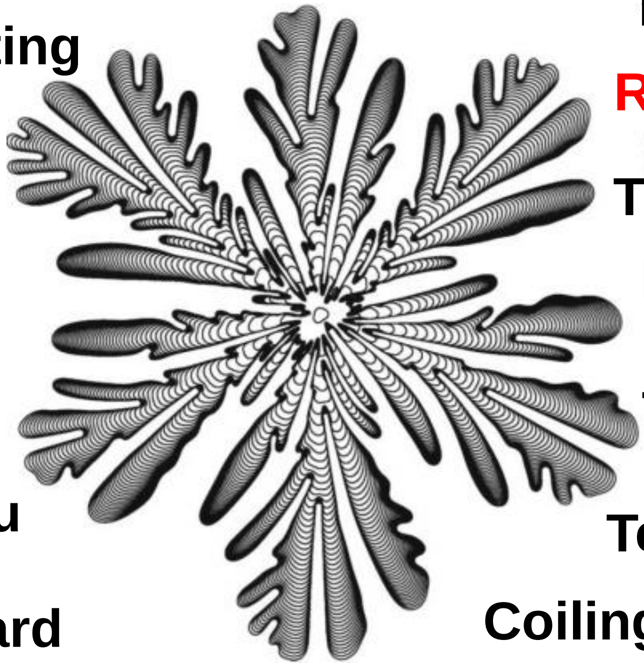
Rayleigh-Taylor instability

Vortex shedding **Saffman-Taylor**

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor



Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

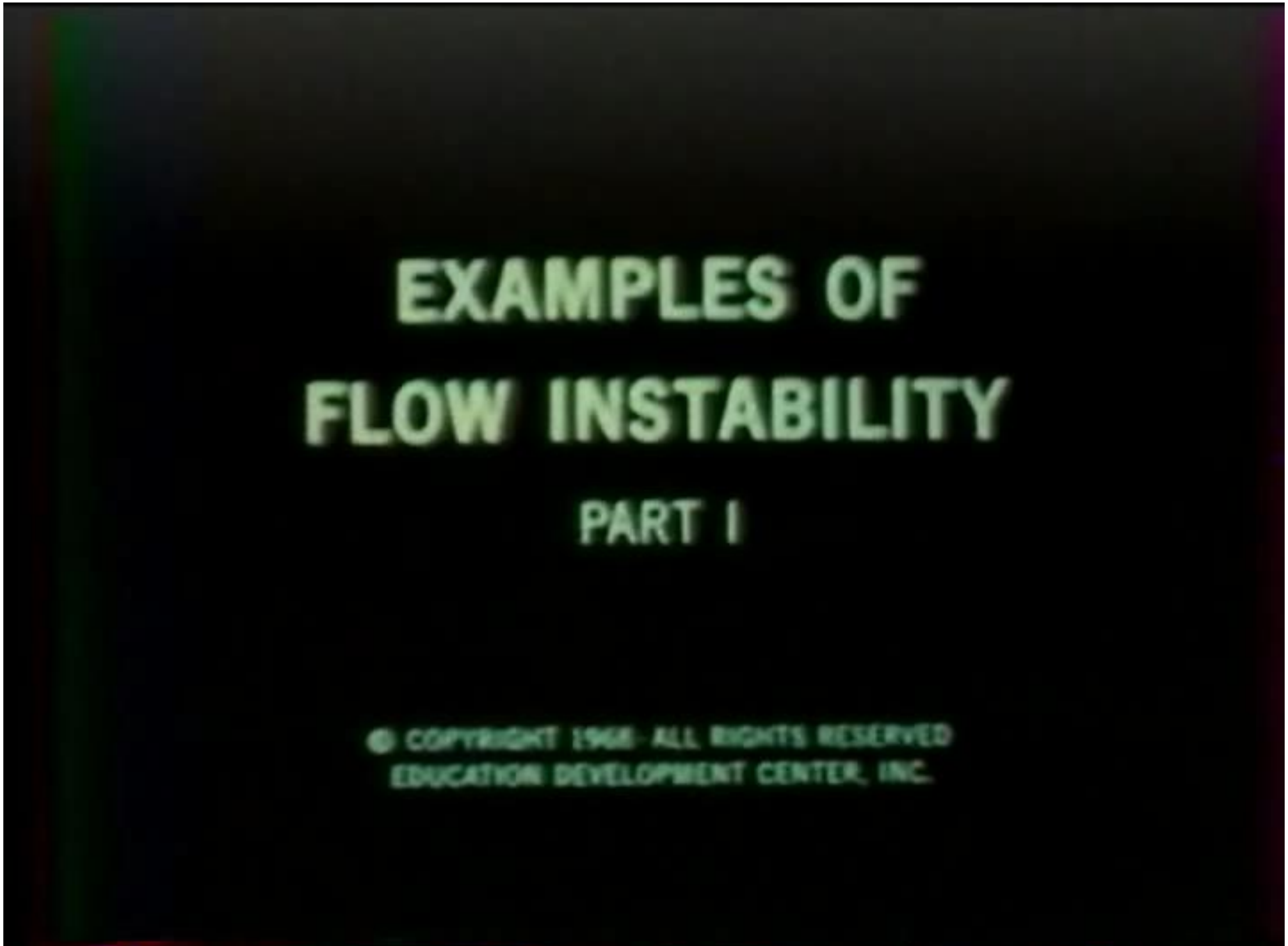
Tearing instability

Rayleigh-Benard

Coiling instability

Benard-Marangoni **Kelvin-Helmholtz**

A movie


EXAMPLES OF
FLOW INSTABILITY
PART I
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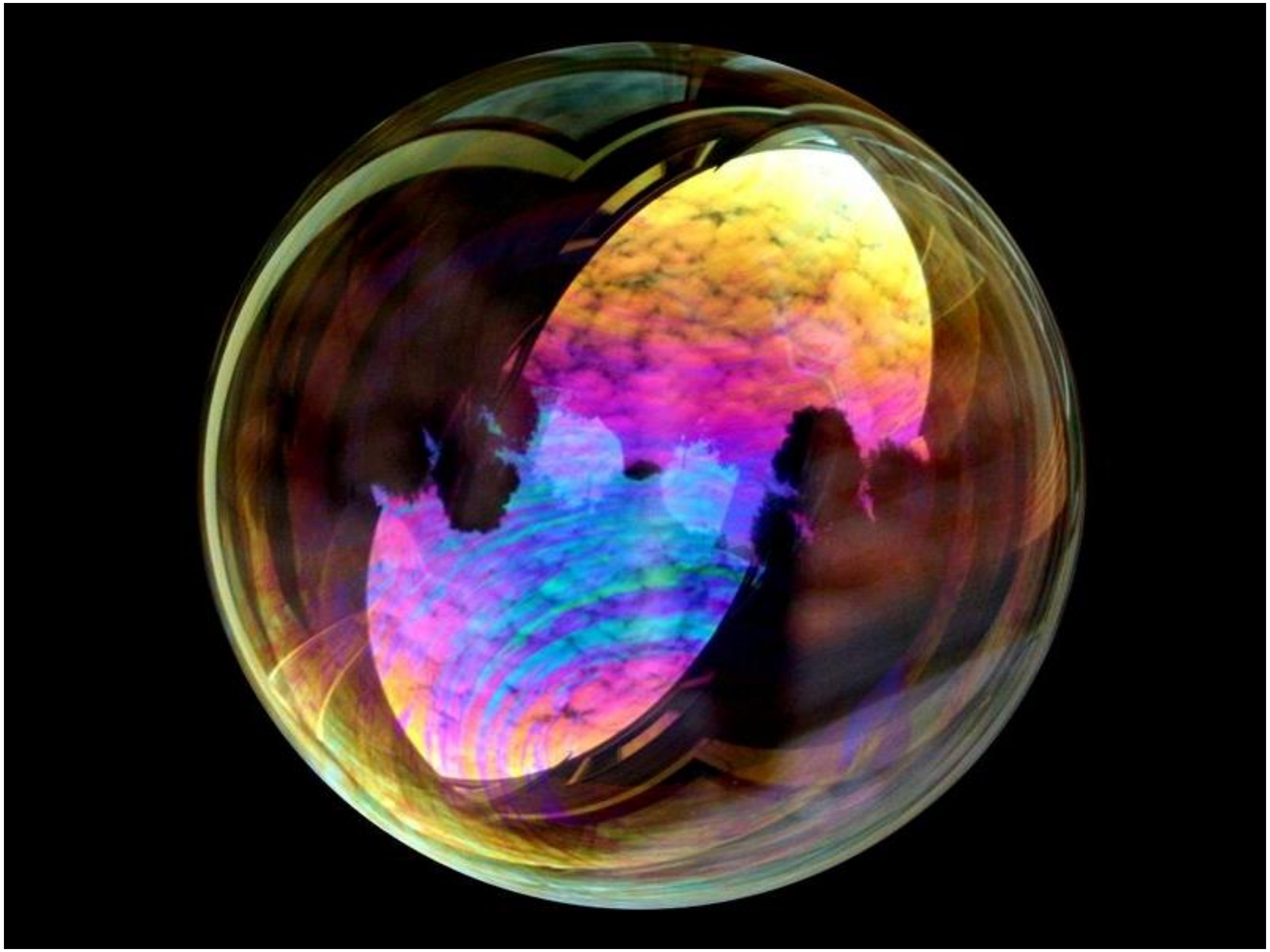
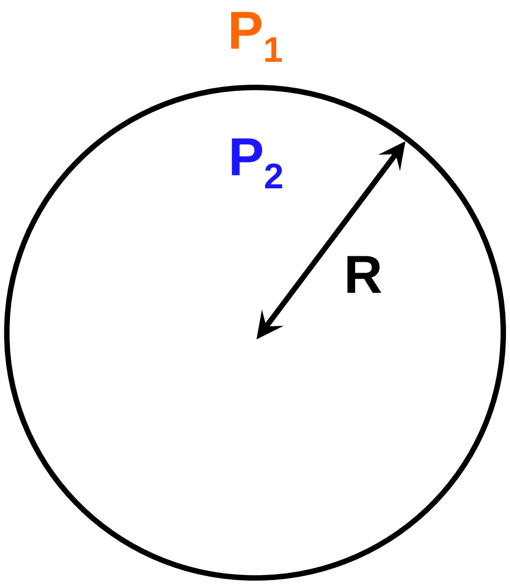
A movie

A movie

Surface tension



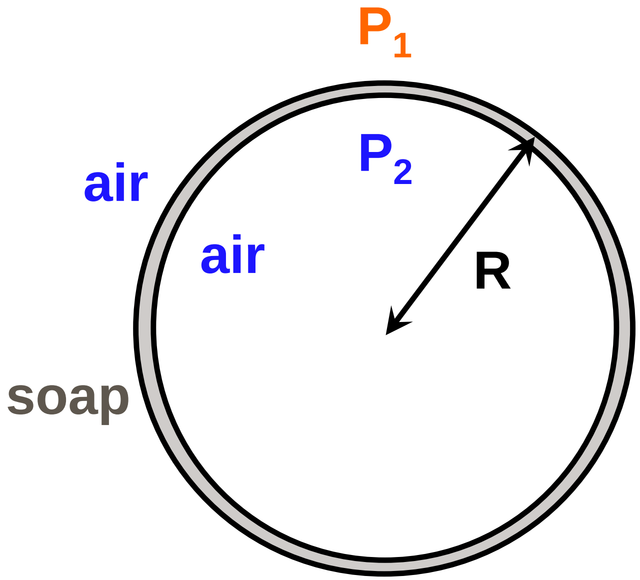
Surface tension
Spherical drop or bubble



$$P_2 - P_1 = 2\gamma / R$$

γ: surface tension

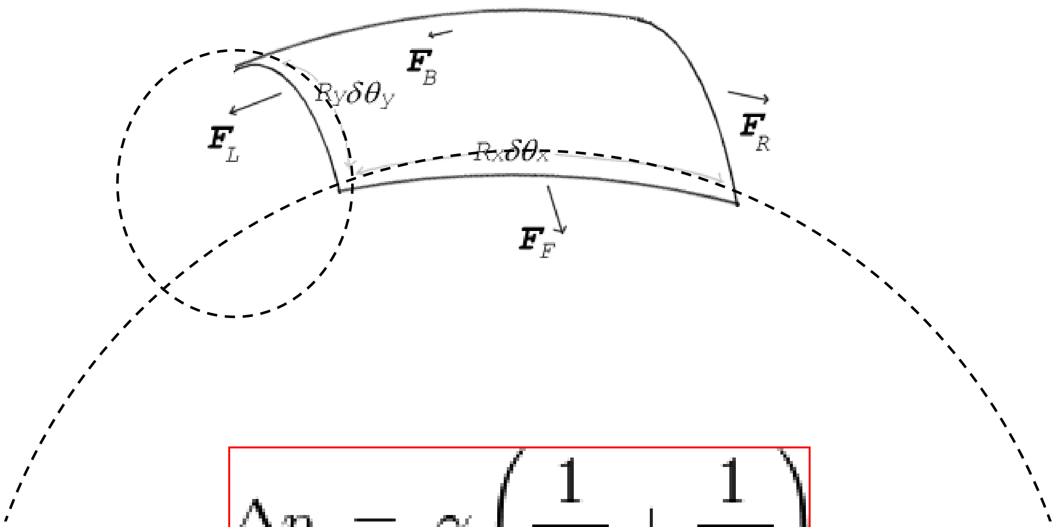
Surface tension
Soap bubble



$P_2 - P_1 = 4\gamma/R$

γ: surface tension

Surface tension
Laplace Law

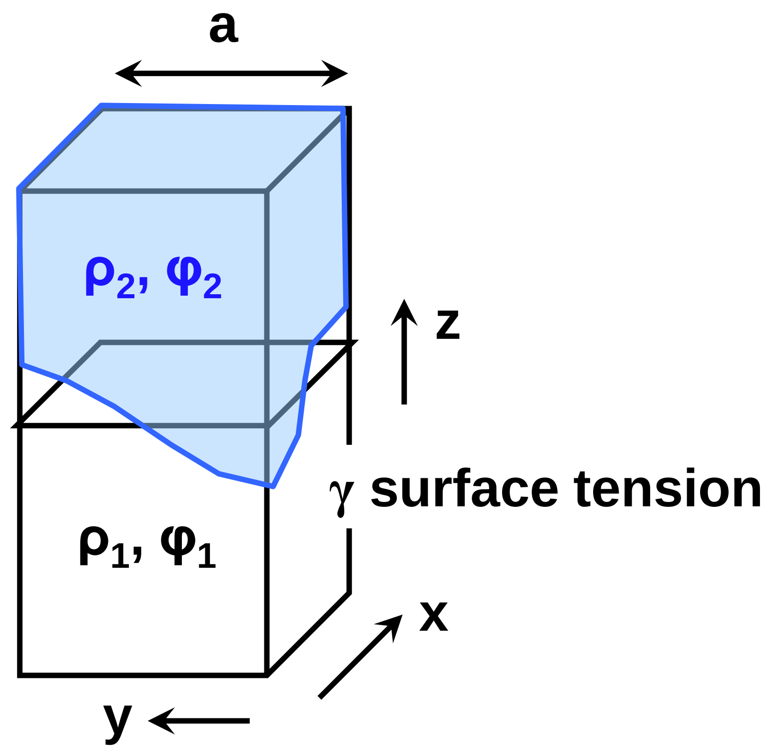


$$\boxed{\Delta p = \gamma\left(\frac{1}{R_x}+\frac{1}{R_y}\right)}$$

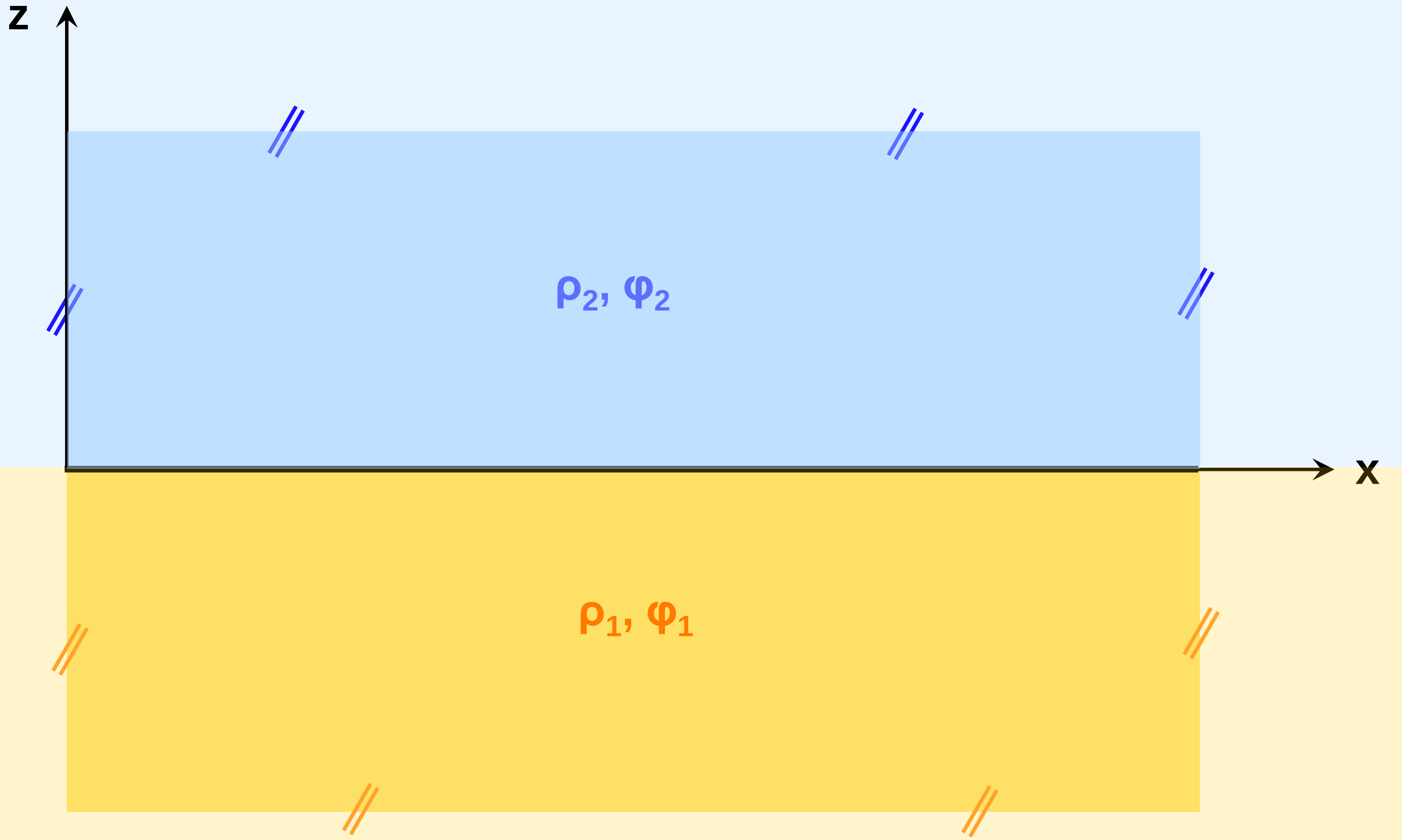
Rayleigh Taylor instability

Dense fluid (ρ_2) above

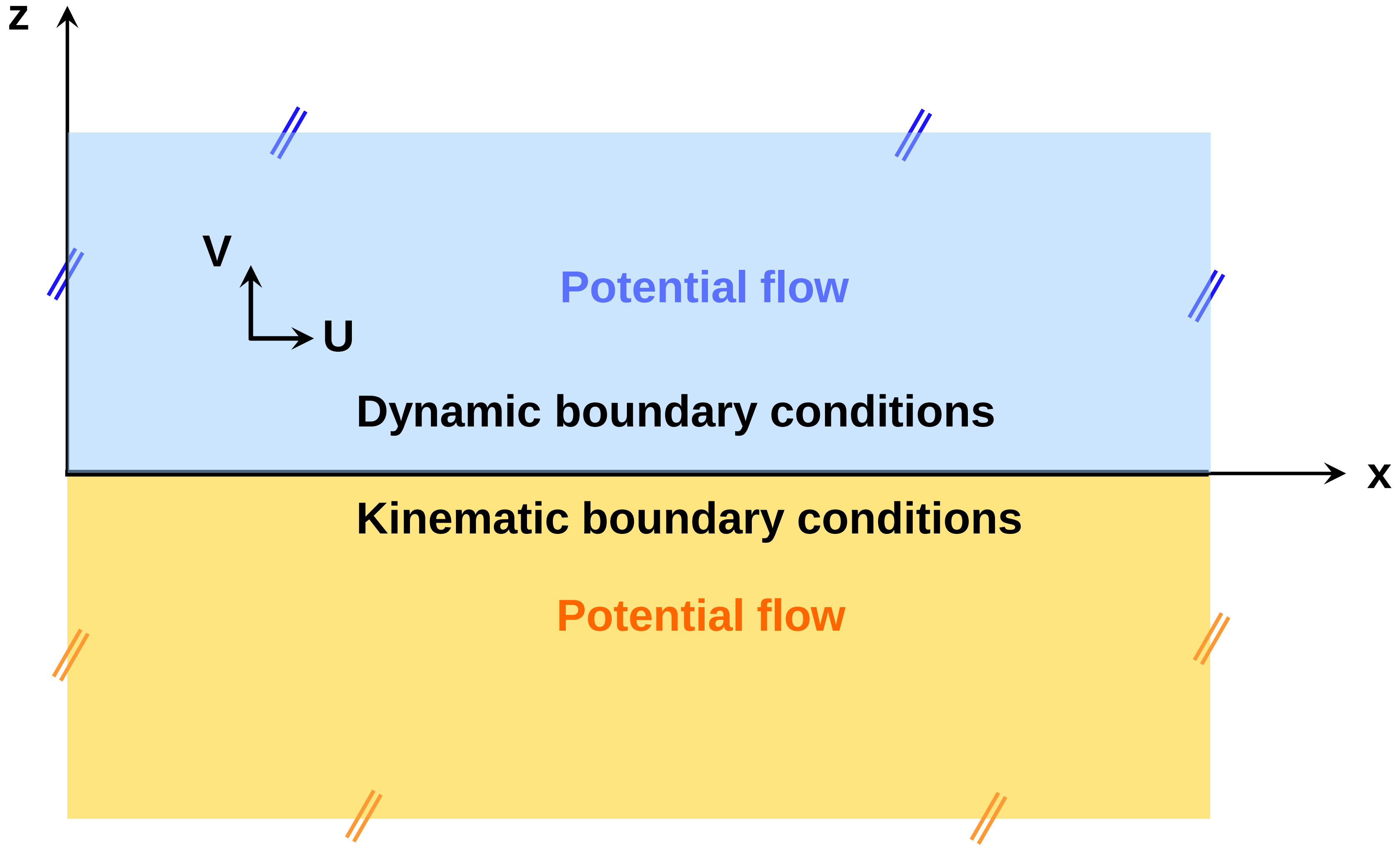
Light fluid (ρ_1) below



Rayleigh Taylor instability
Simplify to 2D : two semi-infinite domains



Rayleigh Taylor instability
Simplify to 2D : two semi-infinite domains



Instability analysis:

1. **Equations and boundary conditions**
2. **Base state**
3. **Linearized equations**
4. **Normal mode expansion**
5. **Dispersion relation**
6. **Analysis of the dispersion relation**

1. Equations

$$\boxed{\begin{aligned} \Delta \Phi_1 &= 0 \\ \Delta \Phi_2 &= 0 \end{aligned}}$$

Potential flow

$$\boxed{\begin{aligned} U_1 = \frac{\partial \Phi_1}{\partial x}, \qquad V_1 = \frac{\partial \Phi_1}{\partial z} \\ U_2 = \frac{\partial \Phi_2}{\partial x}, \qquad V_2 = \frac{\partial \Phi_2}{\partial z} \end{aligned}}$$

Velocity field

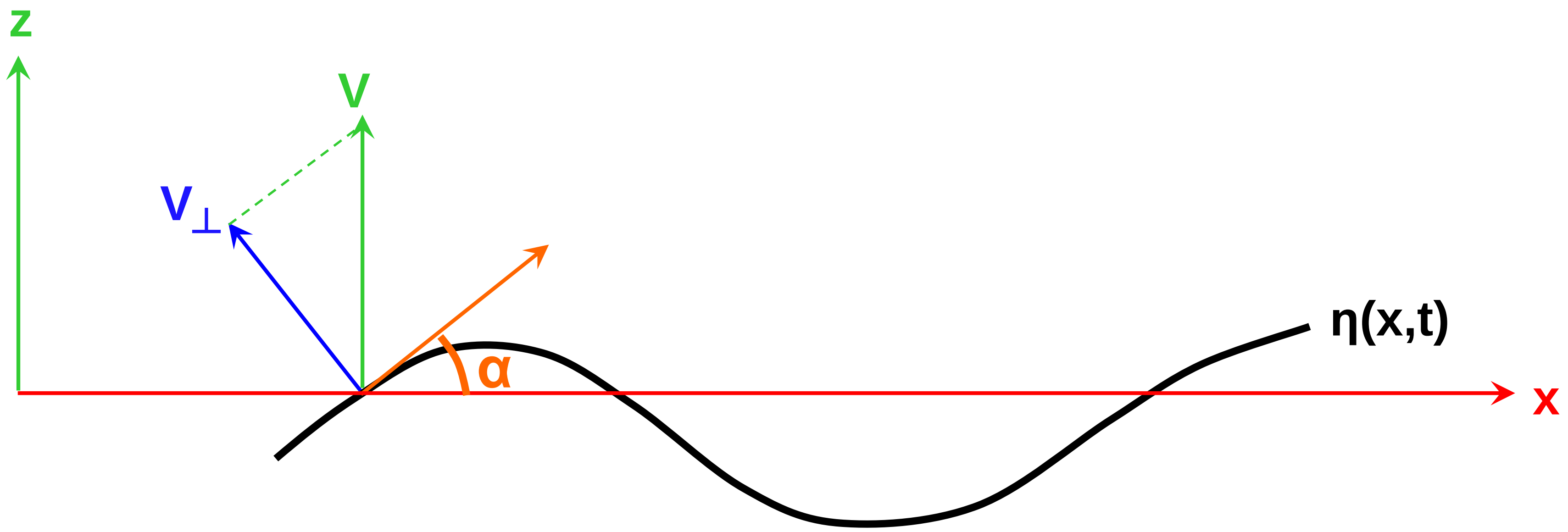
1. Boundary conditions

$$\boxed{\begin{aligned} \Phi_1 &= 0 \text{ at } z = -\infty \\ \Phi_2 &= 0 \text{ at } z = +\infty \end{aligned}}$$

far-field

$\text{at } z = \eta$ **?**

1. Kinematic boundary condition

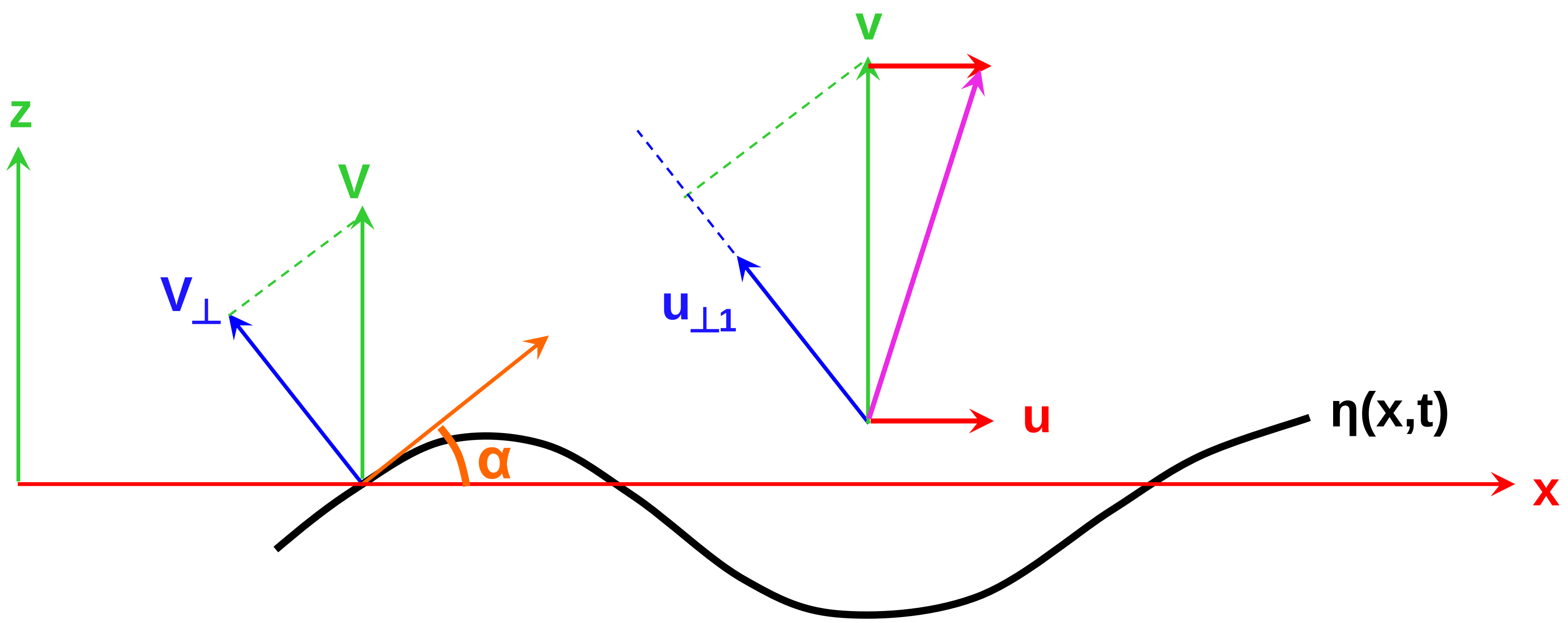


Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$V_{\perp} = \partial\eta/\partial t \cos(\alpha)$

1. Kinematic boundary condition



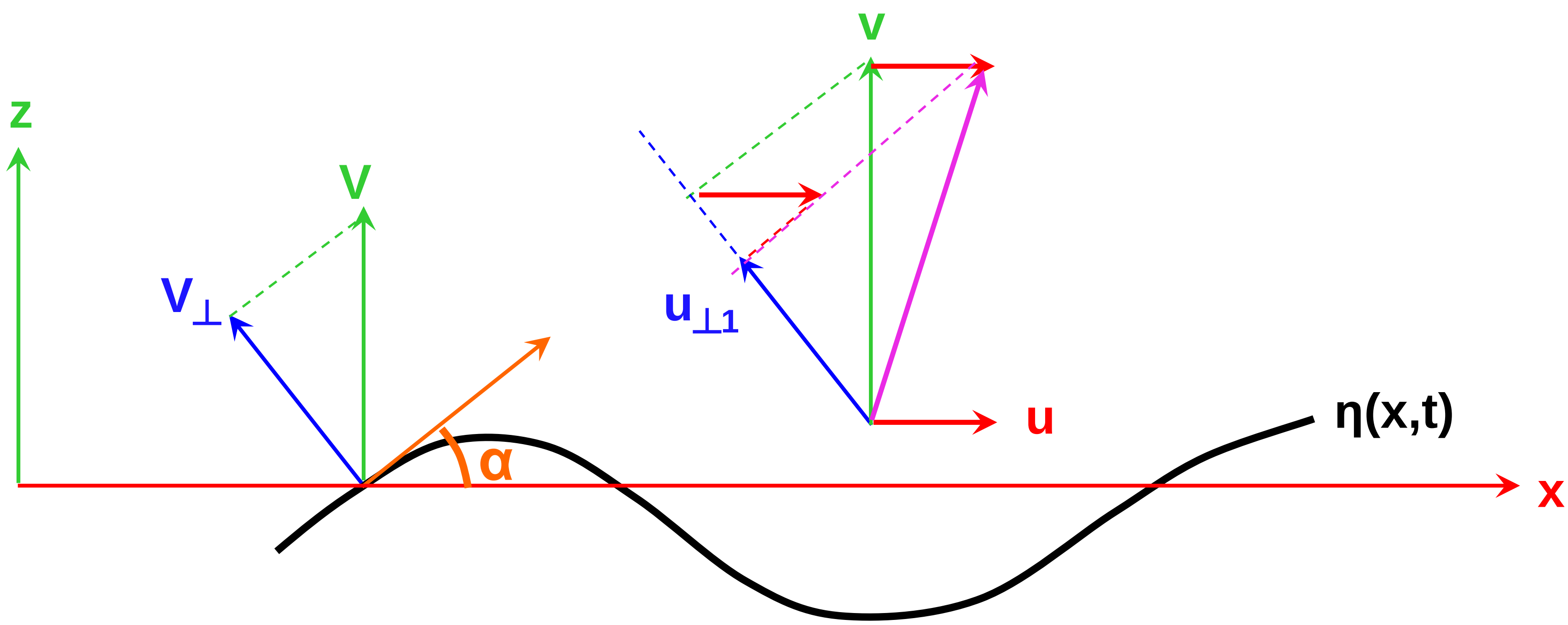
Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$V_{\perp} = \partial\eta/\partial t \cos(\alpha)$

$u_{\perp 1} = v_1 \cos(\alpha) +$

1. Kinematic boundary condition



Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$\left.\begin{aligned} V_\perp &= \partial\eta/\partial t \cos(\alpha) \\ u_{\perp 1} &= v_1 \cos(\alpha) - u_1 \sin(\alpha) \end{aligned}\right\} \partial\eta/\partial t = v_1 - u_1 \tan(\alpha) \Rightarrow \boxed{\partial\eta/\partial t = v_1 - u_1 \partial\eta/\partial x}$$

1. Kinematic boundary conditions

$$\boxed{\begin{array}{l} \Phi_1 = 0 \text{ at } z = -\infty \\ \Phi_2 = 0 \text{ at } z = +\infty \end{array}}$$

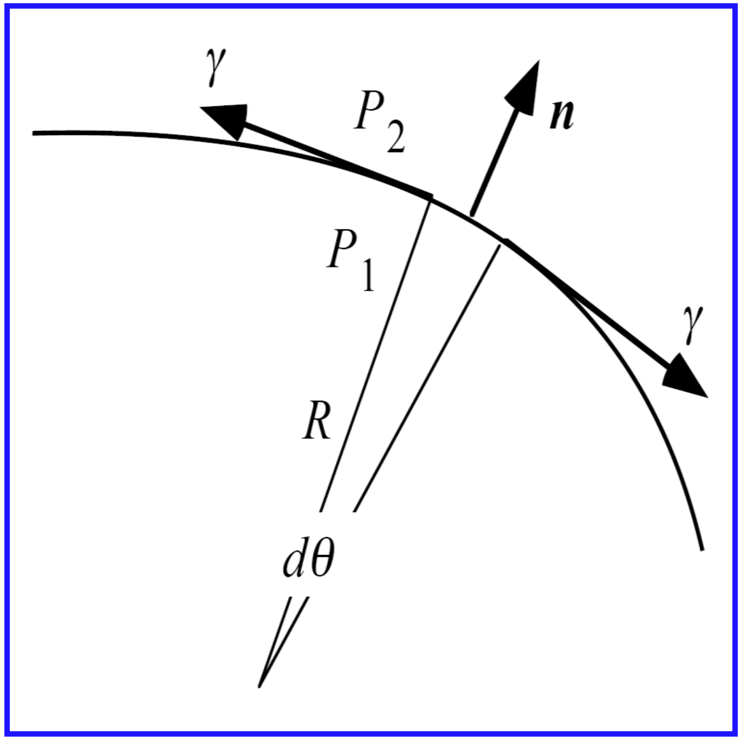
far-field

$$U_1 \frac{\partial \eta}{\partial x} - V_1 = \frac{\partial \eta}{\partial t}$$
$$U_2 \frac{\partial \eta}{\partial x} - V_2 = \frac{\partial \eta}{\partial t}$$

$$\text{at } z = \eta$$

1. Dynamic boundary conditions

$$P_1 - P_2 = -\gamma \frac{\frac{\partial^2 \eta}{\partial x^2}}{\left(1 + \frac{\partial \eta}{\partial x}^2\right)^{3/2}} \text{ at } z = \eta$$



$$\mathbf{n} = \frac{(-\partial_x \eta, 1)}{\sqrt{1 + \partial_x^2 \eta}}$$

$$\mathcal{C} = \nabla . \mathbf{n}$$

1. More equations

$$\boxed{\begin{aligned} \frac{\partial \Phi_1}{\partial t} + \frac{U_1^2 + V_1^2}{2} + \frac{P_1}{\rho_1} + gz &= C_1(t) \\ \frac{\partial \Phi_2}{\partial t} + \frac{U_2^2 + V_2^2}{2} + \frac{P_2}{\rho_2} + gz &= C_2(t) \end{aligned}}$$

= 0

= 0

2nd Bernouilli relations

2. Base state

$$\Phi_1 = 0,$$

$$\Phi_2 = 0$$

$$\eta = 0$$

$$P_1 = -\rho_1 g z$$

$$P_2 = -\rho_2 g z$$

3. Perturb and linearize perturbation expansion

$$
\begin{aligned}
\Phi_1 &= 0 &&+ \epsilon\phi_1 \\
\Phi_2 &= 0 &&+ \epsilon\phi_2 \\
U_1 &= 0 &&+ \epsilon u_1 \\
V_1 &= 0 &&+ \epsilon v_1 \\
U_2 &= 0 &&+ \epsilon u_2 \\
V_2 &= 0 &&+ \epsilon v_2 \\
P_1 &= -\rho_1 g z &&+ \epsilon p_1 \\
P_2 &= -\rho_2 g z &&+ \epsilon p_2 \\
\eta &= 0 &&+ \epsilon\sigma
\end{aligned}
$$

$$\epsilon \ll 1$$

Variables **Base state** **Small perturbation**

3. Linearized equations

$$\boxed{\begin{aligned} \Delta\phi_1 &= 0 \\ \Delta\phi_2 &= 0 \end{aligned}}$$

perturbed potential flow

$$\boxed{\begin{aligned} u_1 &= \frac{\partial \phi_1}{\partial x}, \qquad v_1 = \frac{\partial \phi_1}{\partial z} \\ u_2 &= \frac{\partial \phi_2}{\partial x}, \qquad v_2 = \frac{\partial \phi_2}{\partial z} \end{aligned}}$$

3. Perturbed kinematic boundary conditions

$$\boxed{\begin{aligned} \phi_1 &= 0 \text{ at } z = -\infty \\ \phi_2 &= 0 \text{ at } z = +\infty \end{aligned}}$$

$$\boxed{\begin{aligned} -\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 &= \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma \\ -\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 &= \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma \end{aligned}}$$

3. Perturbed kinematic boundary conditions

$$\boxed{\begin{aligned} \phi_1 &= 0 \text{ at } z = -\infty \\ \phi_2 &= 0 \text{ at } z = +\infty \end{aligned}}$$

$$\cancel{-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x}} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

$$\cancel{-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x}} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

$$\boxed{\begin{aligned} v_1 &= \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma \\ v_2 &= \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma \end{aligned}}$$

3. Flattened kinematic boundary conditions

$$\boxed{\begin{aligned}\frac{\partial \phi_1}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma \\ \frac{\partial \phi_2}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma\end{aligned}}$$

Taylor expansion around 0: $\boxed{\frac{\partial \phi}{\partial z}(\epsilon\sigma) = \frac{\partial \phi}{\partial z}(0) + \epsilon\sigma \frac{\partial^2 \phi}{\partial z^2}(0) + \ldots}$

$$\boxed{\begin{aligned}\frac{\partial \phi_1}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = 0 \\ \frac{\partial \phi_2}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = 0\end{aligned}}$$

⇨transforms a b.c. at an unkwown interface into a fixed place!

3. Perturbed dynamic boundary conditions

$$\boxed{(P_1 + \epsilon p_1 - P_2 - \epsilon p_2)\big|_{\epsilon\sigma} = -\gamma\epsilon\frac{\partial^2\sigma}{\partial x^2}\left(1 - 3/2\epsilon^2\left(\frac{\partial\sigma}{\partial x}\right)^2\right)}$$

Replace P_1=-gρ_1z, … **and linearize**

$$g\,(\rho_2 - \rho_1)\sigma + (p_1 - p_2)\big|_{\epsilon\sigma} = -\gamma\frac{\partial^2\sigma}{\partial x^2}$$

flatten

$$\boxed{(\rho_2 - \rho_1)g\sigma + (p_1 - p_2)\big|_0 = -\gamma\frac{\partial^2\sigma}{\partial x^2}}$$

3. Perturbed and linearized Bernouilli

Perturbed 2nd Bernouilli relations

$$\begin{aligned}
\epsilon\frac{\partial \phi_1}{\partial t} + \cancel{\epsilon^2\frac{u_1^2+v_1^2}{2}} + \epsilon\frac{p_1}{\rho_1} &= 0 \\
\epsilon\frac{\partial \phi_2}{\partial t} + \cancel{\epsilon^2\frac{u_2^2+v_2^2}{2}} + \epsilon\frac{p_2}{\rho_2} &= 0
\end{aligned}$$

Linearized 2nd Bernouilli relations

$$\begin{aligned}
\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} &= 0 \\
\frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} &= 0
\end{aligned}$$

4. Normal mode expansion

Fourier transform in x and t

$$\boxed{\begin{aligned} \phi_1 &= f_1(z)\exp(i(kx-\omega t)), \\ \phi_2 &= f_2(z)\exp(i(kx-\omega t)), \\ \sigma &= C\exp(i(kx-\omega t)), \end{aligned}}$$

k is the wavenumber and ω the frequency (in rad/s)

$$\boxed{\lambda = 2\pi/k} \qquad \boxed{T = 2\pi/\omega}$$

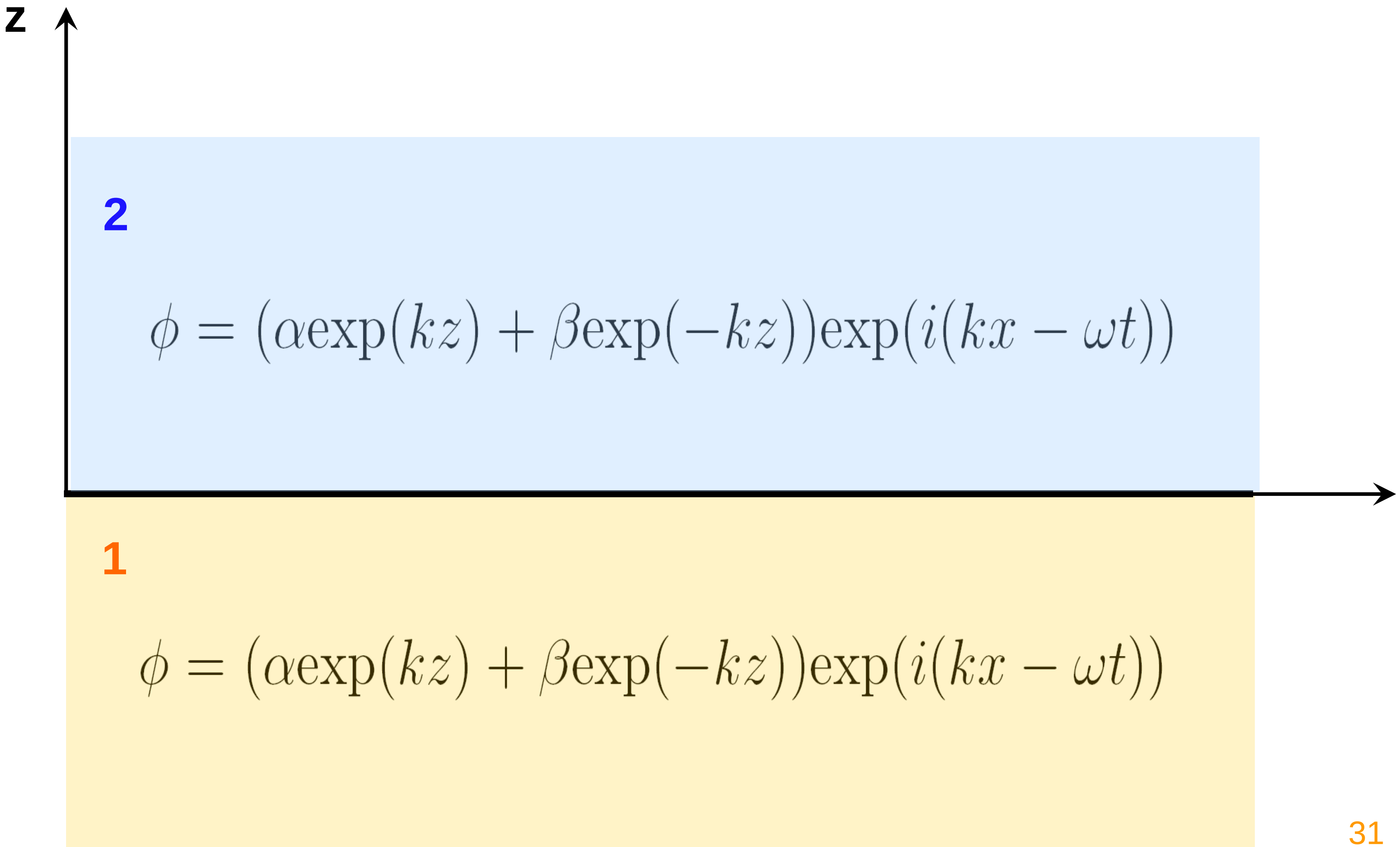
$$\boxed{f = \omega/(2\pi)}$$

4. Normal mode expansion

Solution to Laplace equation:

4. Normal mode expansion

Solution to Laplace equation:



4. Normal mode expansion

Solution to Laplace equation:

$$\boxed{\begin{aligned} \phi_1 &= A\exp(kz)\exp(i(kx-\omega t)), \\ \phi_2 &= B\exp(-kz)\exp(i(kx-\omega t)), \\ \sigma &= C\exp(i(kx-\omega t)). \end{aligned}}$$

4. Normal mode expansion

Replace in boundary conditions

$$\boxed{\begin{aligned} g(\rho_2-\rho_1)C+i\omega\rho_1 A-i\omega\rho_2 B &= \gamma k^2 C \\ kA &= -i\omega C \\ -kB &= -i\omega C \end{aligned}}$$

This is an eigenvalue problem iωX=MX!

$$kg(\rho_2-\rho_1)C+\omega^2\rho_1 C+\omega^2\rho_2 C=\gamma k^3 C.$$

5. Dispersion relation

$$\boxed{\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}}$$

- **Unstable if there exists one ω, Im(ω)>0**
- **Neutral if for all ω, Im(ω)=0:**
- **Stable (or damped) if for all ω, Im(ω)<0:**

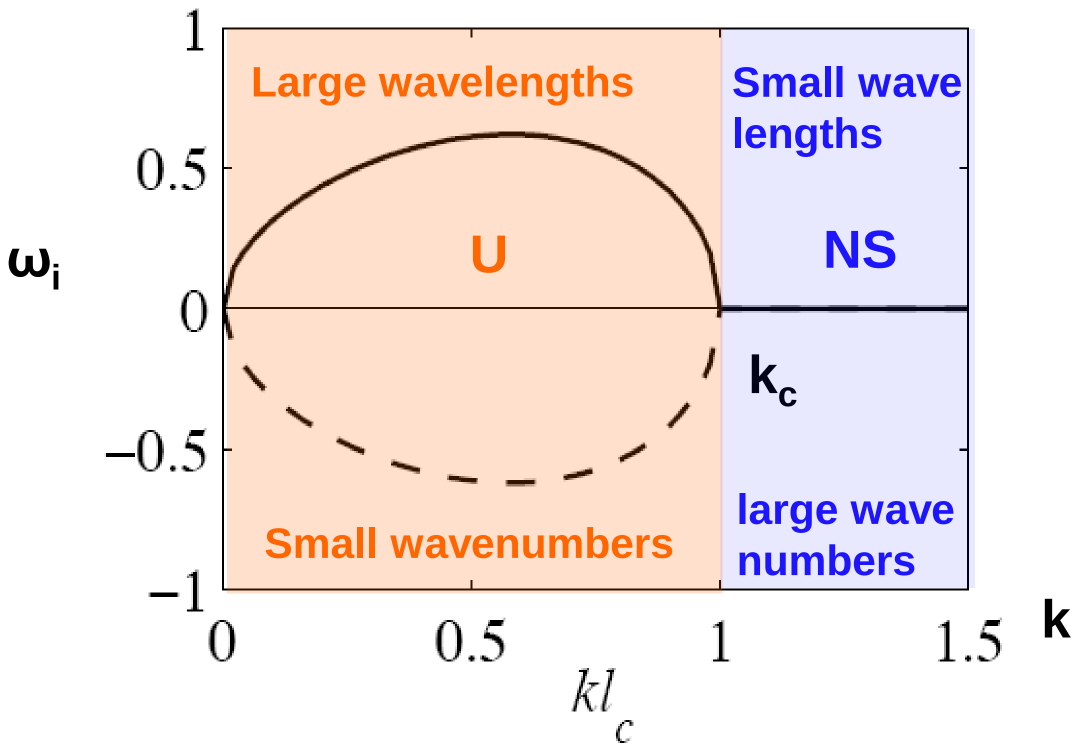
5. Dispersion relation

$$\boxed{\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}}$$

- **Unstable if there exists one ω, Im(ω)>0** $\boxed{\rho_2 > \rho_1}$
- **Neutral if for all ω, Im(ω)=0:** $\rho_1 > \rho_2$
- **Stable (or damped) if for all ω, Im(ω)<0:**

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

4. Dispersion relation



$$k_c = \sqrt{g|\rho_2 - \rho_1|/\gamma}$$

$$\boxed{l_c = \sqrt{\gamma/(g|\rho_2 - \rho_1|)}}$$

capillary length: 2.7mm for air/water

Instability analysis:

1. **Equations and boundary conditions**
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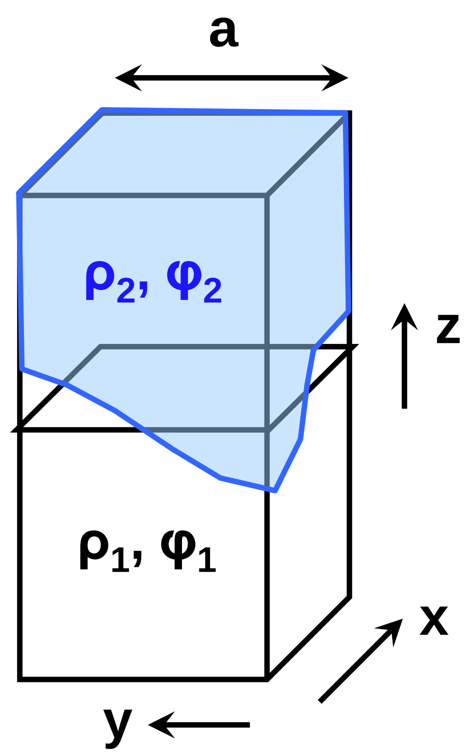
5. **Dispersion relation**
6. **Analysis of the dispersion relation**

Rayleigh Taylor instability

Dense fluid (ρ_2) above

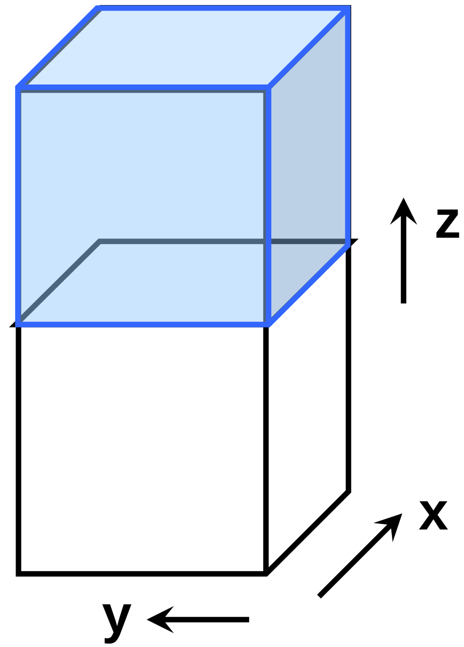
Light fluid (ρ_1) below



Base state

Flat Interface at z=0 ⇨ h(x,y)=0

No flow ⇨ $\varphi_1(x,y)=0$, $\varphi_2(x,y)=0$



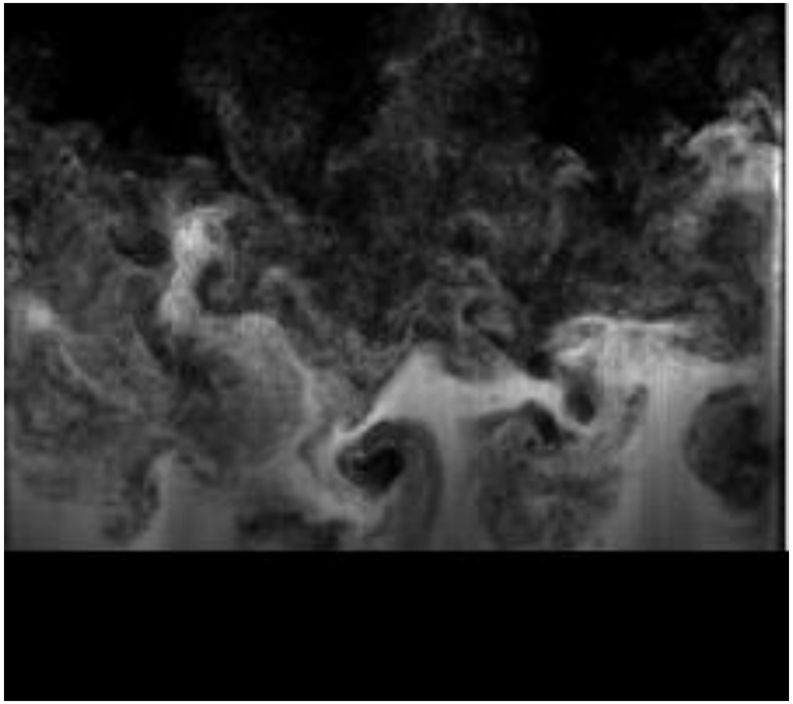
$$C=\frac{\left(1+\left(\frac{\partial S}{\partial x}\right)^2\right)\frac{\partial^2 S}{\partial y^2}-2\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}\frac{\partial^2 S}{\partial x\partial y}+\left(1+\left(\frac{\partial S}{\partial y}\right)^2\right)\frac{\partial^2 S}{\partial x^2}}{\left(1+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2\right)^{3/2}}$$

Global instability analysis:

1. **Equations and boundary conditions**
2. **Base state**
3. **Linearized equations**
4. **Fourier series expansion**
5. **Countable number of admissible frequencies: discretized dispersion relation**
6. **Analysis of the linear stability conditions**

Rayleigh-Taylor instability

This is just the linear part of the story!

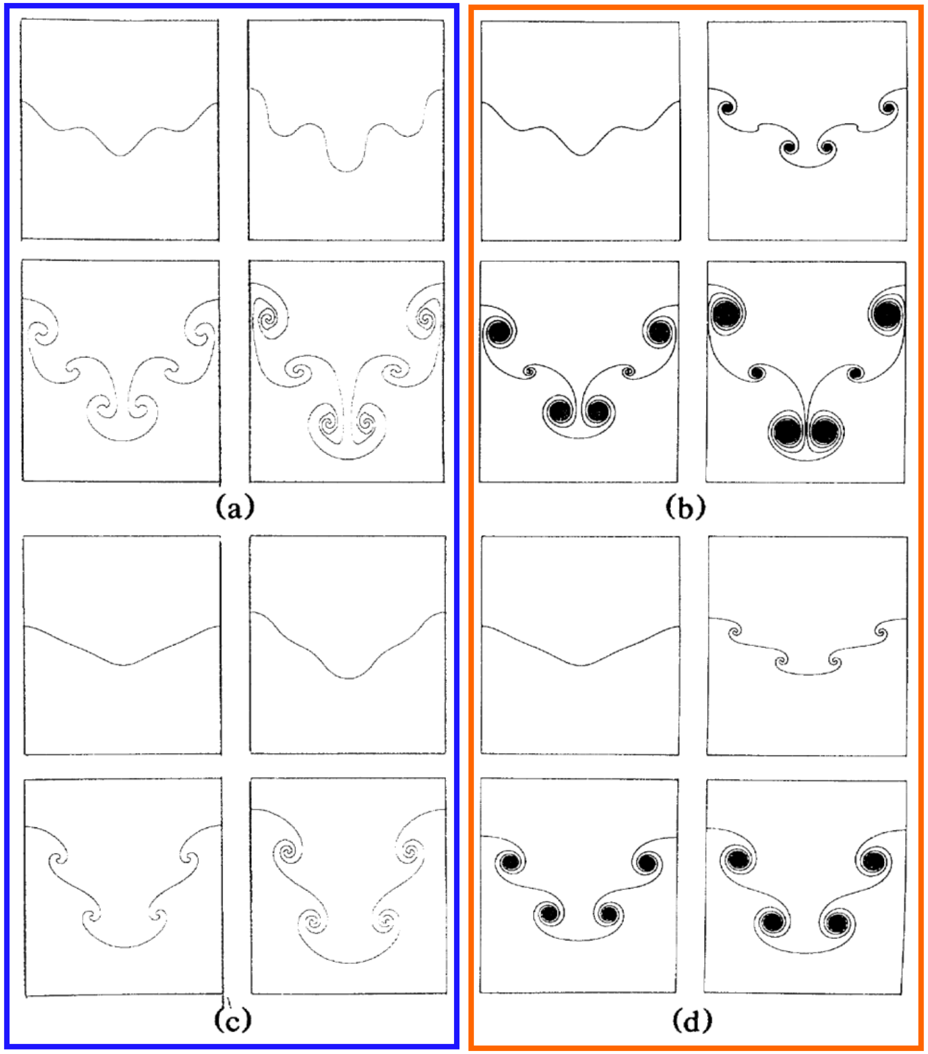


Jones & Jacobs 1997

Rayleigh-Taylor instability

This is just the linear part of the story!

DNS

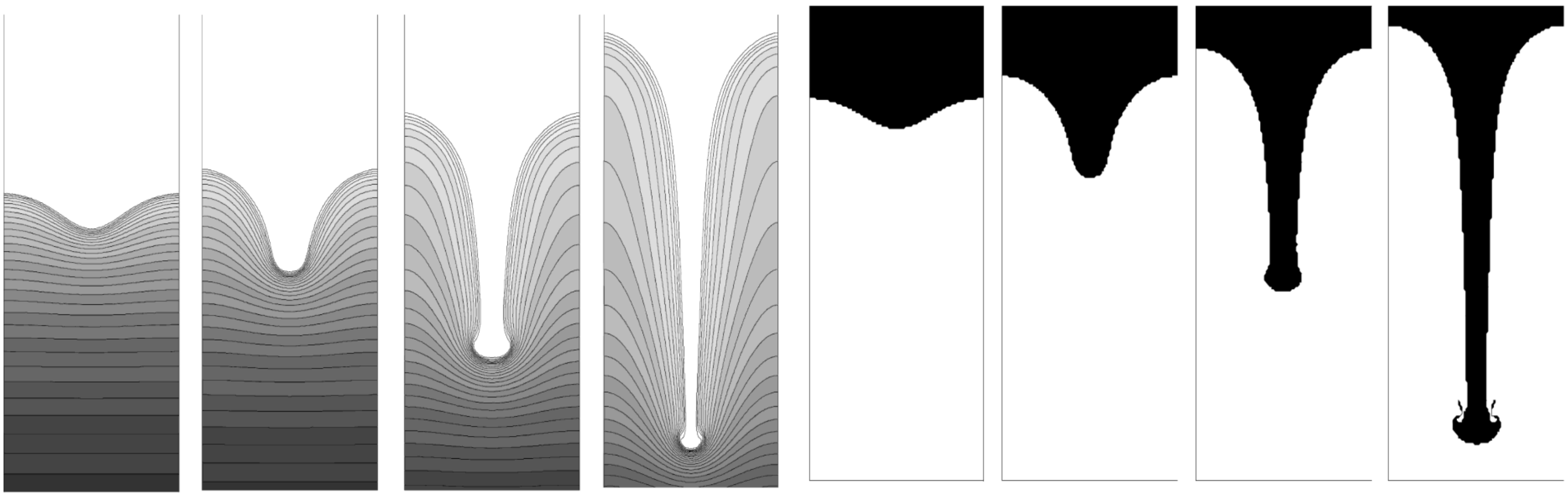


Model

Aref & Tryggvason 1989

Rayleigh-Taylor instability

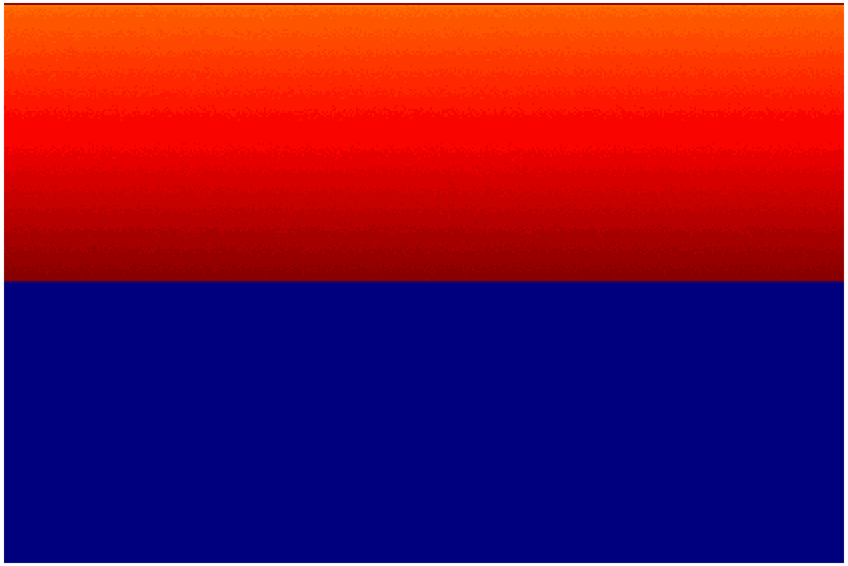
This is just the linear part of the story!



DNS Aref & Tryggvason 1989 **Model**

Rayleigh-Taylor instability

This is just the linear part of the story!



www.LBmethod.org
Jonas Lätt, EPFL

Rayleigh-Taylor instability

A nonlinear paradox: pooring syrup

Unstable but not mixed

Stable but mixed

Instability of suspended thin films

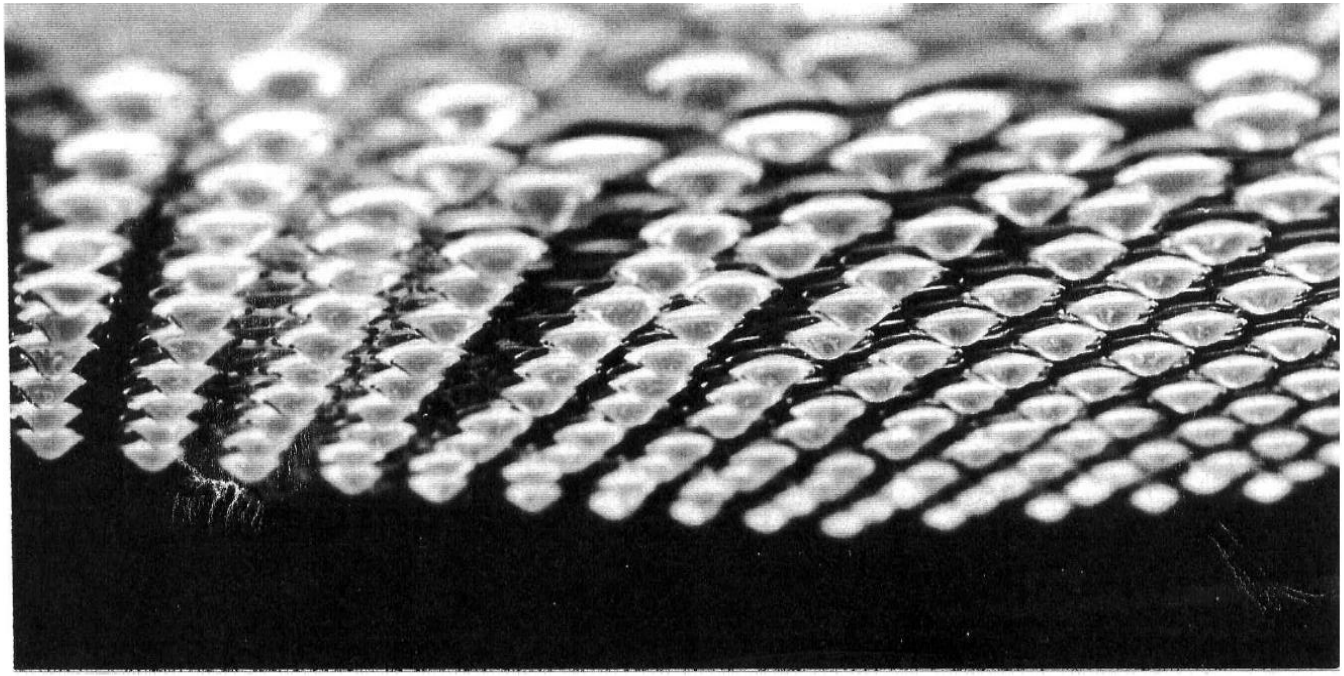


FIG. 2.6 – *Une couche mince d'huile est déposée sur une plaque plane, puis la plaque est retournée. le film d'huile est instable et l'instabilité se manifeste par l'apparition de gouttes pendantes disposées sur un réseau ici hexagonal (Fermigier et al. 1990).*

Instability analysis:

- **A physical mechanism (handwaving arguments)**
- **Scaling laws**
- **A dispersion relation that links wavelengths and frequencies**
- **An experimental validation**