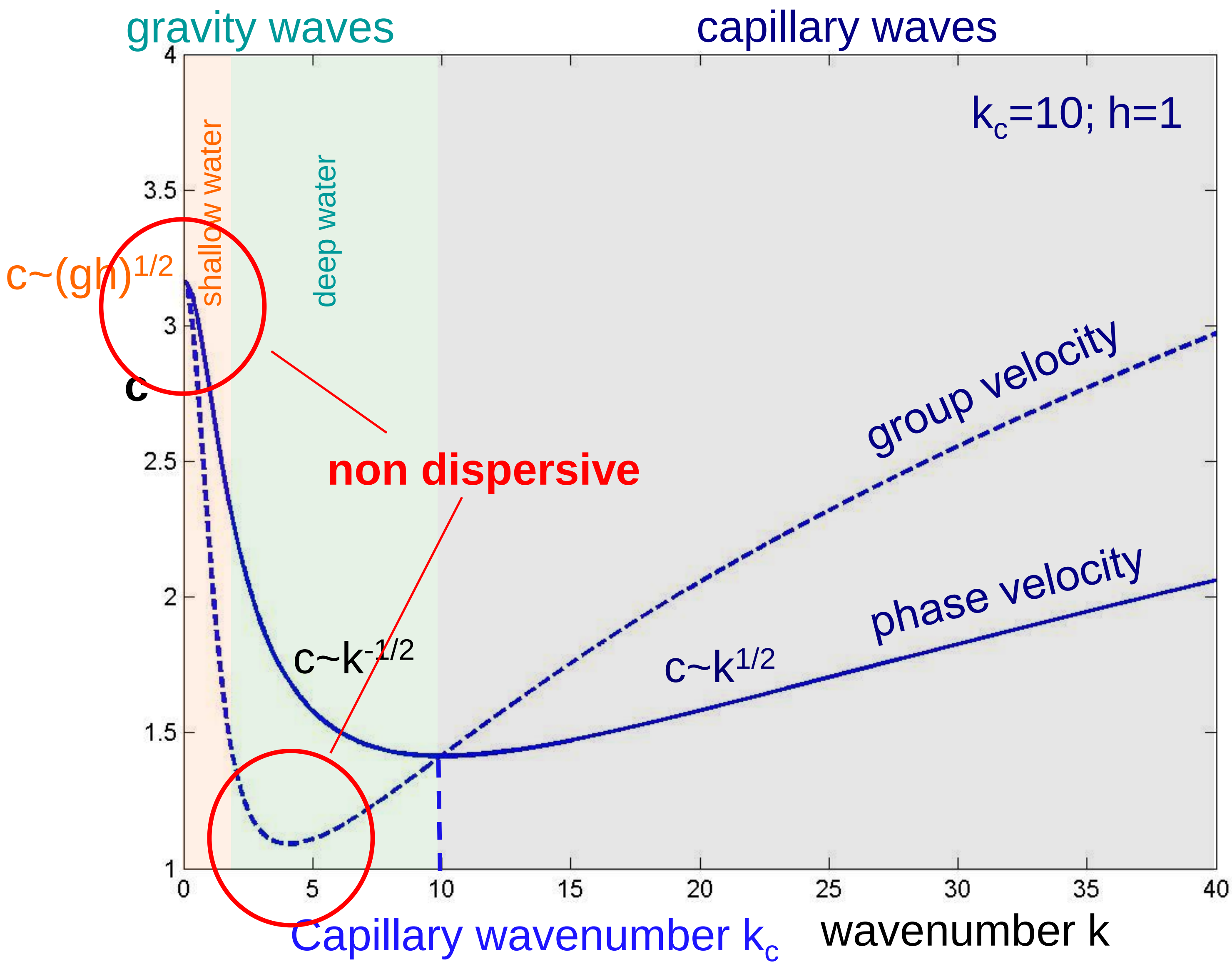
$$A(x,t) = \frac{u_0}{2}\sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}\mathrm{i}\omega_0'' t}}\,\exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}\mathrm{i}\omega_0'' t)}\right)$$

Wave packet dispersion

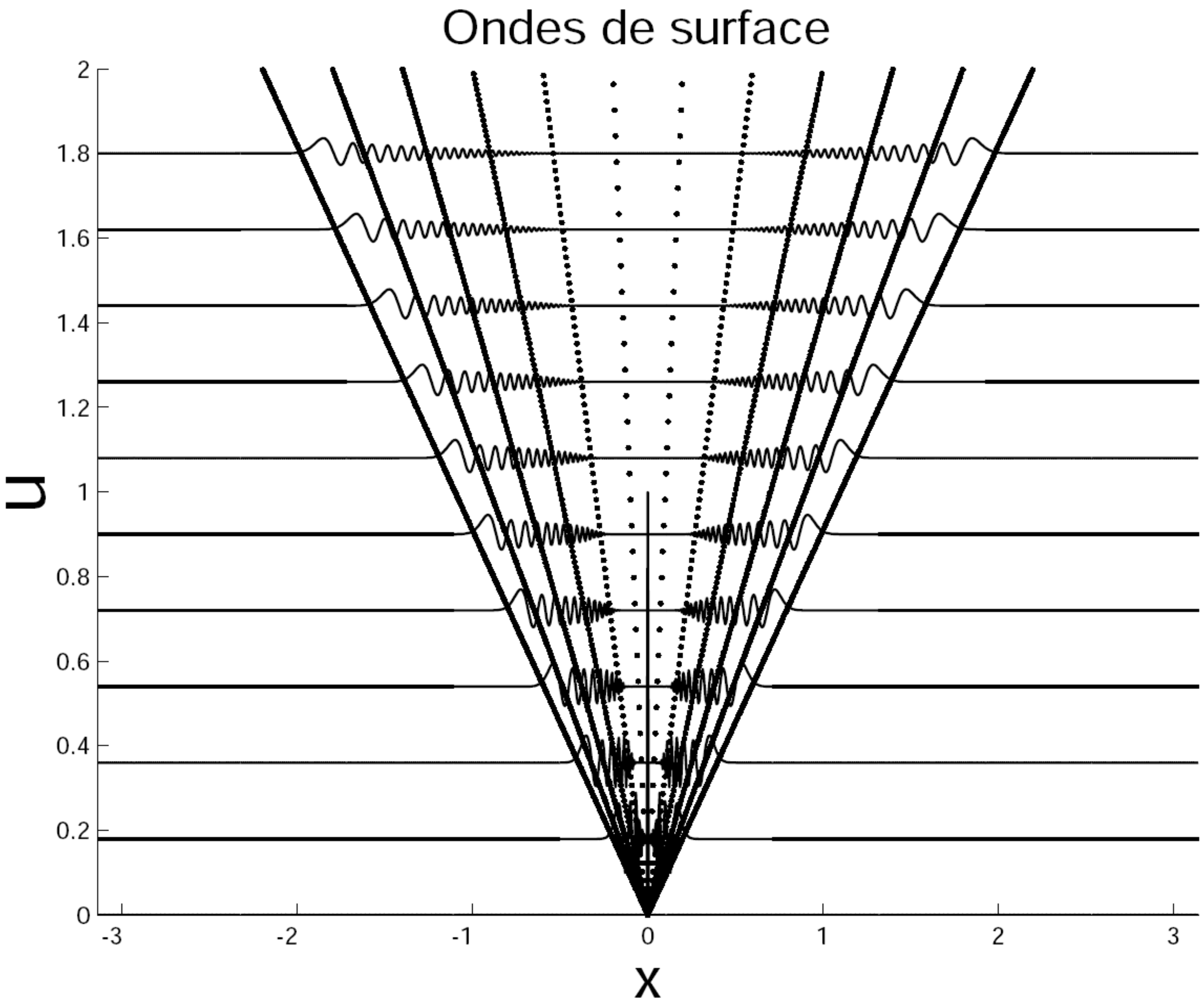


Onde correspondant à l'enveloppe ( ) *pour* $\sigma^{-1}k_0 = 0{,}1$ *et* $\omega_0'' = 4c_g/k_0$: (a), *instant initial* $t = 0$; (b), $c_g t = 100/k_0$.

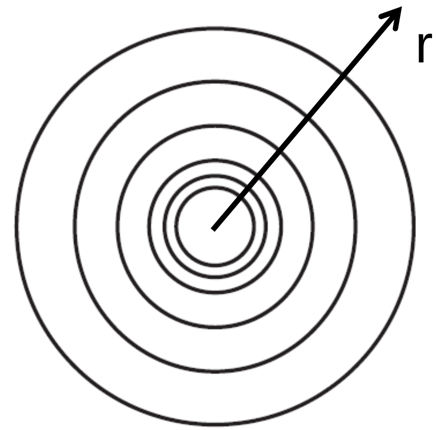
Relation de dispersion



Dispersion



Dispersion



Waves with k reach r at time t=r/v(k)

For deep gravity waves: $v_{deep/gravity} \sim \pm\frac{1}{2}\sqrt{\frac{g}{k}}$

$k=gt^2/4r^2$

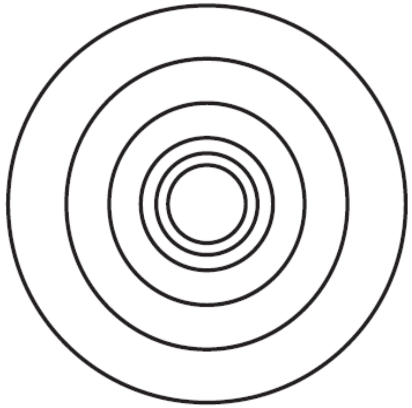
Since $\omega_{deep/gravity} \sim \pm\sqrt{gk}$

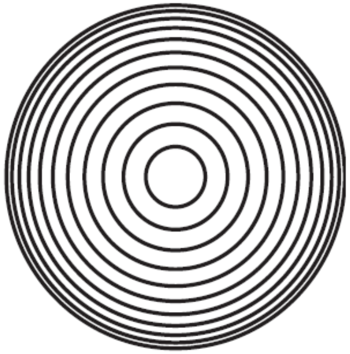
$\omega=gt/2r$

⇨The frequency increases with time

Rings in water

Gravity waves

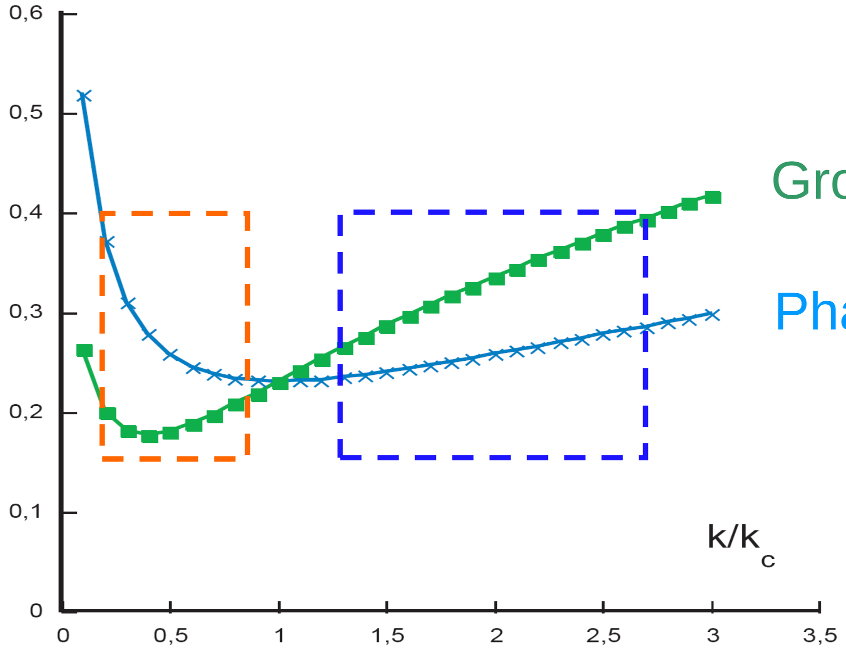




Capillary waves

stone>l_c

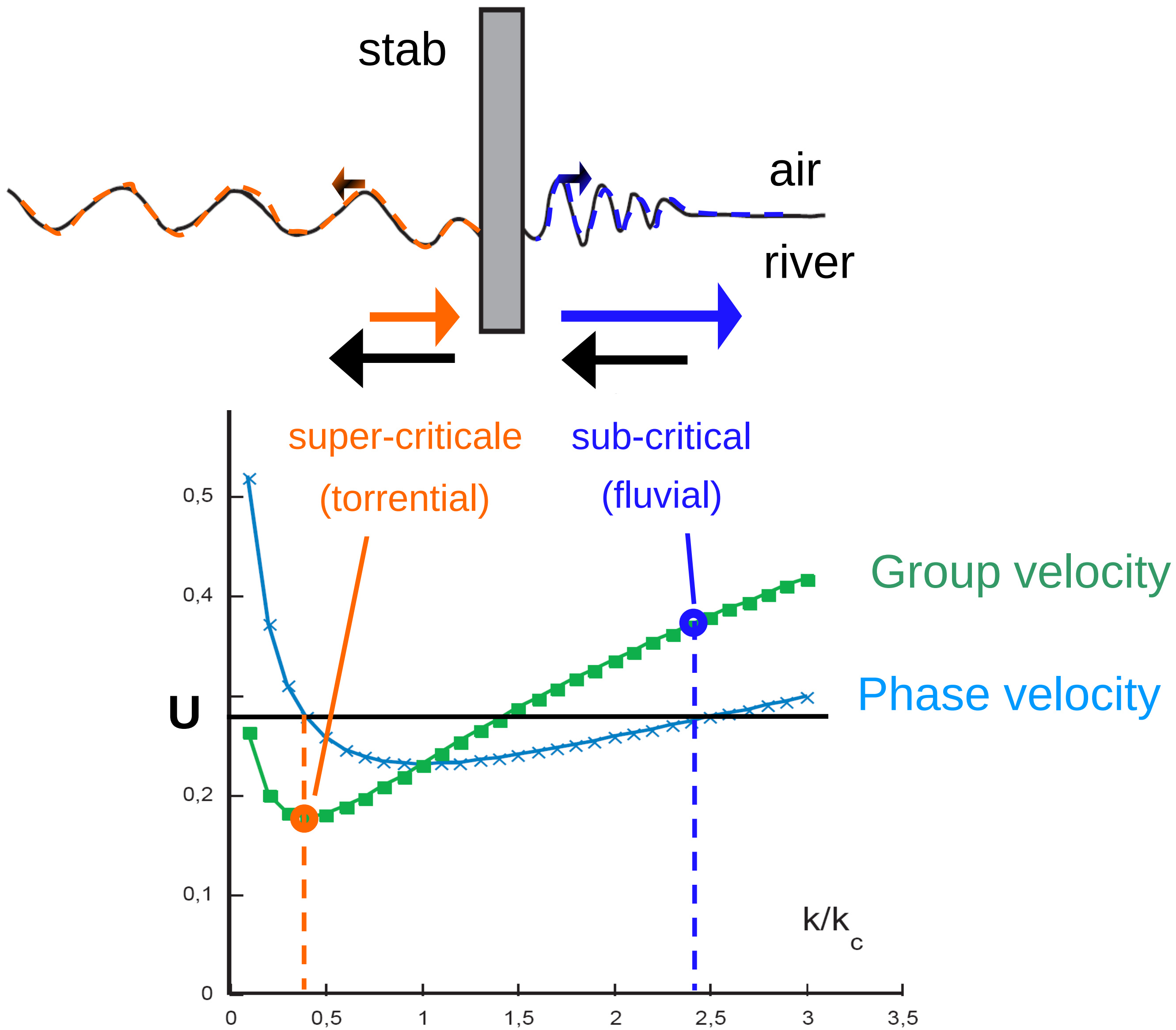
droplet<l_c



Group velocity

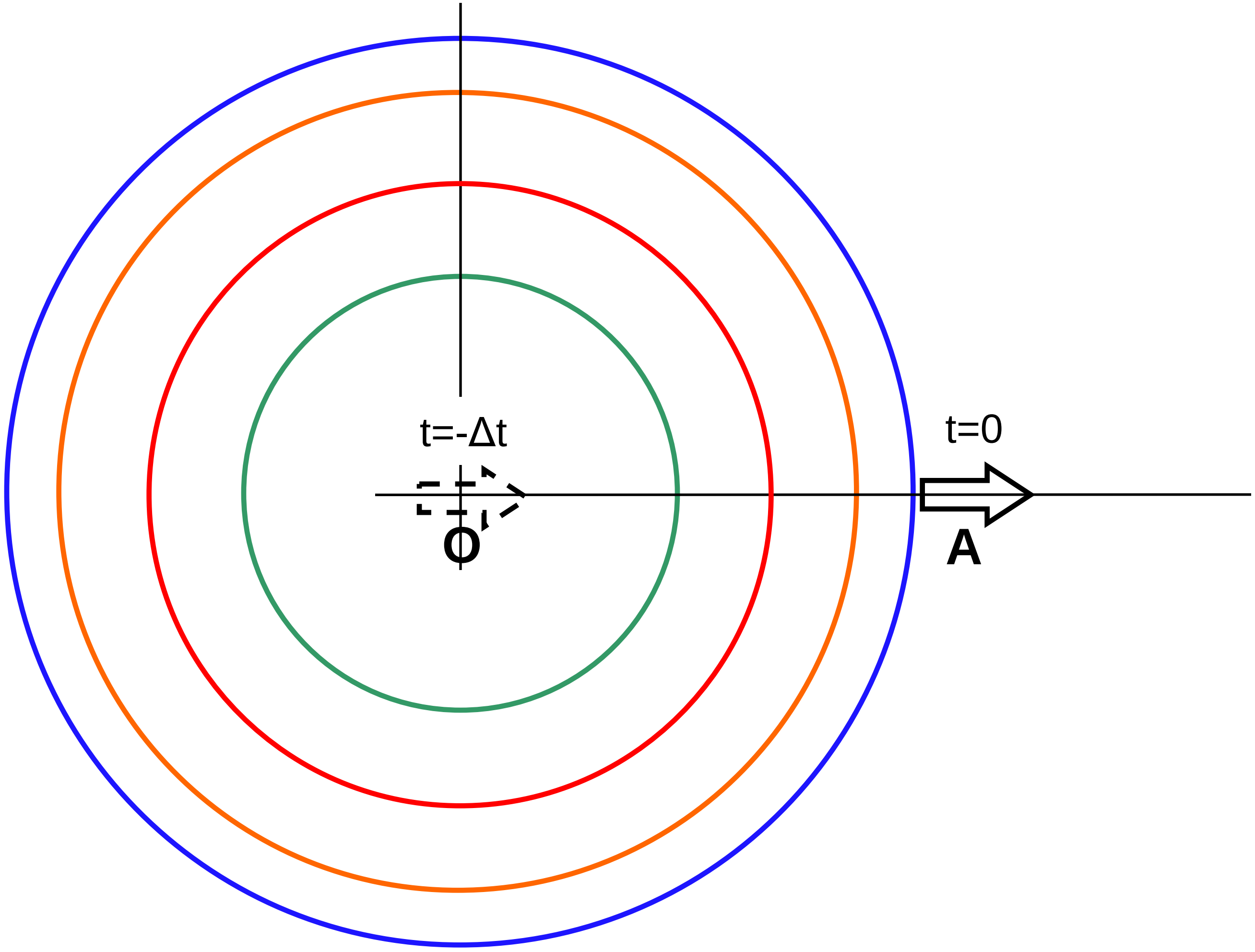
Phase velocity

Waves created by obstacle in a river

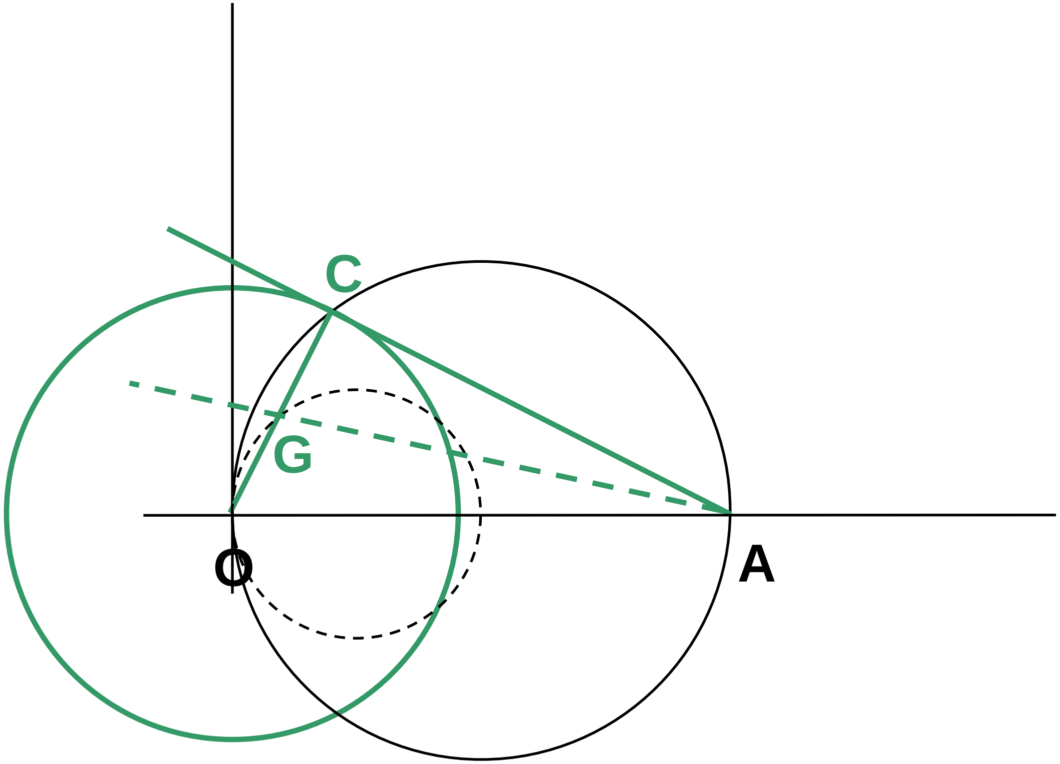


Kelvin’s wake

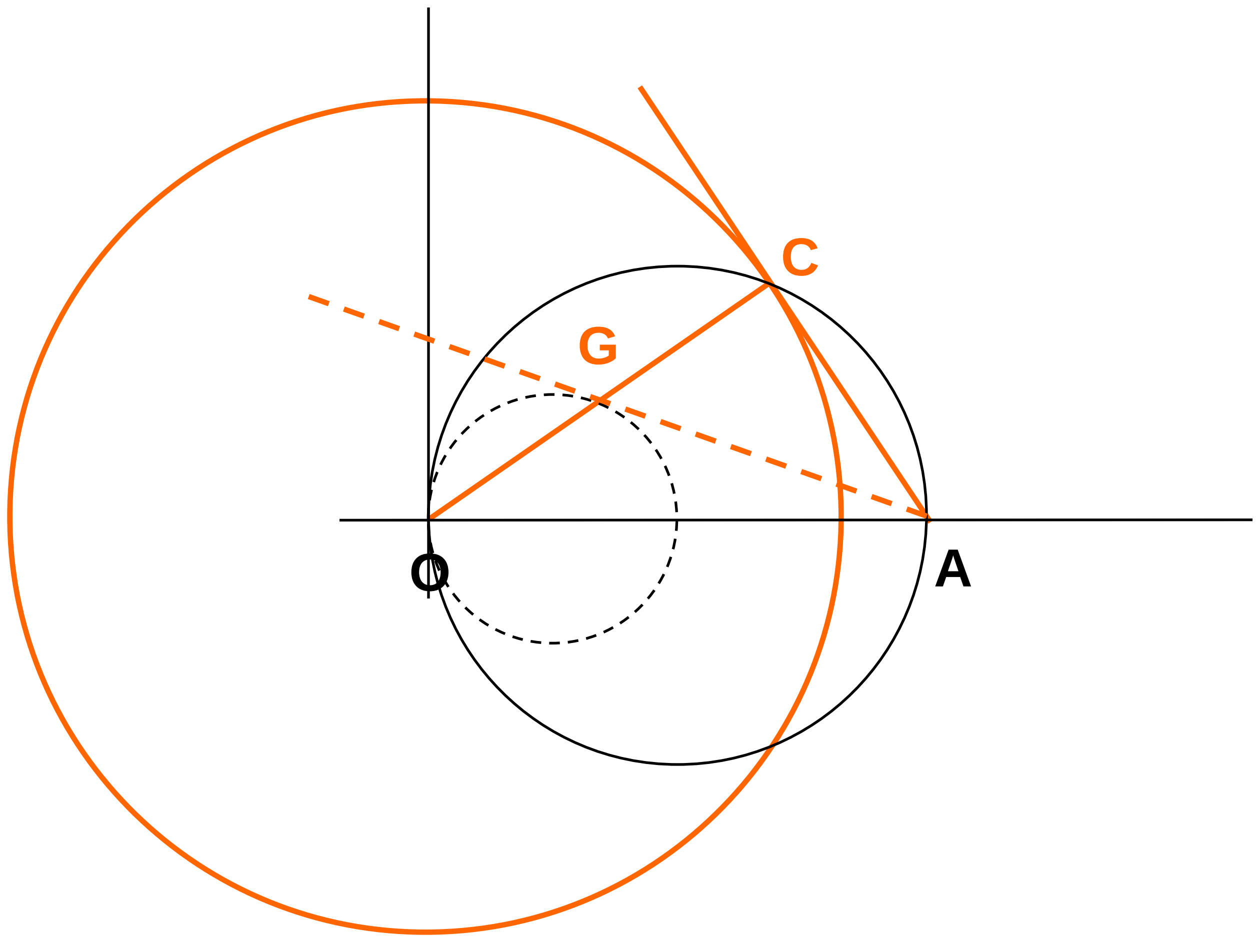
Kelvin's wake



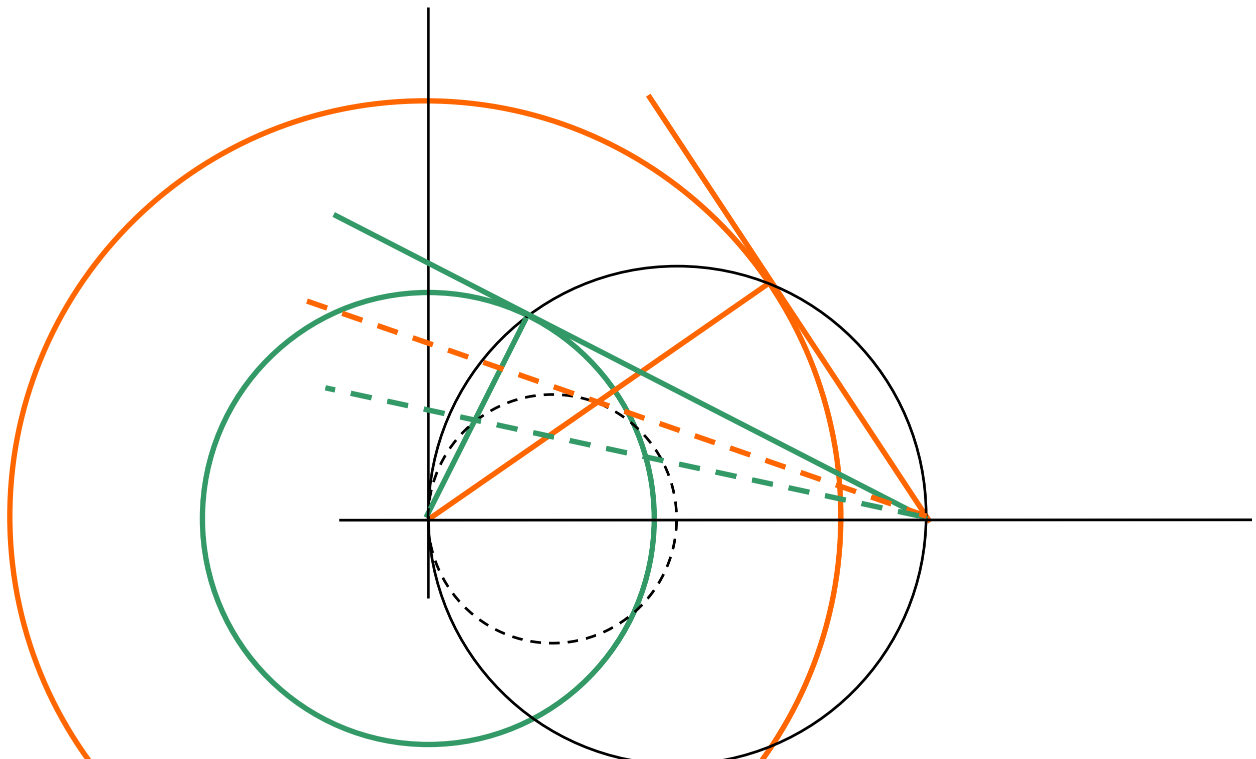
Gravity waves created by a ship



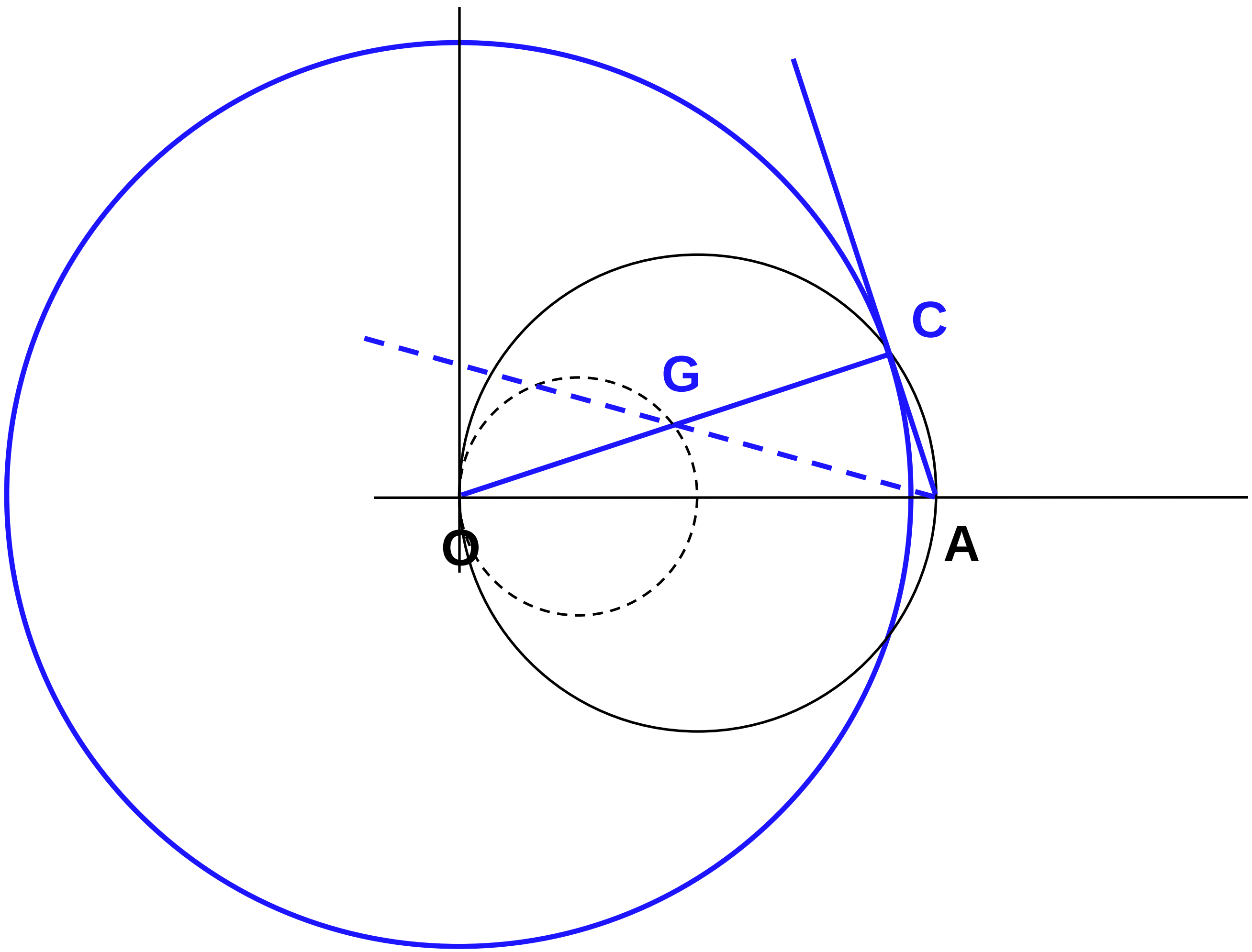
Gravity waves created by a ship



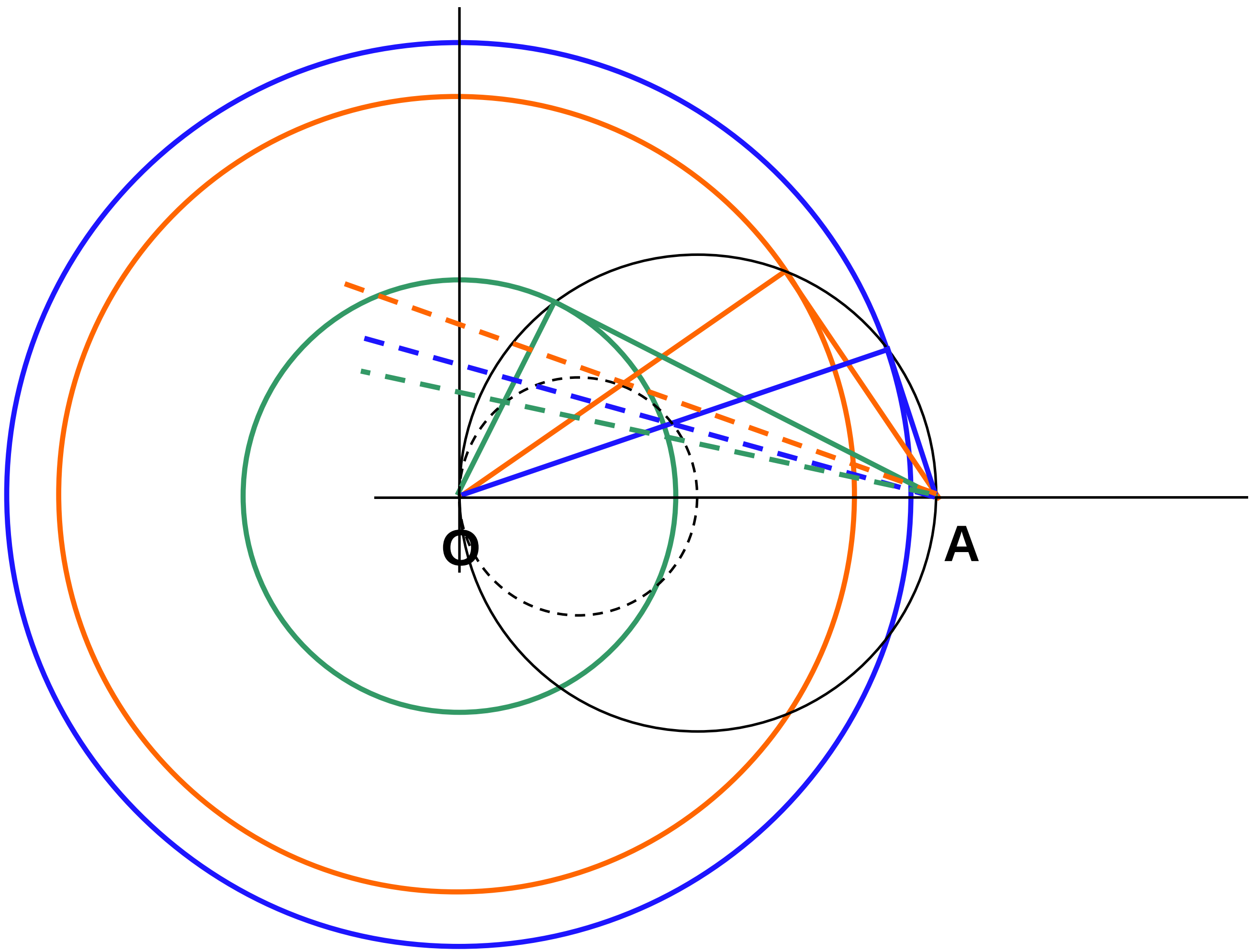
Gravity waves created by a ship



Gravity waves created by a ship



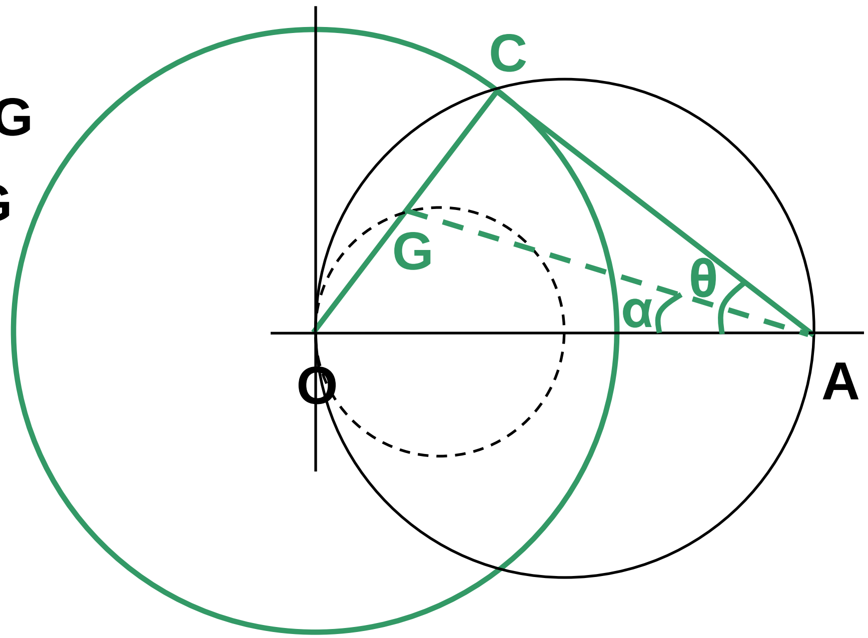
Gravity waves created by a ship



Gravity waves created by a ship

$\sin(\alpha)/OG = \cos(\theta)/AG$

$\sin(\theta-\alpha)AG = GC = OG$

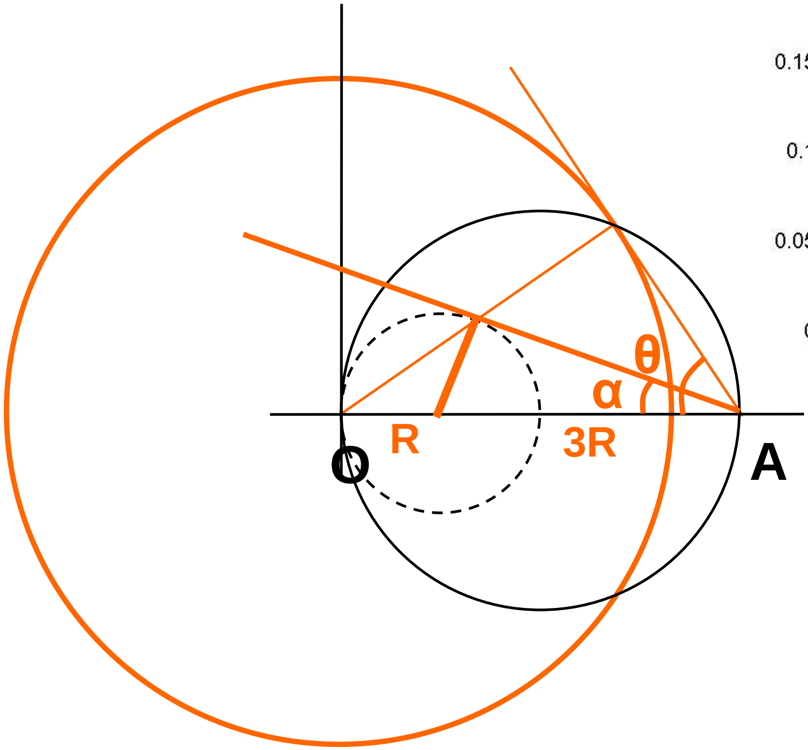


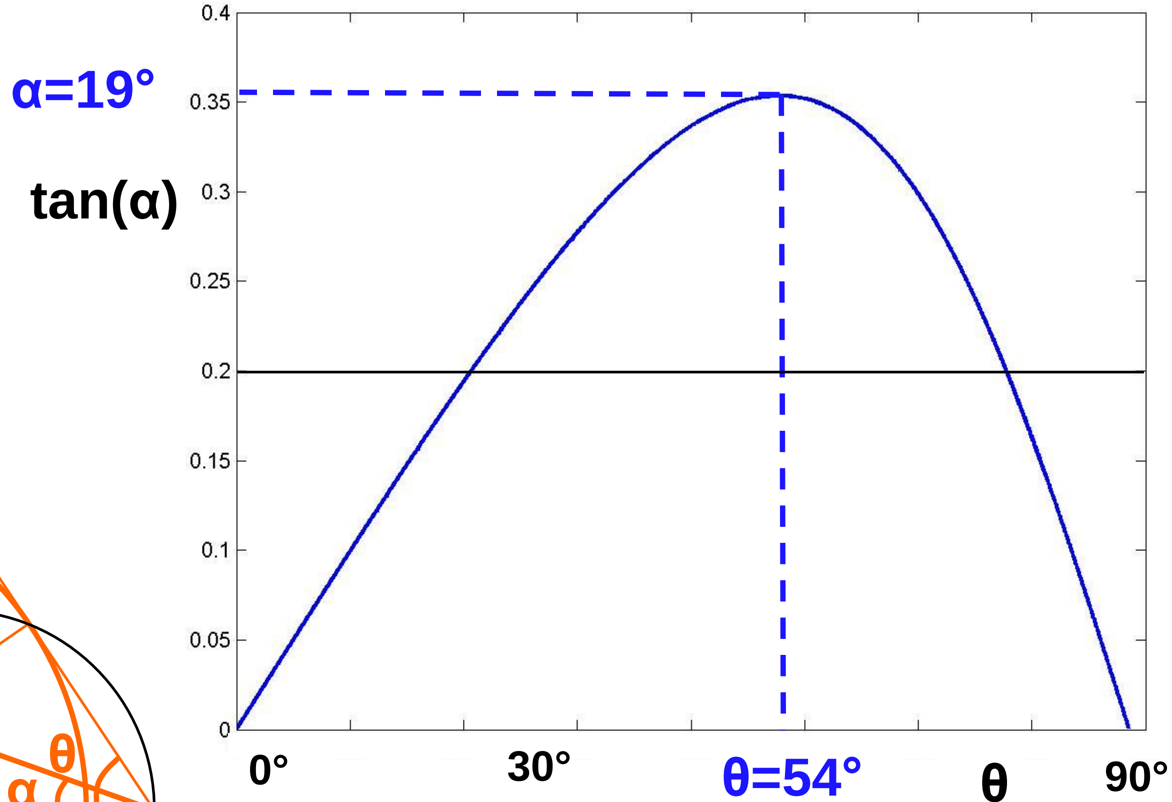
$\Rightarrow \sin(\alpha) = \cos(\theta)\sin(\theta-\alpha)$

$\Rightarrow \sin(\alpha) = \cos(\theta)(\sin(\theta)\cos(\alpha)+\cos(\theta)\sin(\alpha))$

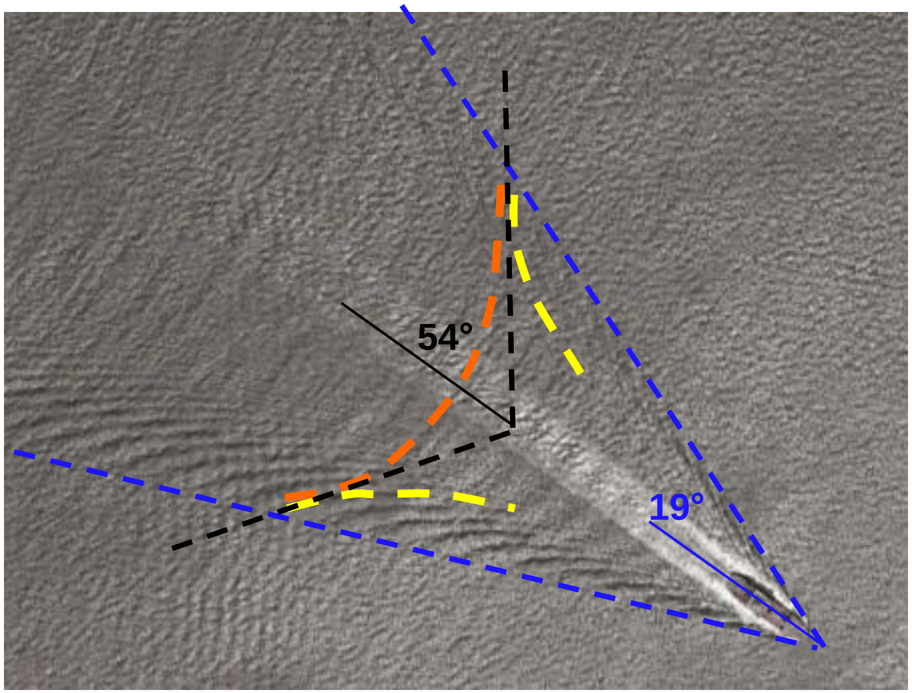
$\Rightarrow \tan(\alpha) = \cos(\theta)\sin(\theta)/(1+\cos^2(\theta))$

Gravity waves created by a ship

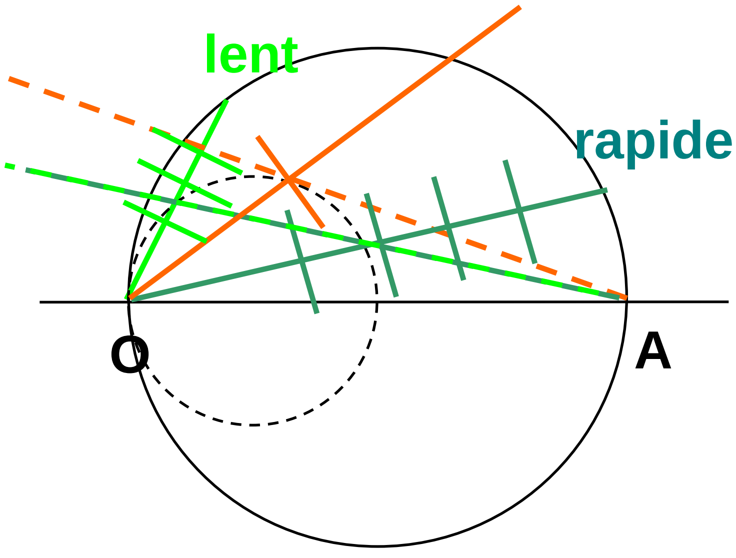


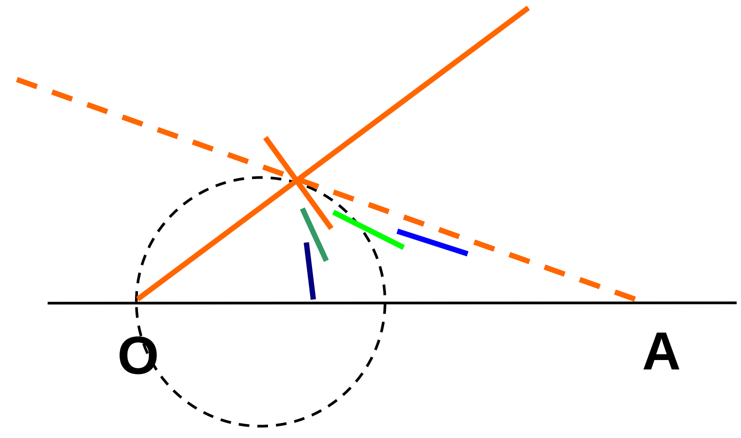


Gravity waves created by a ship

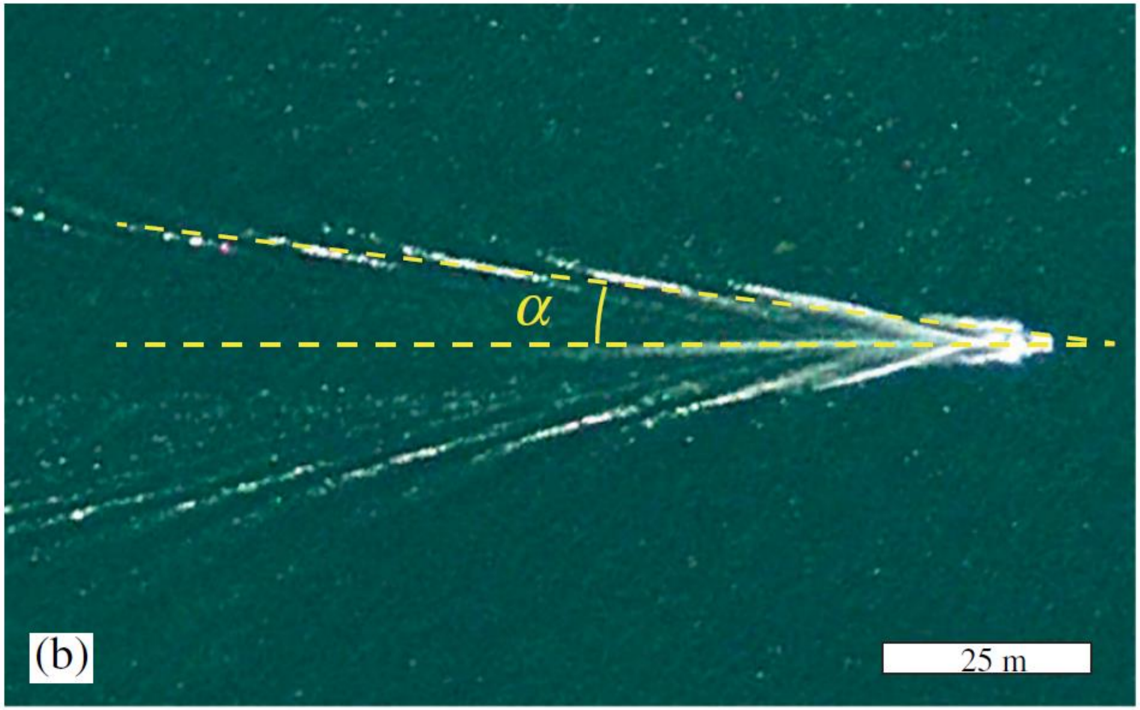


Gravity waves created by a ship



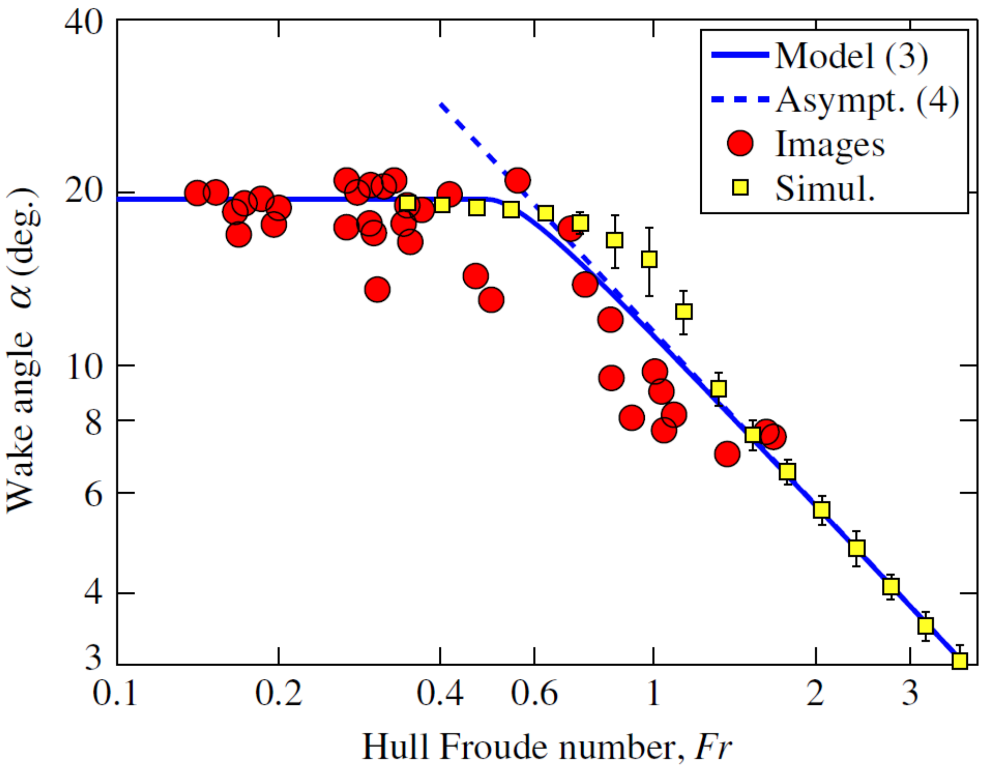


But observations show



Moisy and Rabaud 2013

But observations show



Moisy and Rabaud 2013