Most flows are unstable...

Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

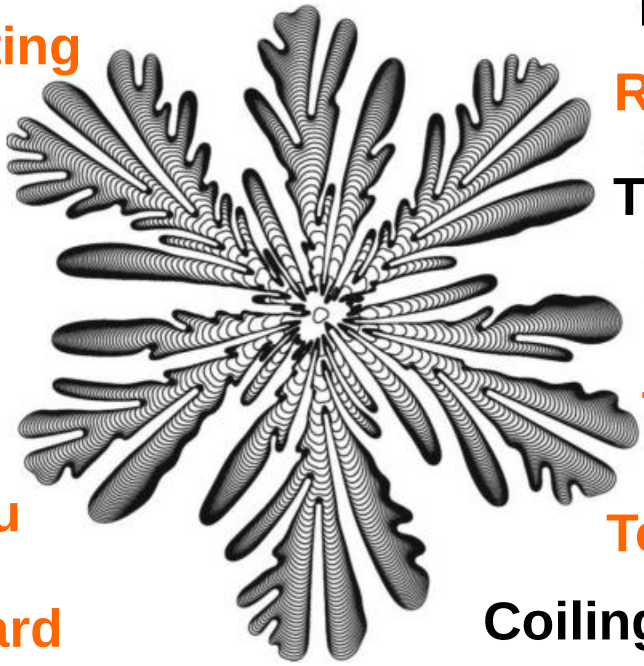
Tearing instability

Rayleigh-Benard

Coiling instability

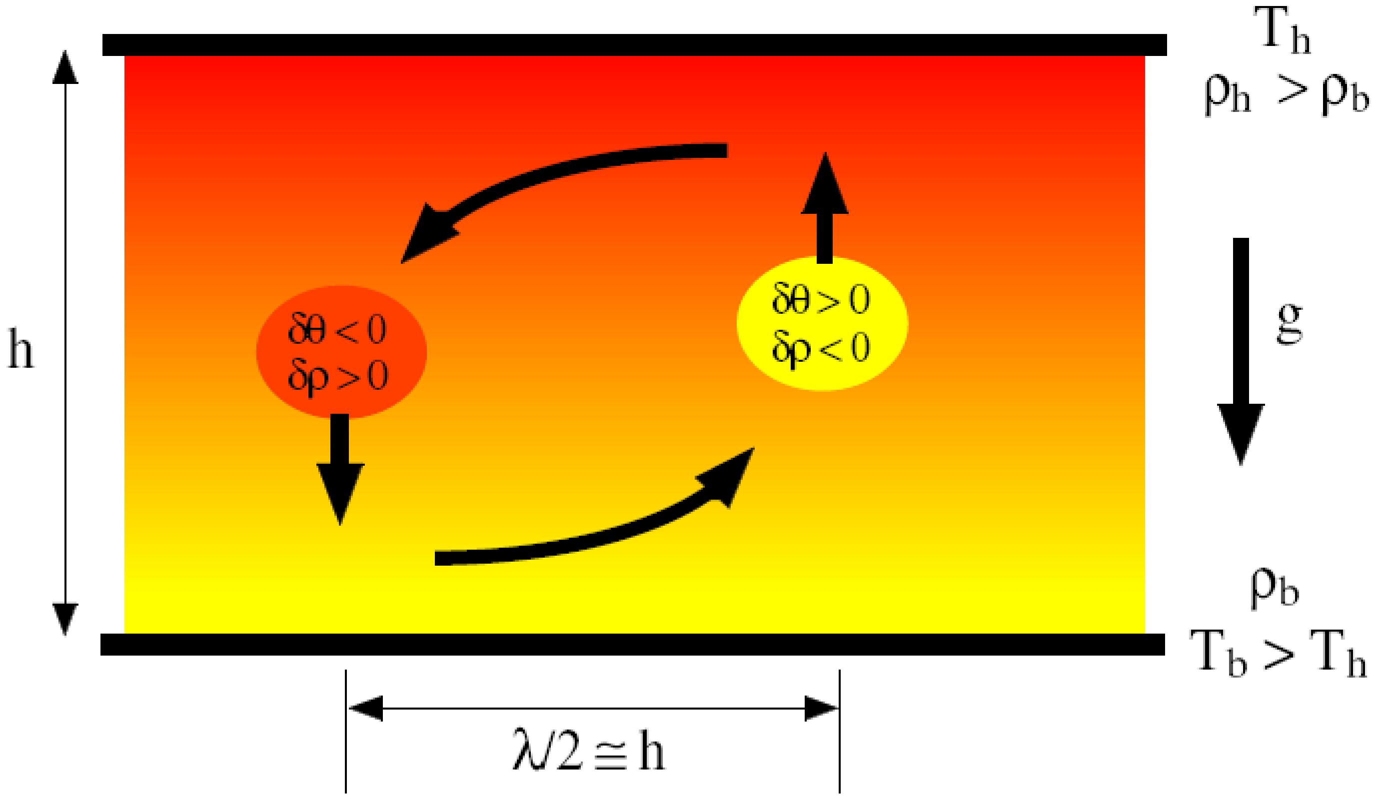
Benard-Marangoni

Kelvin-Helmholtz



Chaos

Rayleigh Bénard convection



Lessons learned from linear stability analysis

1. The pure conductive solution becomes unstable above a threshold, which for fixed depth h and fluid properties is simply proportional to the temperature gradient.

2. We therefore introduced a control parameter which equals 0 at threshold

$$r = \frac{\Delta T - \Delta T_c}{\Delta T_c}$$

3. Close to threshold, when linearity can be assumed, the perturbation writes

$$v_z = X(t) f_v(z) \sin(kx) \,, \qquad k = 2\pi/\lambda \,.$$

4. Temperature and vertical velocity are in phase

5. The unstable wavelength at threshold is roughly 2h

Heuristic equation for an unstable system

$$\frac{d}{dt}X = \sigma X$$

$$\sigma = r/\tau_0$$

1. We retrieve the critical slowing at threshold
2. The system is stable (damped in fact!) for r<0 and unstable for r>0

Heuristic equation for an unstable system

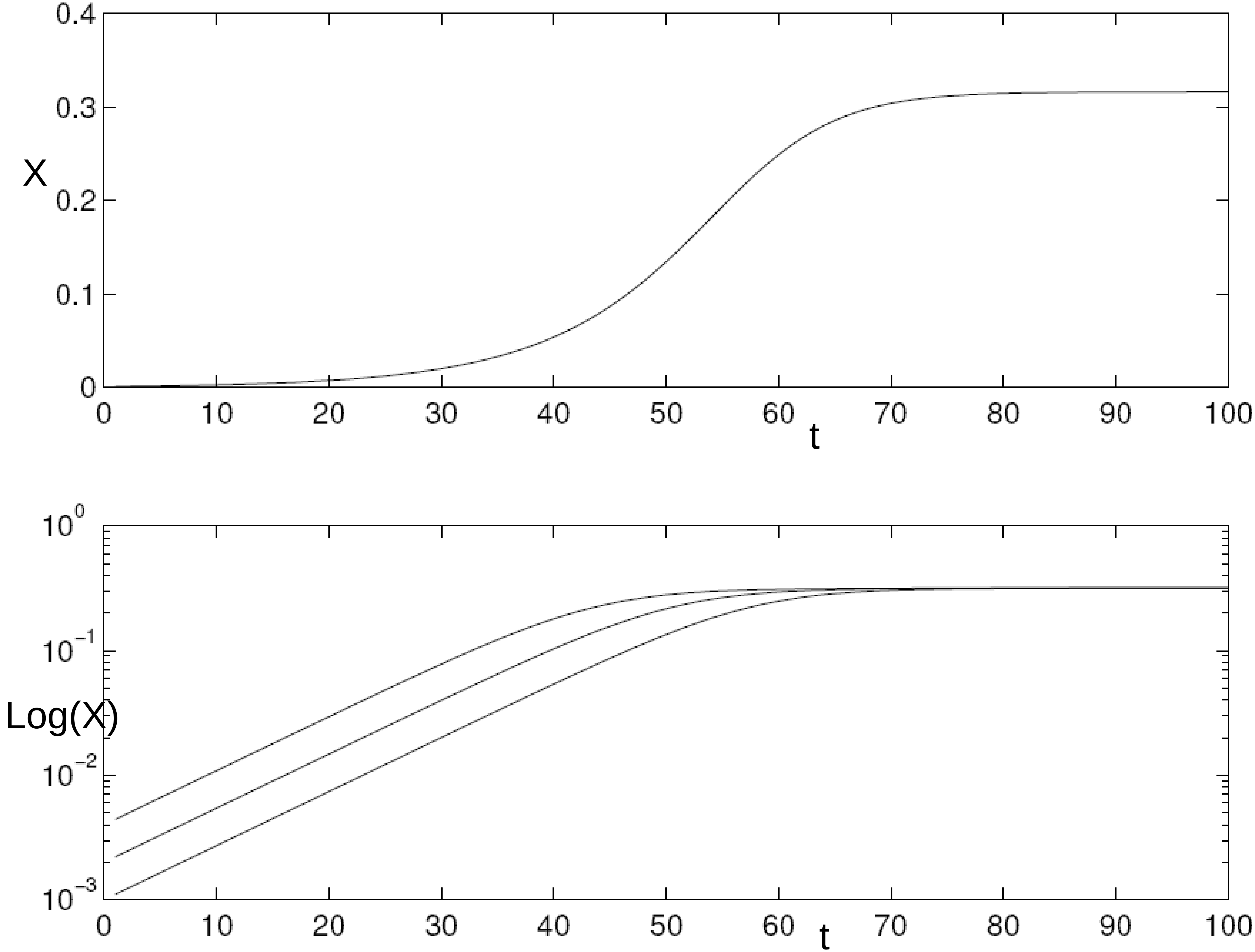
1. Since for r>0, there is exponential growth, X cannot assumed to remain small for long…
2. Nonlinear effects have to be introduced

$$\frac{d}{dt}X = \sigma_{\text{eff}} X$$

3. where σ_{eff} is now a function of X!

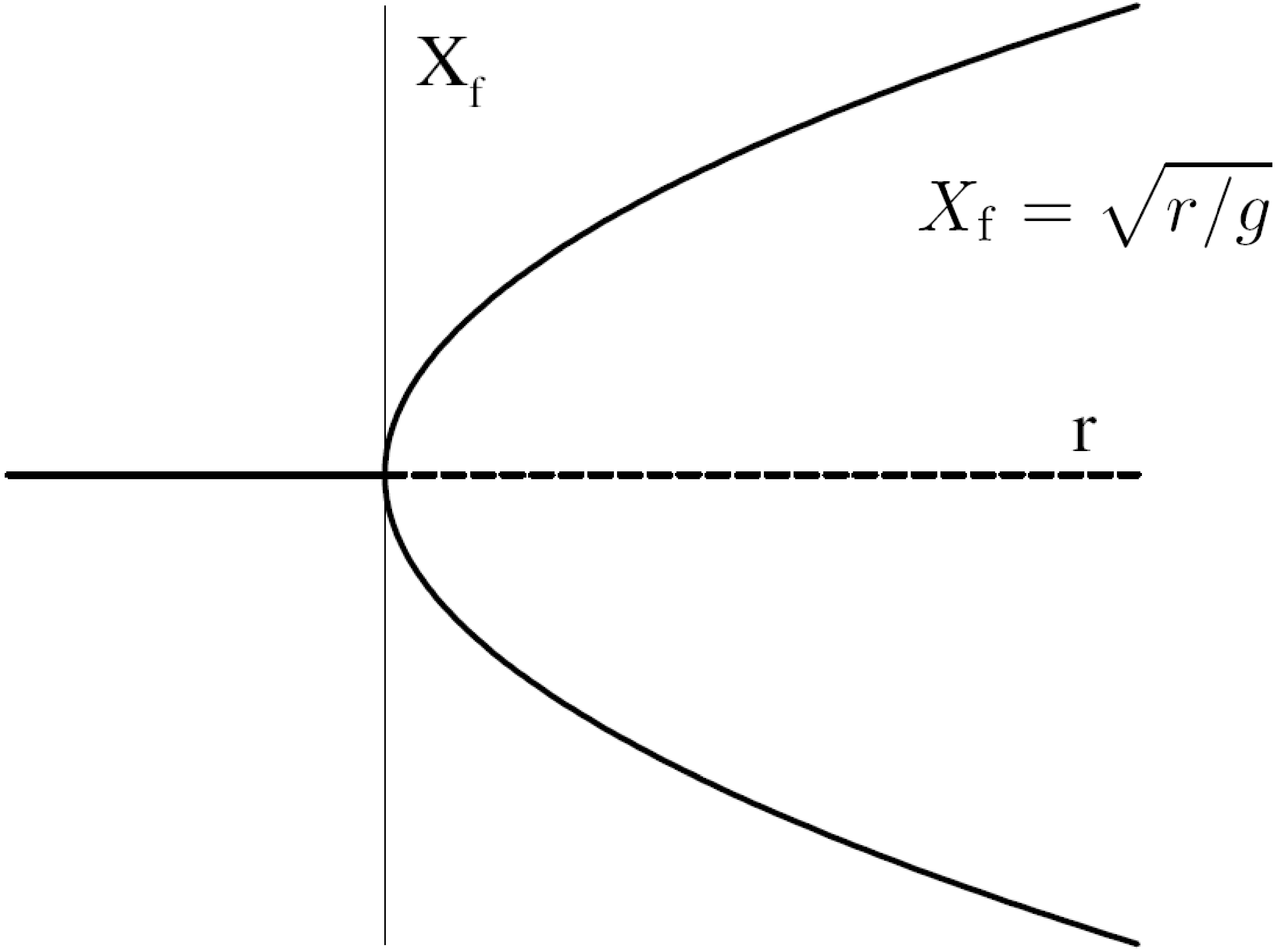
4. By symmetry, there is no reason that a positive velocity anomaly would behave differently than a negative one…

$$\tau_0 \sigma_{\text{eff}} = r - gX^2$$



0.4
0.3
0.2
0.1
0
X
0
10
20
30
40
50
60
70
80
90
100
t
10^0
10^-1
10^-2
10^-3
Log(X)
0
10
20
30
40
50
60
70
80
90
100
t

$$\tau_0 \frac{d}{dt} X = F_r(X) = rX - gX^3$$


X_f
$X_f = \sqrt{r/g}\,.$
r

Stability of the bifurcated state?

$$\begin{aligned}\tau_0 \tfrac{d}{dt}(X_\mathrm{f} + X') &= \tau_0 \tfrac{d}{dt} X' = r(X_\mathrm{f} + X') - g(X_\mathrm{f} + X')^3 \\ &= \left[r X_\mathrm{f} - g X_\mathrm{f}^3\right] + \left(r - 3g X_\mathrm{f}^2\right) X' + \mathcal{O}\left({X'}^2\right)\end{aligned}$$

$$\tau_0 \frac{d}{dt} X' = -2r X'$$

Stable since r>0!

Now we include more physics

X: vertical velocity
Y : temperature anomaly
P: Prandtl number

Driving force Archimedes

Viscous resistance

$$\frac{d}{dt}X = P(Y - X)$$

Now we include more physics

X: vertical velocity
Y : temperature anomaly
P: Prandtl number
R: Rayleigh number

Driving force Archimedes

Viscous resistance

$$\frac{d}{dt}X = P(Y - X)$$

$$\frac{d}{dt}Y = RX - Y$$

Heat transport

Heat diffusion

Stability of equilibrium state (X=0,Y=0)

$$X = X_0 \exp(st)\,, \qquad Y = Y_0 \exp(st)\,,$$

$$\begin{aligned} sX_0 &= P(Y_0 - X_0)\,, \\ sY_0 &= RX_0 - Y_0\,, \end{aligned}$$

$$(s+P)(s+1) - RP = 0$$

$$s^{(\pm)} = \tfrac{1}{2}\left(-(P+1) \pm \left[(P-1)^2 + 4PR\right]^{1/2}\right)$$

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>0

Unstable if $\left[(P-1)^2 + 4PR\right] > (P+1)^2$

$$4PR > 4P$$

$$R > 1$$

There is also a mean temperature correction Z

X $\propto \sin(kx)$

Y $\propto \sin(kx)$

$$\frac{d}{dt}Z = XY - bZ$$

Lorentz model

X: vertical velocity

Y : temperature anomaly

Z: mean temperature correction

Driving force Archimedes

Viscous resistance

$$\frac{d}{dt}X = P(Y - X)$$

Heat transport

$$\frac{d}{dt}Y = (R - Z)X - Y$$

Heat diffusion

$$\frac{d}{dt}Z = XY - bZ$$

Relaxation to linear temperature profile

Mean effect on temperature

Limiting equation for R close to 1 and X=Y

$$Z = XY/b$$

$$\frac{d}{dt}X = (R-1)X - X^3/b$$

Now consider Lorentz system at large R and b=8/3, P=10

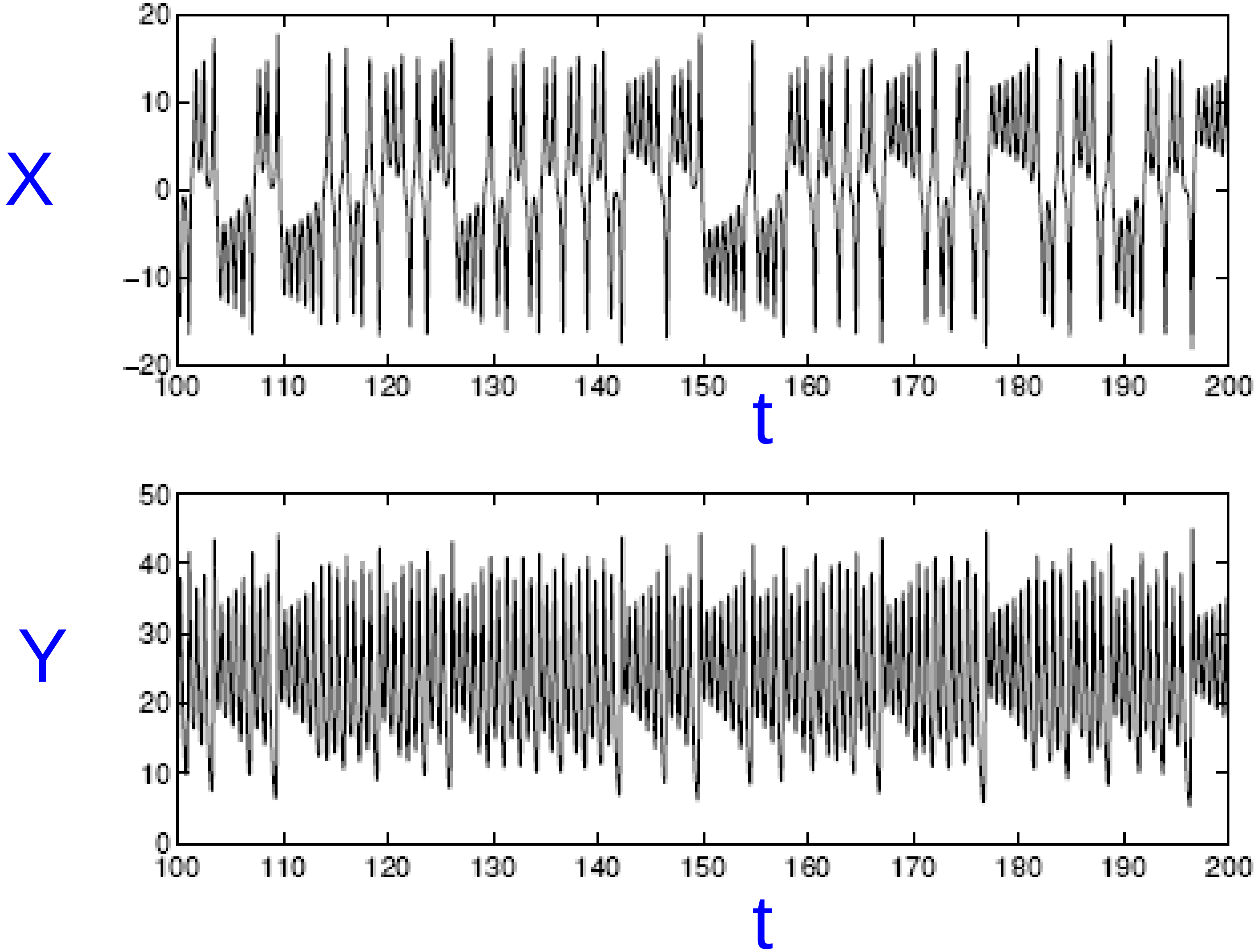
$$\frac{d}{dt}X = P(Y - X)$$

$$\frac{d}{dt}Y = (R - Z)X - Y$$

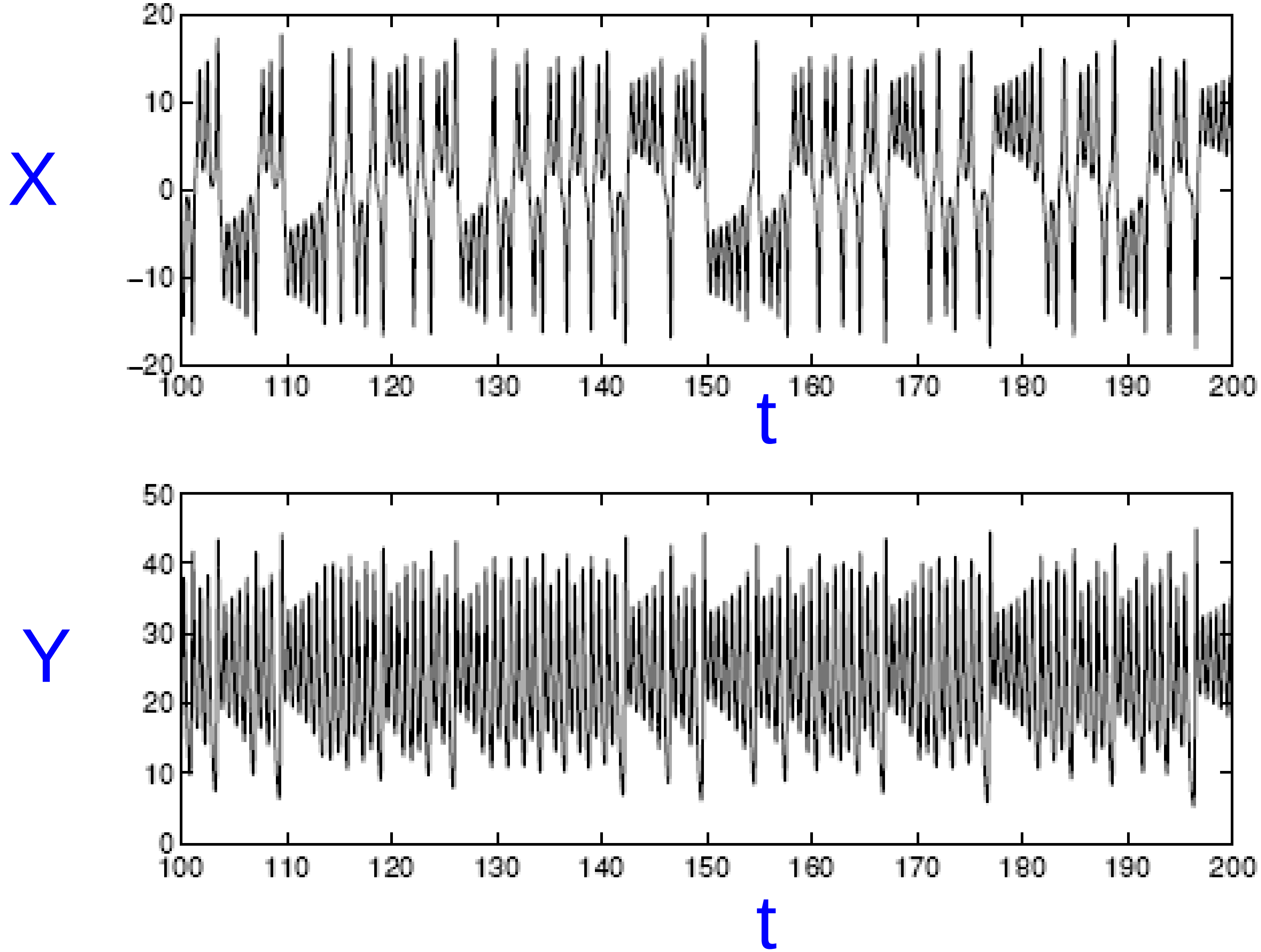
$$\frac{d}{dt}Z = XY - bZ$$

One can show that the steady convection roll solution becomes unstable for R>24.74

Here R=28, b=8/3, P=10



Here R=28, b=8/3, P=10
This is chaotic!



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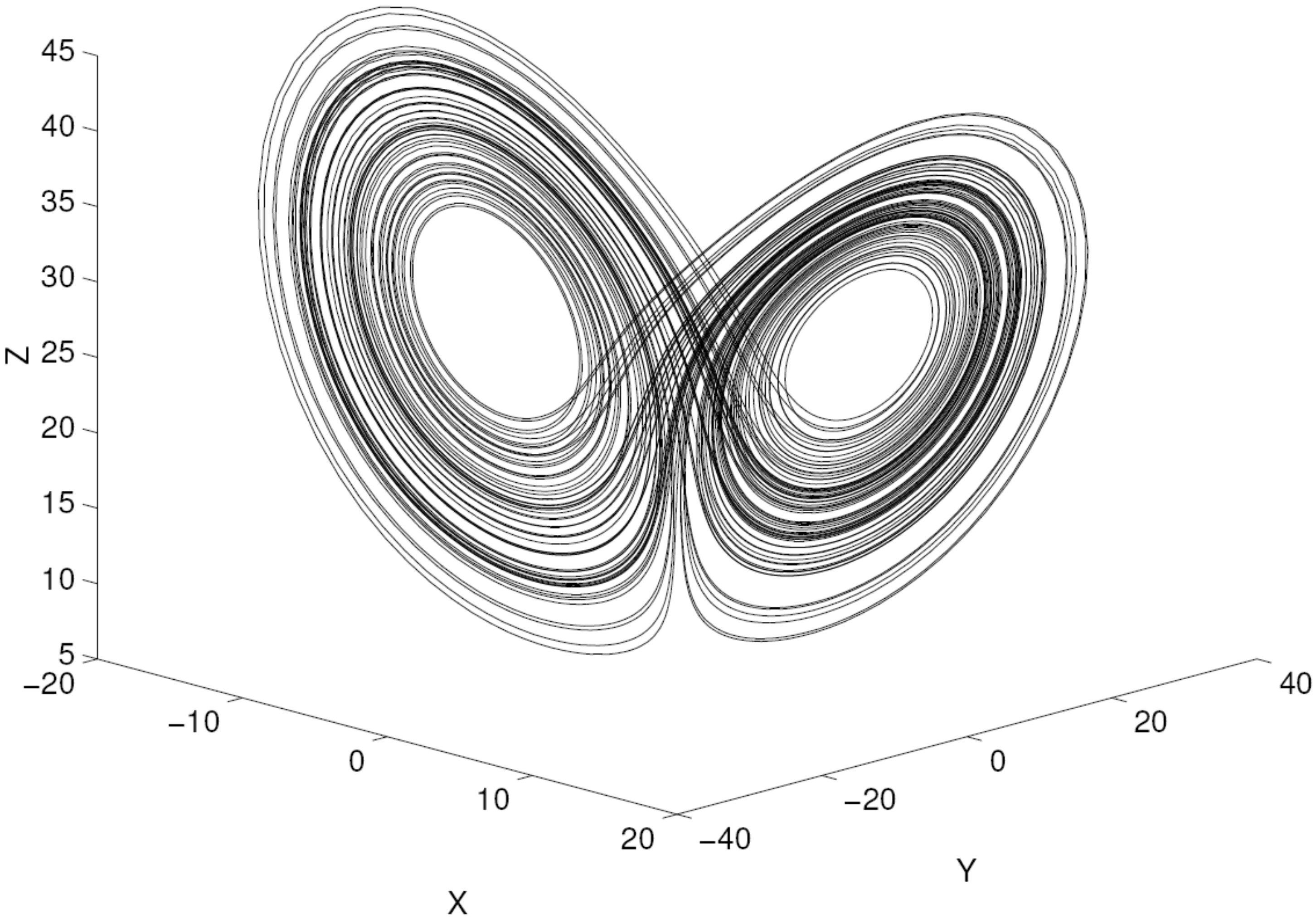
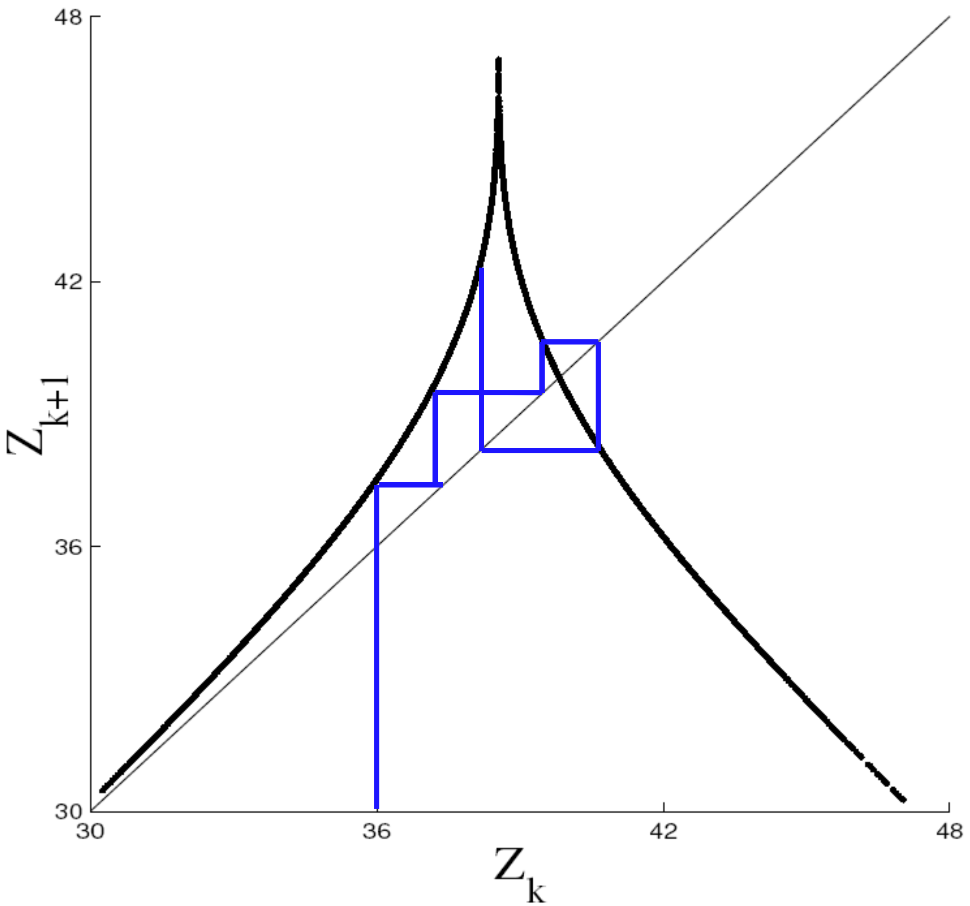


What happens?

Lorentz plot the successive maxima of Z, Z_k

In fact he plots Z_{k+1} as a function of Z_k



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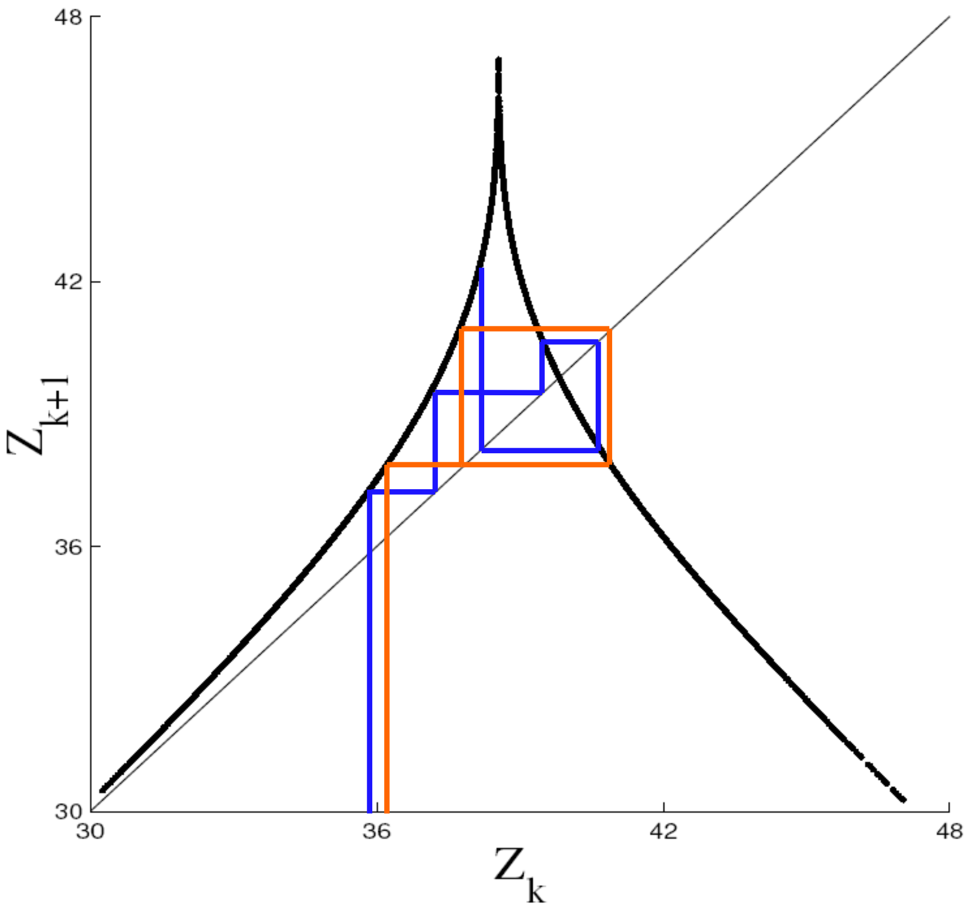


Discrete time iterations

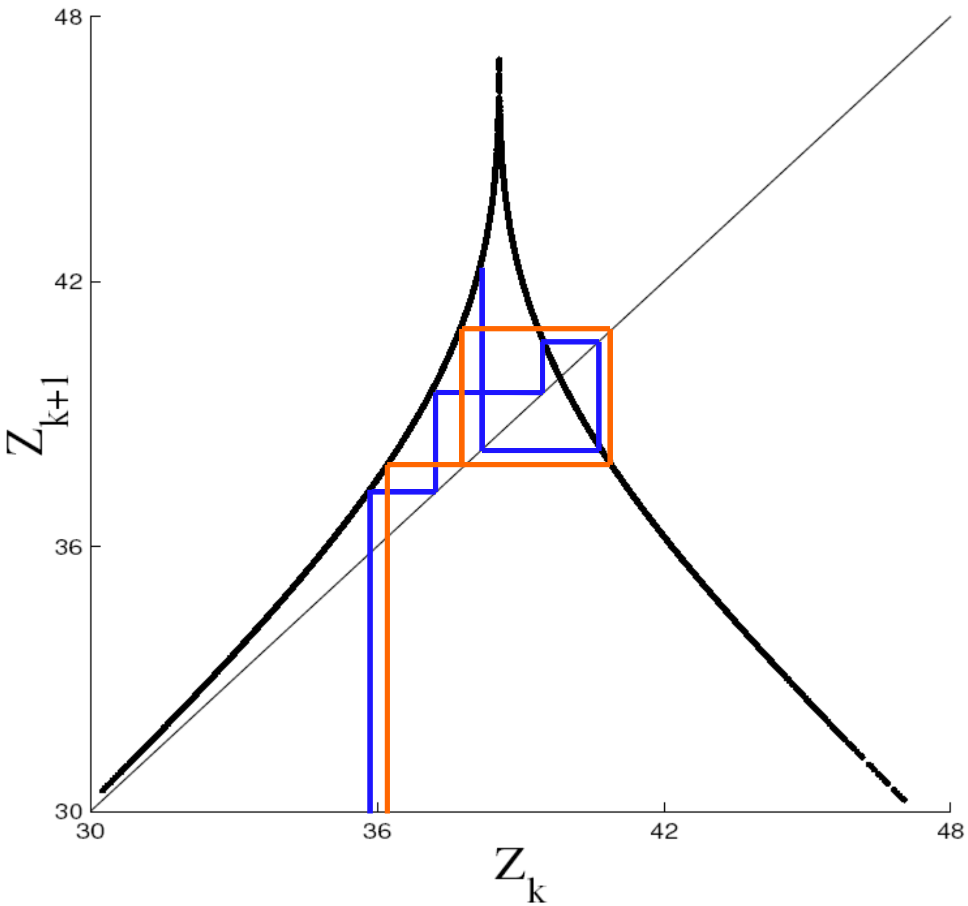
$$Z_{k+1}=f(Z_k)$$

Consider $Z_k'=Z_k+\delta Z$

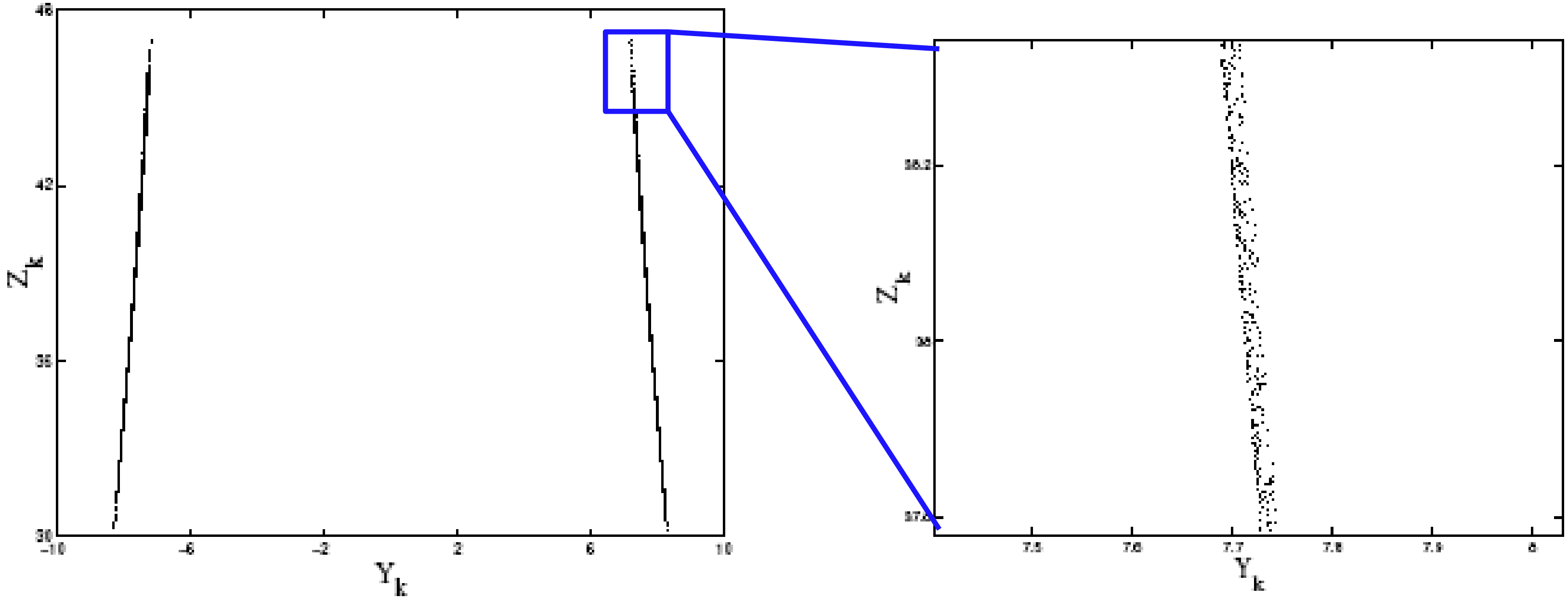
$$Z_{k+1}'=Z_{k+1}+df/dZ(Z_k)\delta Z$$

The divergence depends on the the sign of $|df/dZ|-1$

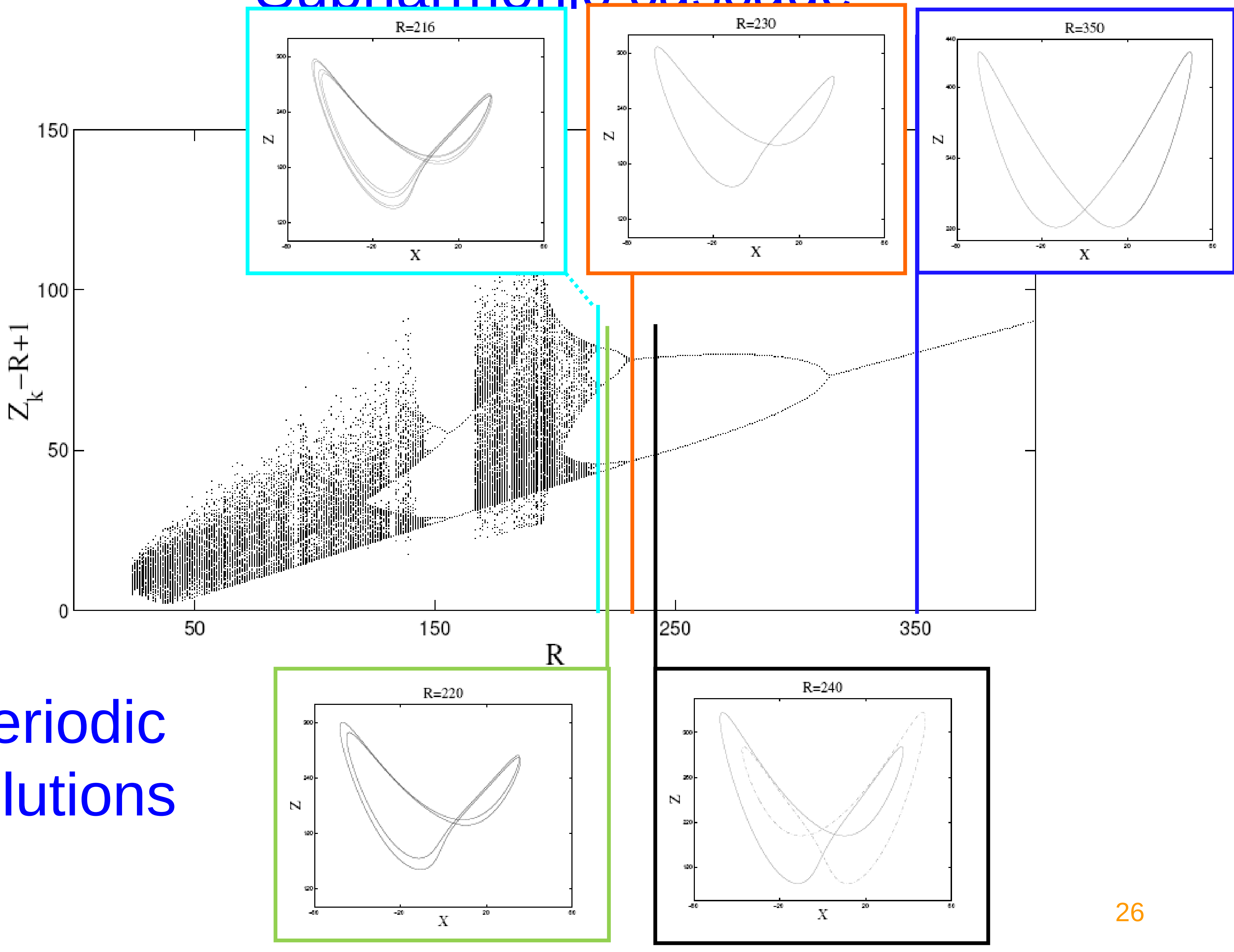
Here |df/dZ|>1 everywhere!
Small initial differences get exponentially amplified
Loss of memory of initial condition
Butterfly effect



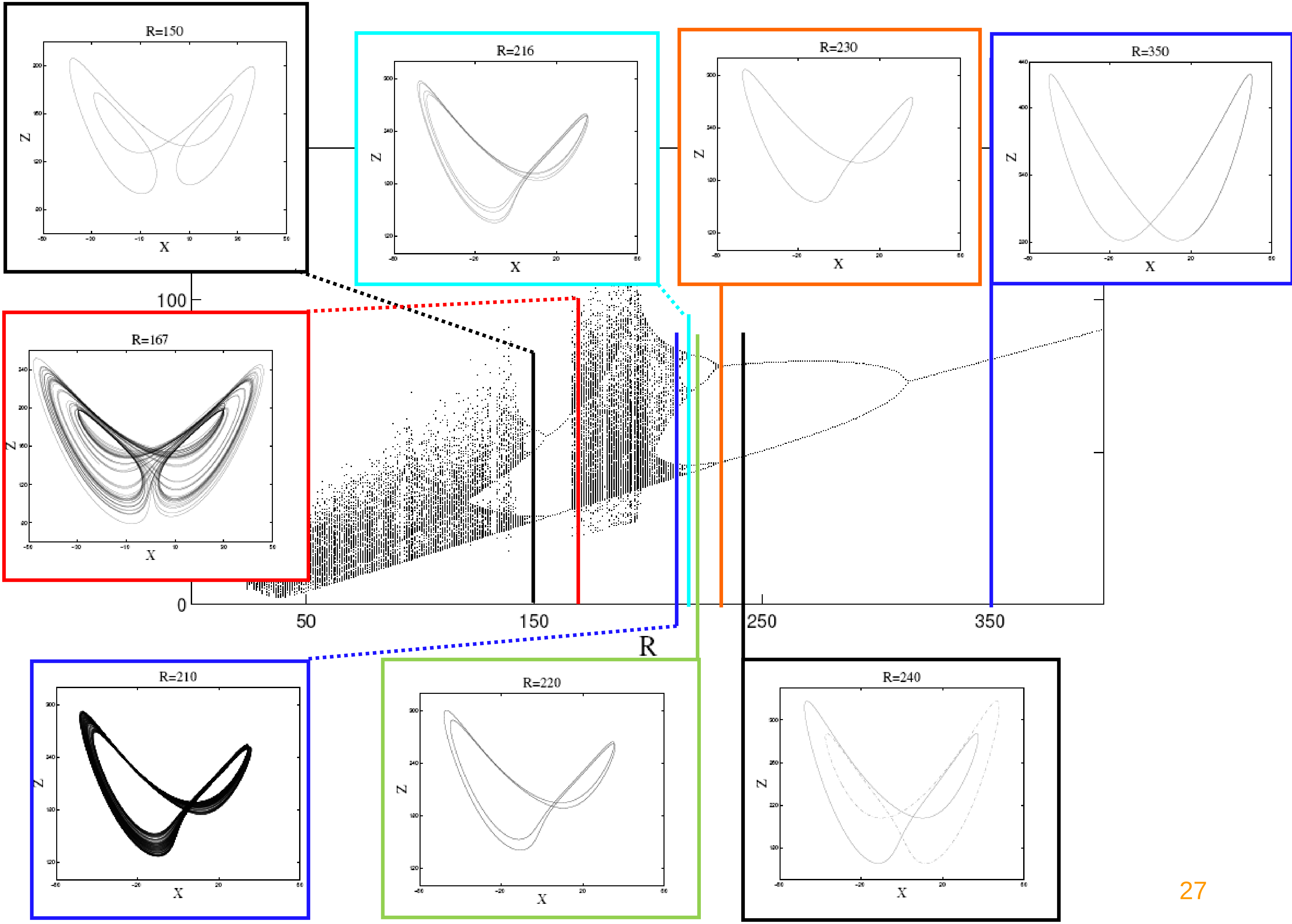
Beware, this is not a function



Subharmonic cascade



Periodic solutions


R=150
R=216
R=230
R=350
R=167
R=210
R=220
R=240
100
0
50
150
250
350
R
Z
X

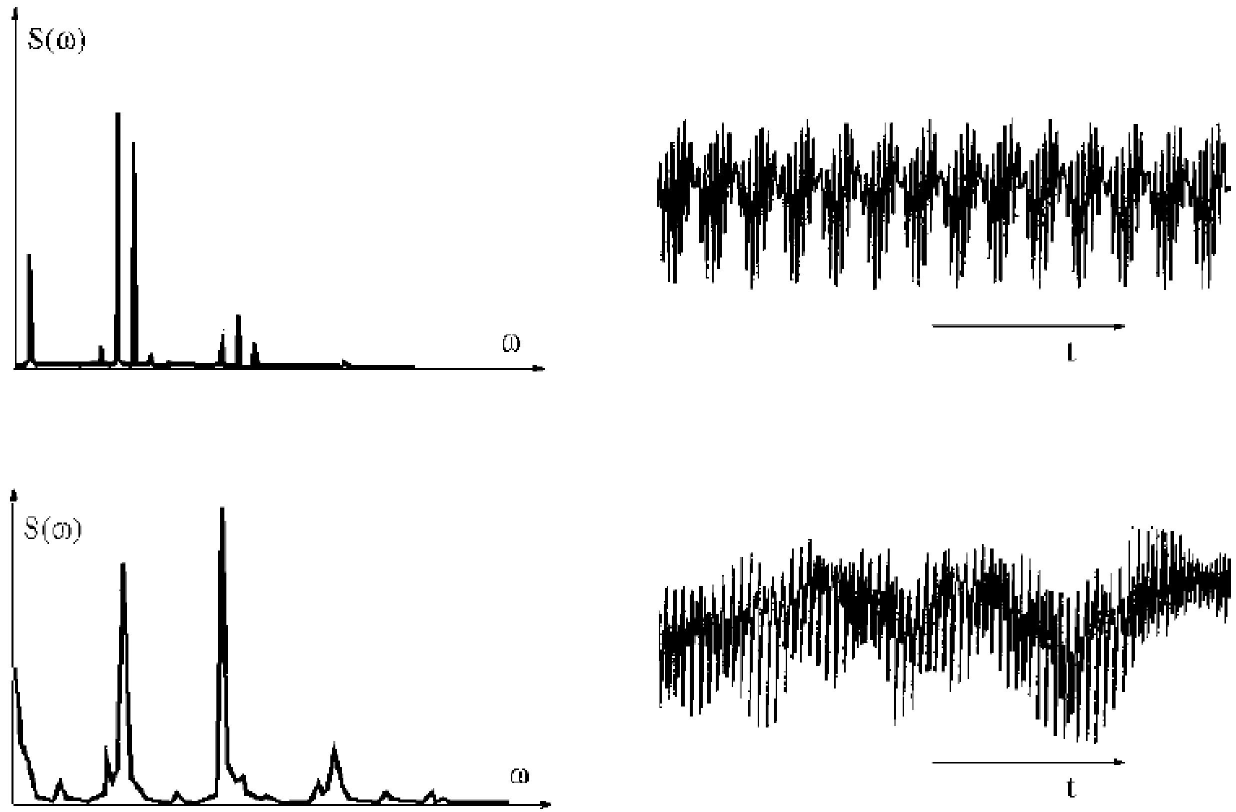
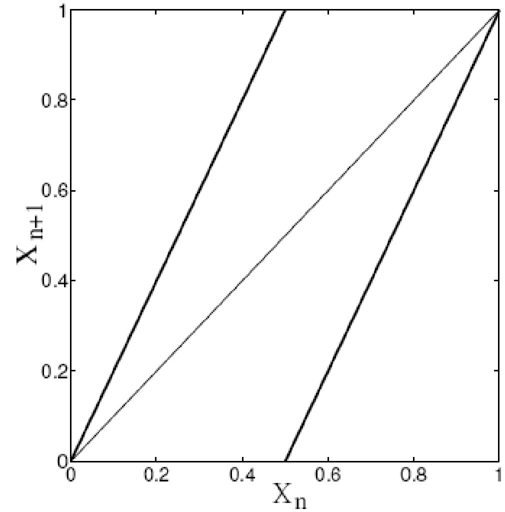


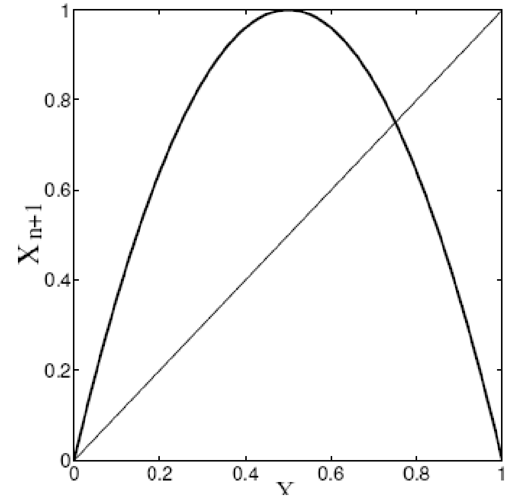
Figure 6.14: Spectres du signal de vitesse mesuré en un point de la cellule. En régime bipériodique (en haut), le spectre est composé de deux raies principales et de leurs combinaisons simples. Le régime chaotique est caractérisé par des raies au pied élargi et par la présence de bruit de basse fréquence ($\omega \approx 0$) dans le spectre (d'après Bergé et Dubois, 1981).

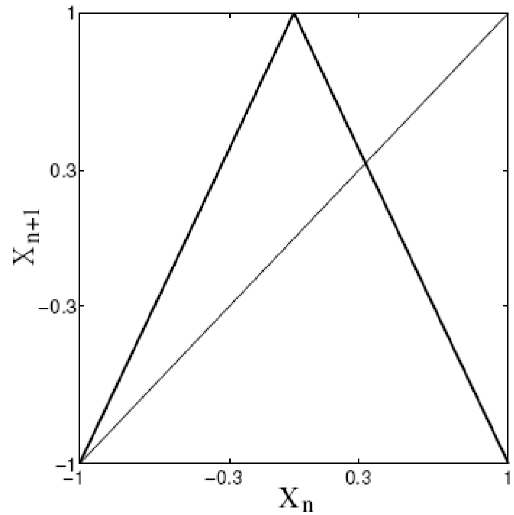
Different maps

dyadic

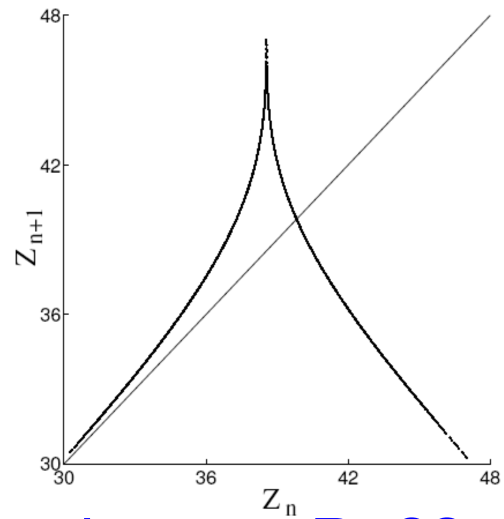


logistic





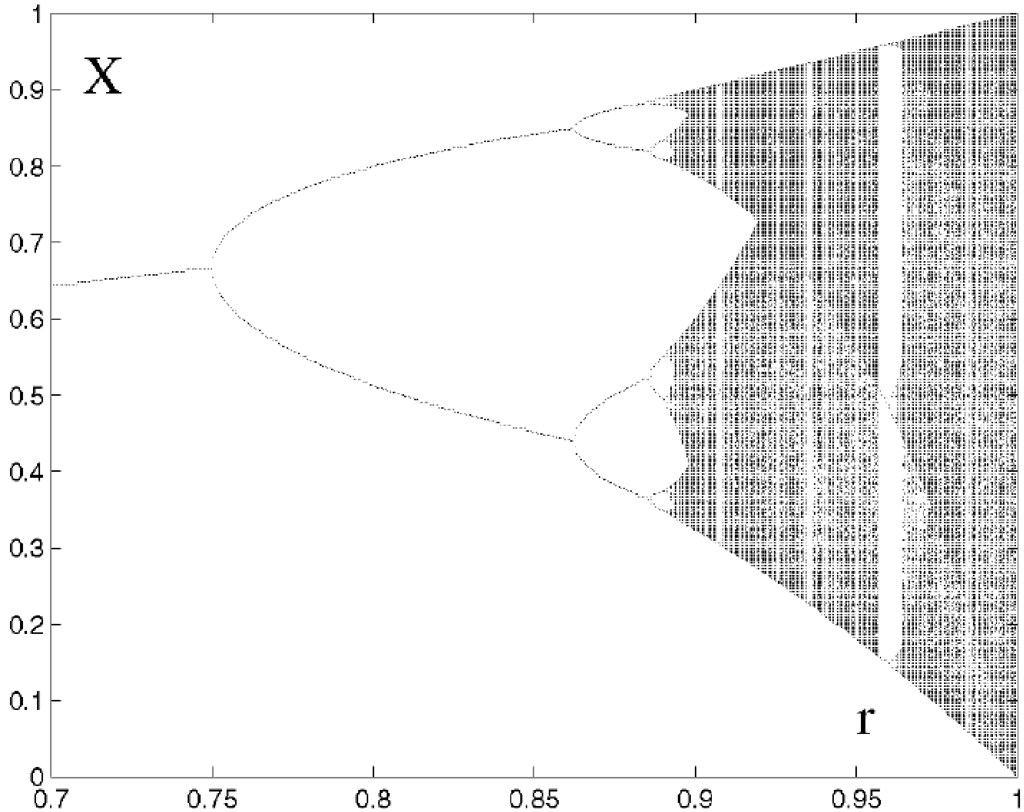
accent circonflexe



Lorentz R=28

Logistic map

$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



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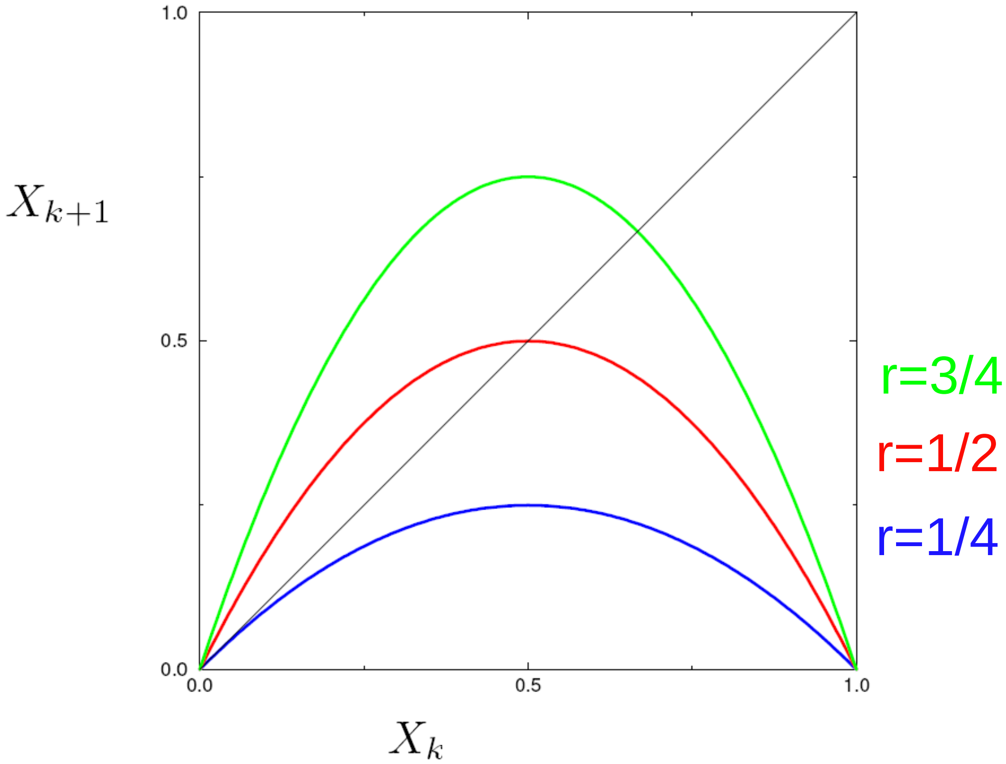
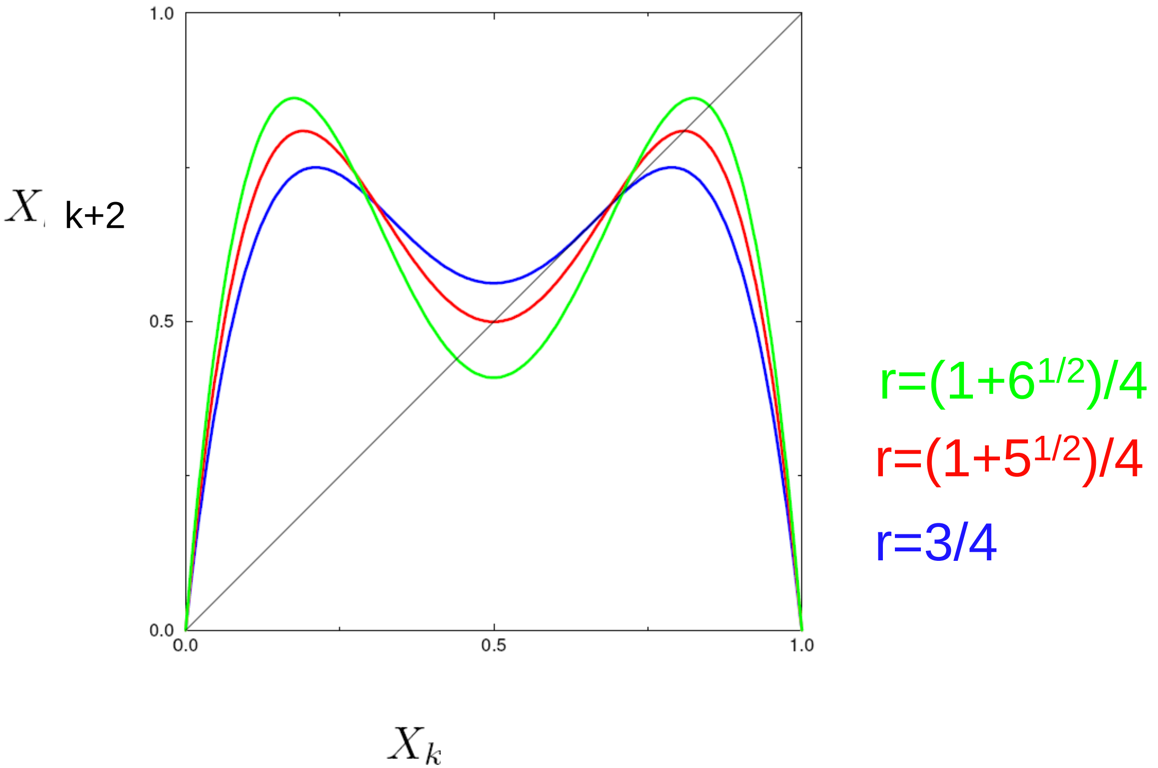


Logistic map

$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



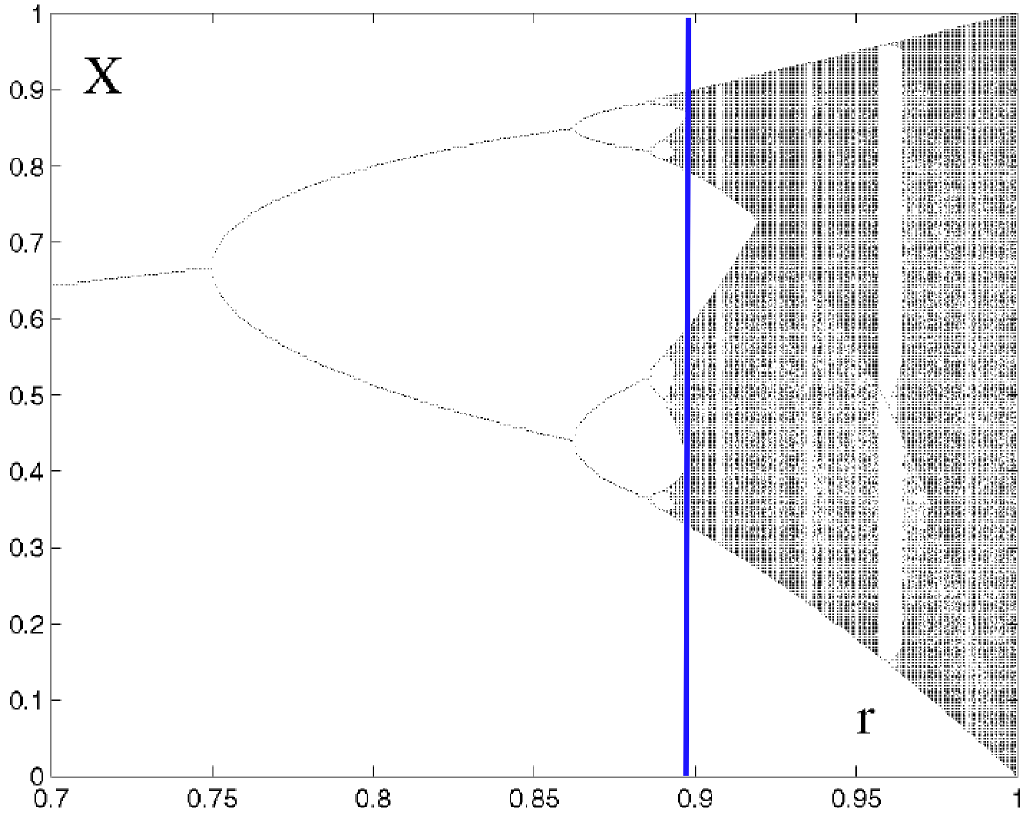
Logistic map

critical bifrucation parameters for period doubling

n	$T_{(n)}$	$r_{(n)}$	$r_{(n)}^{\mathrm{ss}}$
0	1	$1/4$	$1/2$
1	2	$3/4$	0,80901 ...
2	4	0,86237 ...	0,87464 ...
3	8	0,88602 ...	0,88866 ...
4	16	0,89218 ...	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	0,89248 ...	0,89248 ...

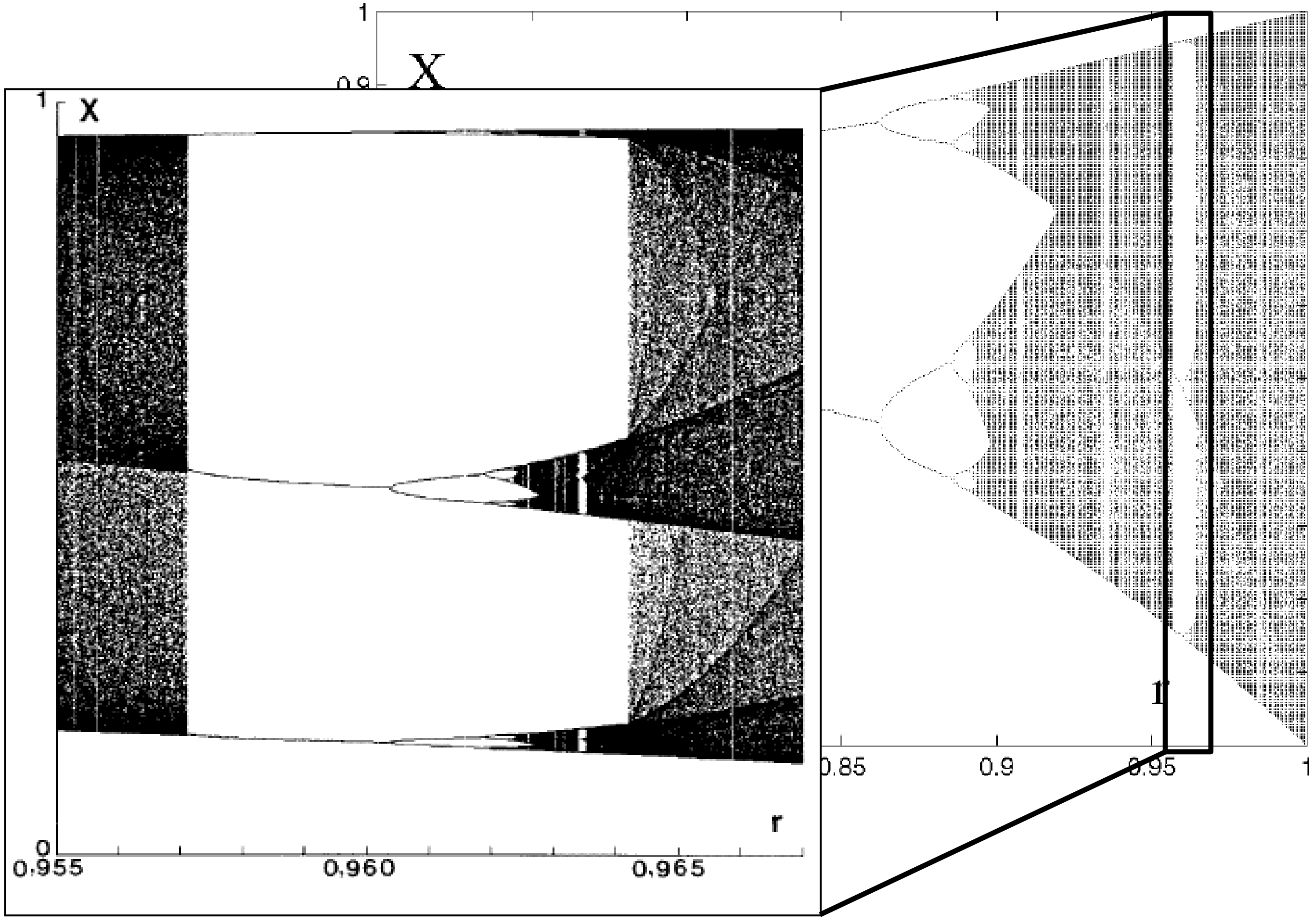
Periodic window

$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$

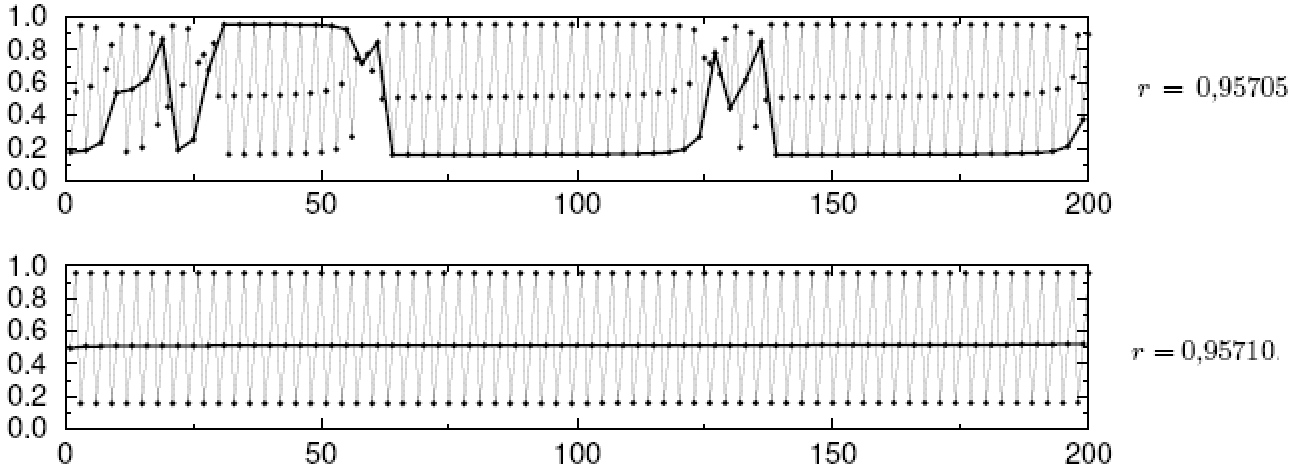


Logistic map
Periodic window

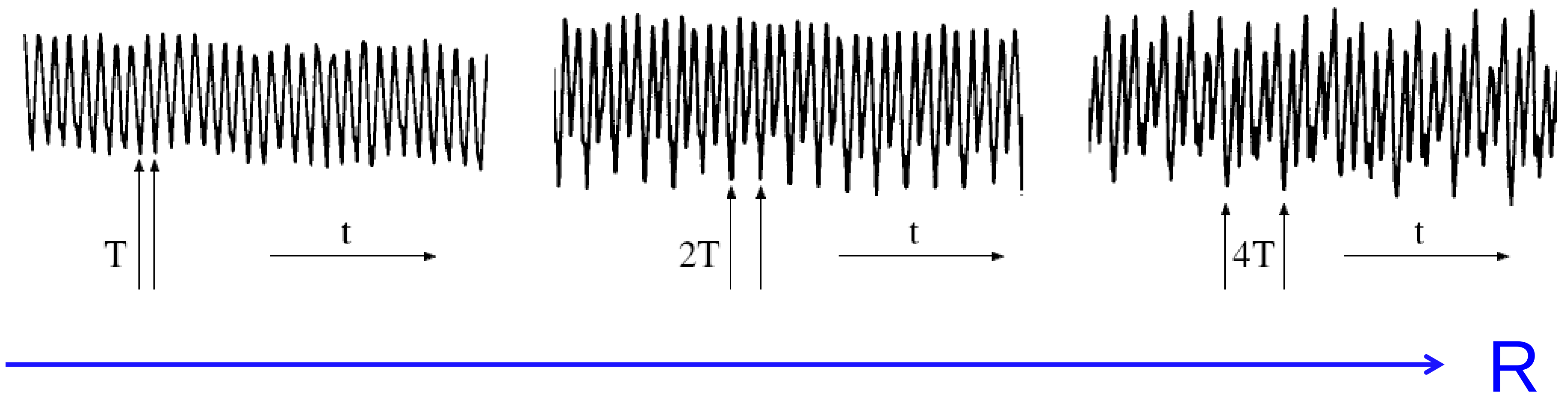
$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



Very very sensitive system



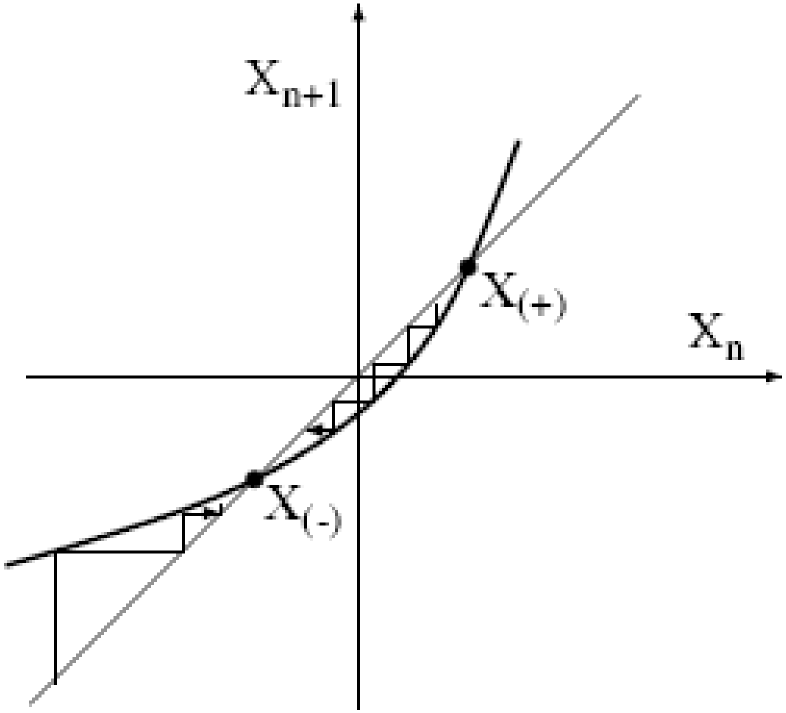
Subharmonic cascade in convection



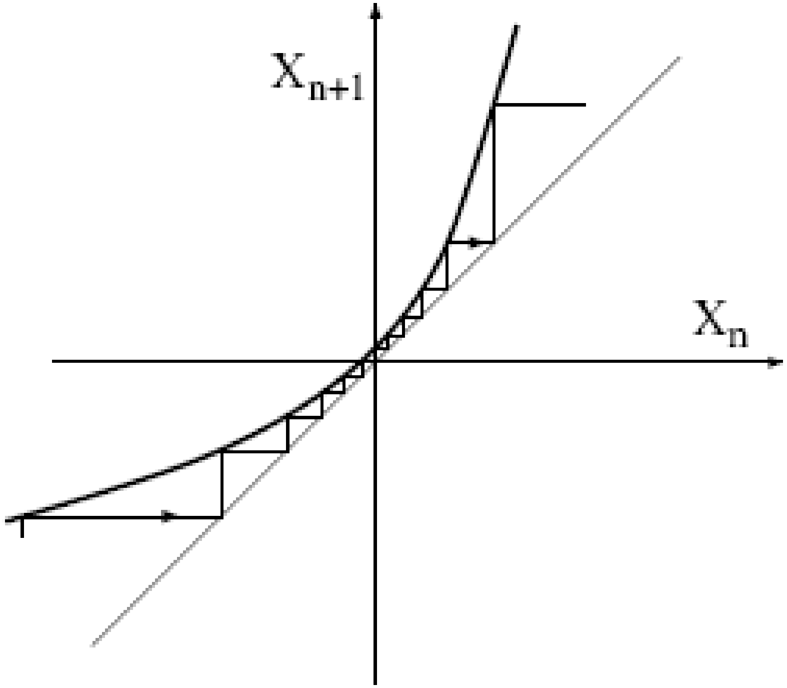
Libchaber and Maurer ‘80

Liquid Helium

Intermittency

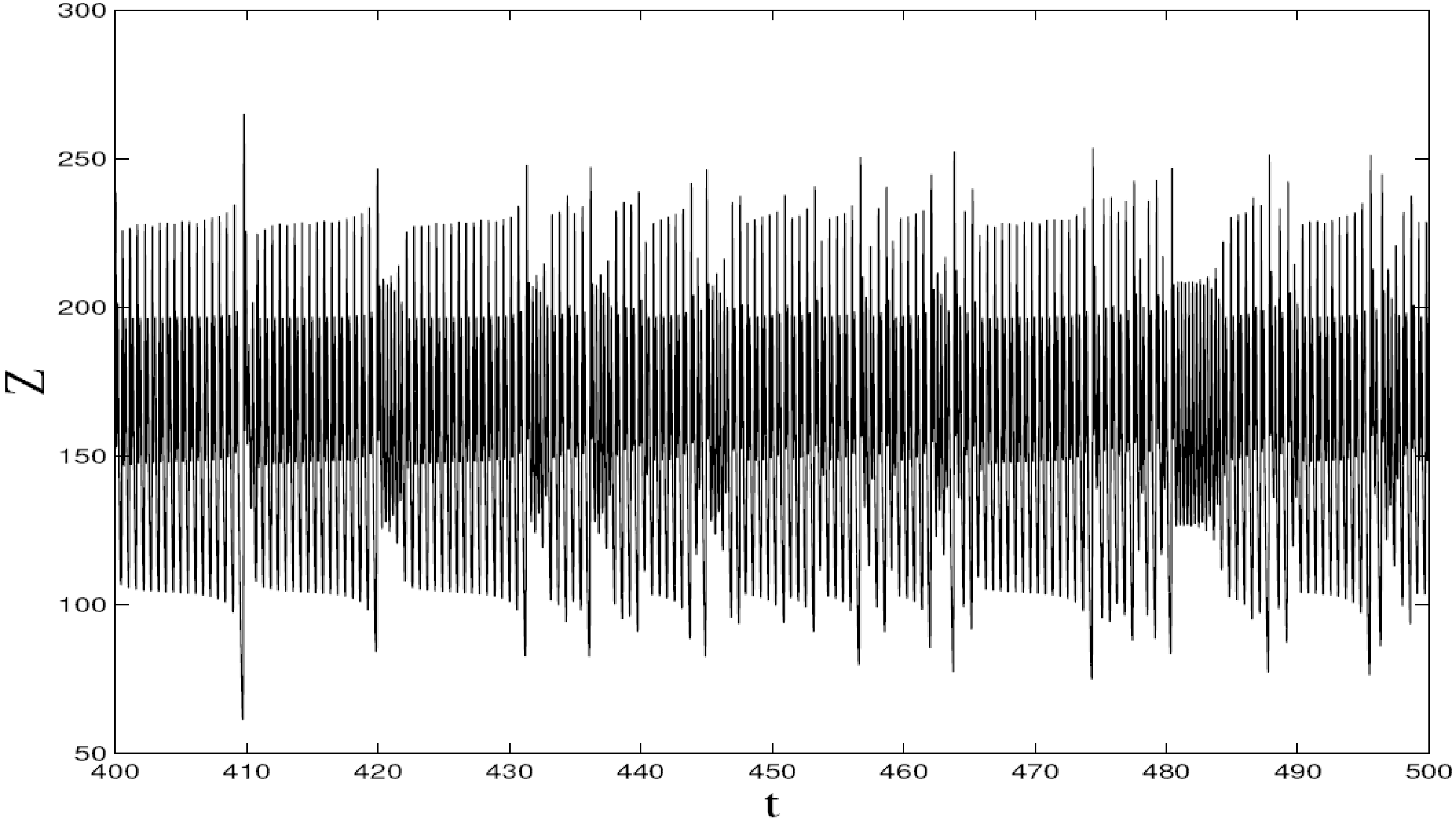


Stable



No fixed point-
Long attracting periods
Intermittent regime

Intermittency



Lorentz at R=166