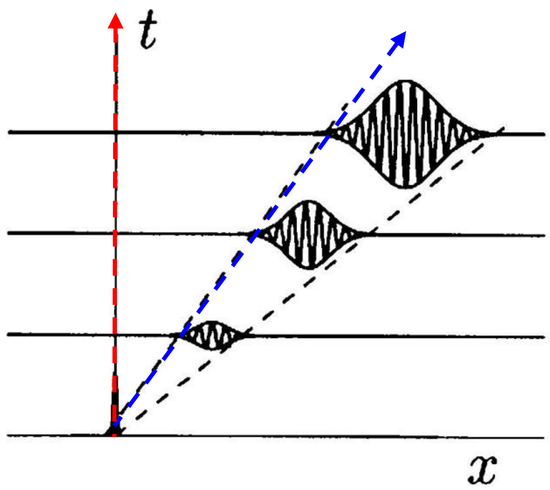
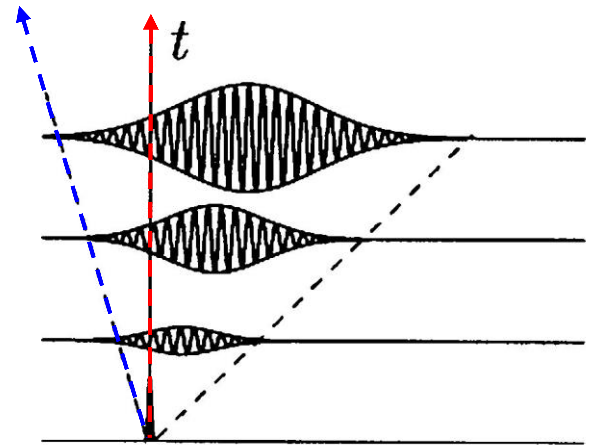
Most flows are unstable...

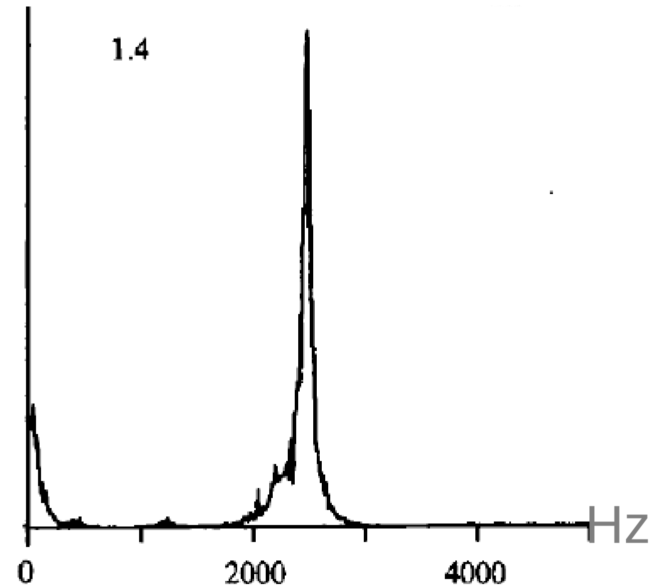
1. Intro-definitions
2. Rayleigh-Taylor
3. Rayleigh Plateau (destabilization through surface tension)
4. Rayleigh-Benard (convection)
5. Benard-Marangoni
6. Taylor Couette-Centrifugal instability
7. Kelvin-Helmholtz
8. Inflection point theorem Rayleigh
9. Orr sommerfeld, transient growth
10. Spatial growth
11. Spatio-temporal growth
12. Global stability analysis and resolvent

☞Convective instability

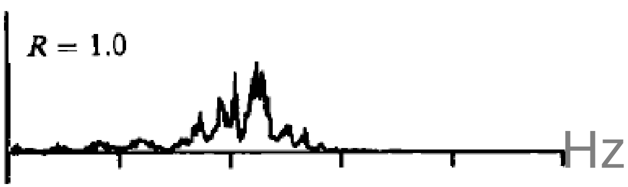
☞Absolute instability







Spectrum

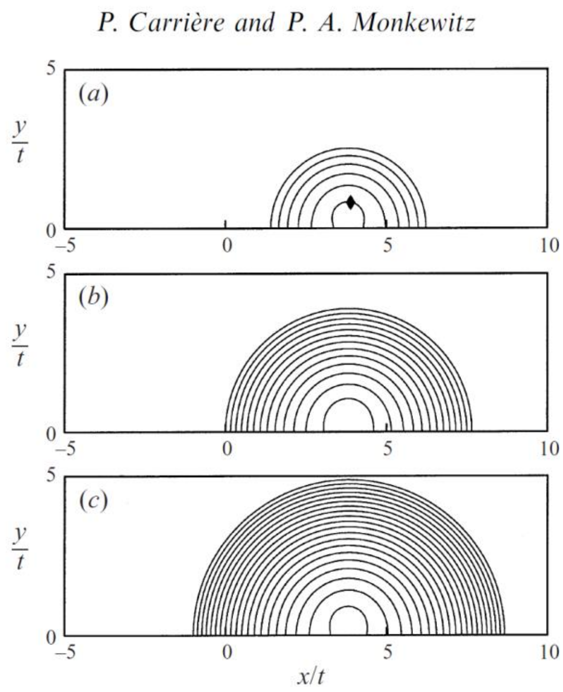


☞amplifier

☞oscillator

Mixing layer experiments by Strikowsky and Niccum (1991)

A/C analysis in Rayleigh-Benard Poiseuille
Carrière and Monkewitz (1999)



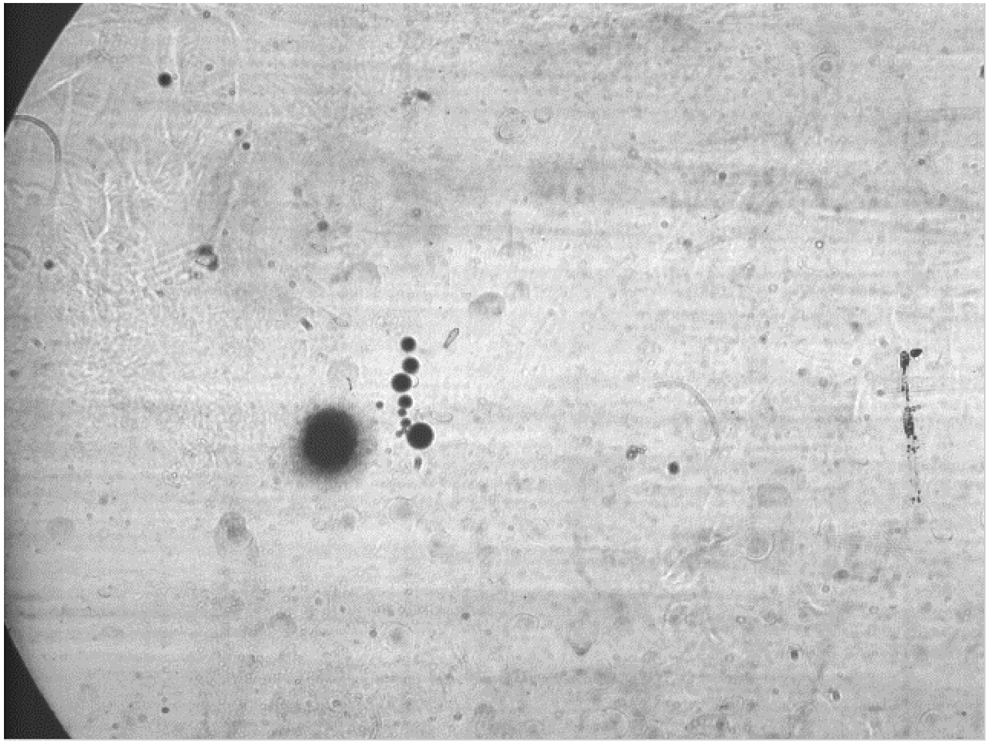
Ra=1760

Ra=1828

Ra=1890

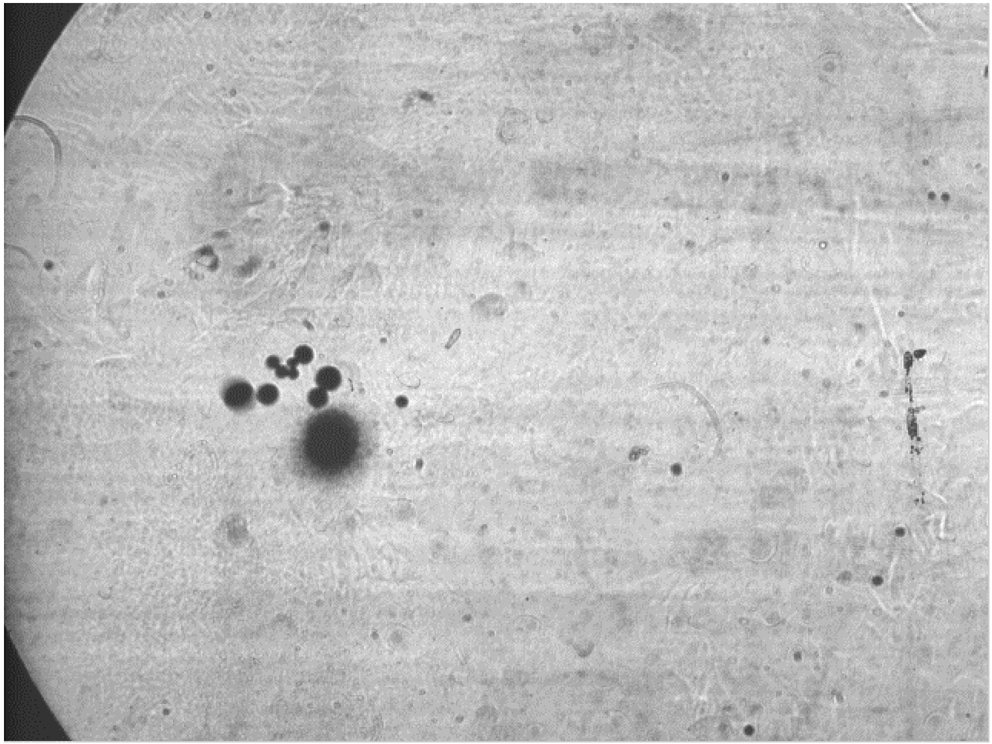
Re=0.63

Impulse in Rayleigh-Benard Poiseuille
Grandjean and Monkewitz (2009)



Ra>1707, Re=0.03

Impulse in Rayleigh-Benard Poiseuille
Grandjean and Monkewitz (2009)



Ra>1707, Re=0.003

Rayleigh-Plateau

$$\boxed{\omega = Uk \pm \sqrt{\frac{\gamma k^2}{\rho}\left(k^2 - \frac{1}{R_0^2}\right)\frac{I_0'(kR)}{I_0(kR)}}}$$

- **Unstable if there exists one ω, Im(ω)>0** **at $k<1/R_0$**
- **Neutral if for all ω, Im(ω)=0** **at $k>1/R_0$**
- **Stable (or damped) if for all ω, Im(ω)<0:**

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

Surface tension is destabilizing as a consequence of the radial curvature

Surface tension is stabilizing as a consequence of the axial curvature

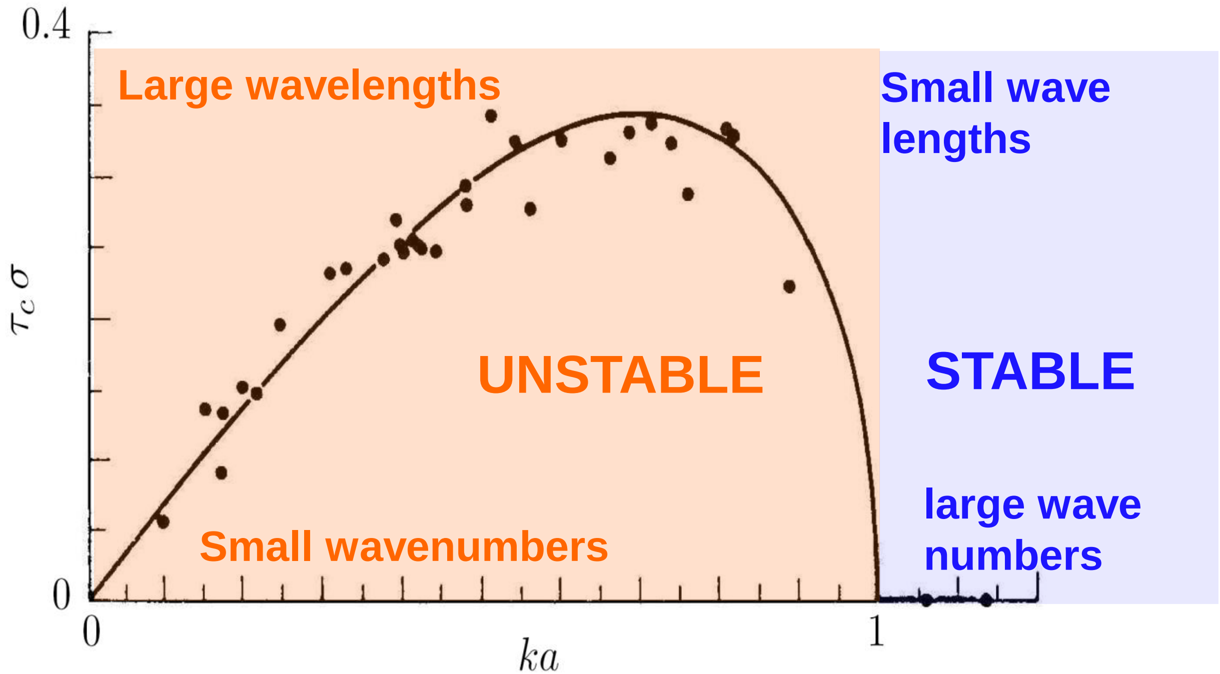


FIG. 2.10 – *Taux de croissance* $\tau_c\sigma$, *avec* $\tau_c = \sqrt{\rho a^3/\gamma}$, *de l'instabilité d'un filet fluide non visqueux, et points expérimentaux. D'après (Drazin & Reid 2004).*

Oh >> 1 – Viscosity dominated

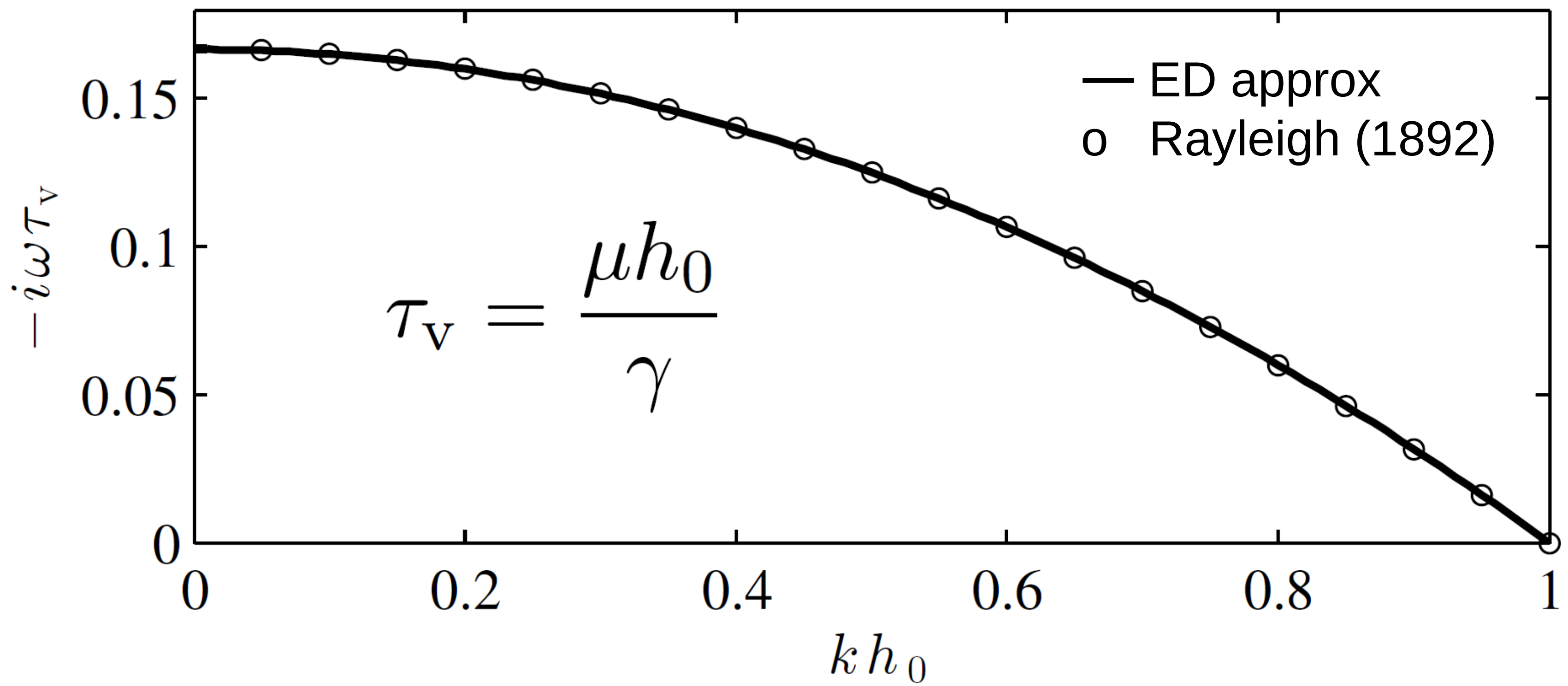
A very similar calculation yields

(Rayleigh)

$$\omega = kU_0 + \mathrm{i}\frac{\gamma}{2\mu R_0}\frac{(1-(kR_0)^2)}{(kR_0)^2(I_0(kR0)^2/I_1(kR_0)^2)-(1+(kR_0)^2)}$$

$$Oh = \frac{\mu}{\sqrt{\rho\gamma R_0}}$$

Eggers and Dupont (1994) equations Oh>>



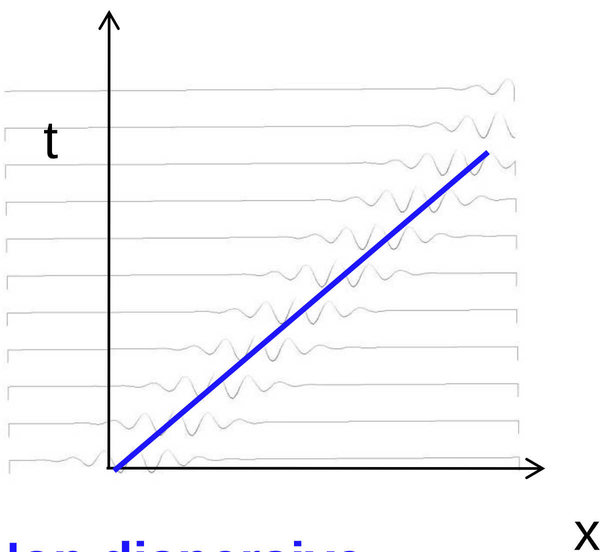
$$\boxed{\omega = u_0 k + \frac{1}{6\tau_v} i \left(1 - (kh_0)^2\right)}$$

Nondimensionalize

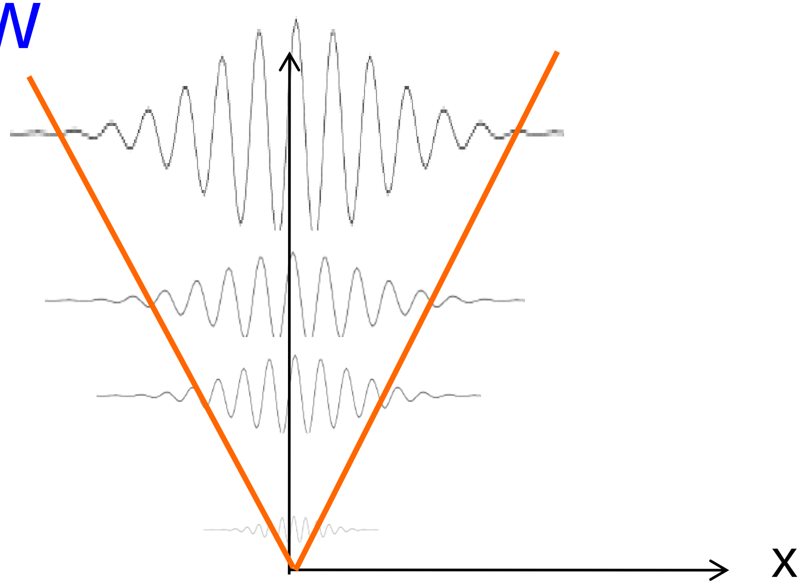
$$\omega = \mathrm{Ca}\, k + \frac{i}{6}(1 - k^2)$$

$\mathrm{Ca} = \mu u_0/\gamma$ is the capillary number (a reduced velocity)

The instability waves simultaneously travel and grow



Non dispersive propagation at the velocity of the interface



Propagating symmetric growing wavepacket

Find k_0

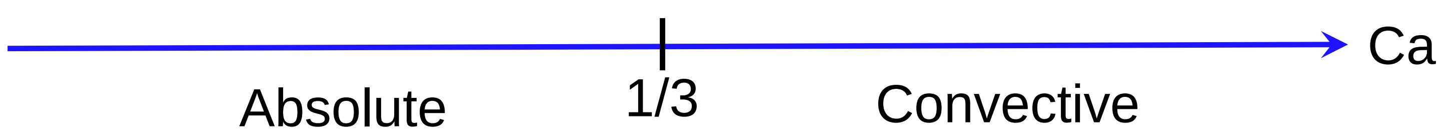
$$\omega = \mathrm{Ca}\, k + \frac{i}{6}(1 - k^2)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0) = 0 \quad \Rightarrow \quad k_0 = -3i\mathrm{Ca}$$

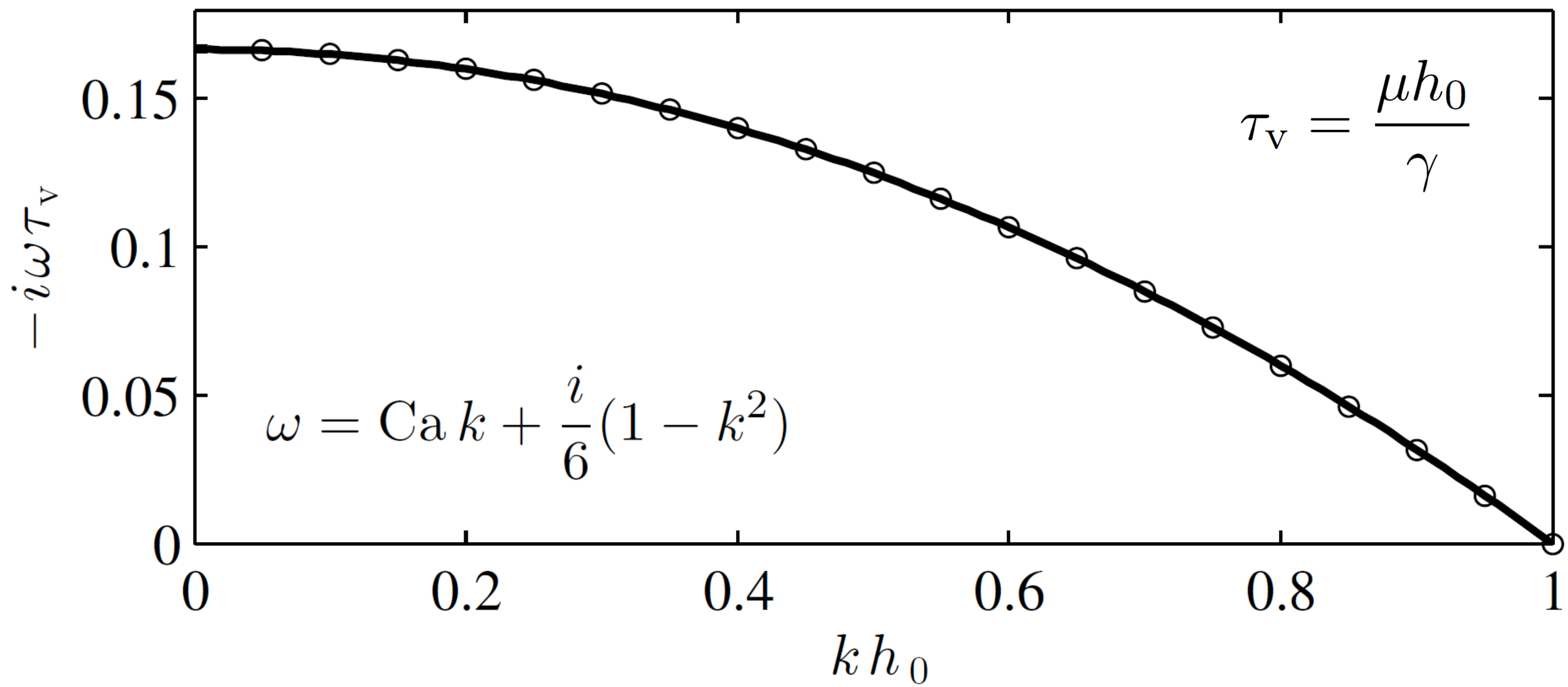
Evaluate ω_0

$$\omega_0 = \omega(k_0) = -\frac{3}{2} i\mathrm{Ca}^2 + \frac{i}{6}$$

$\mathrm{Im}(\omega_0) = (1-9\mathrm{Ca}^2)/6$



Consider the governing equation of the viscous jet and impose boundary conditions



$$\frac{\partial \eta}{\partial t} = -\mathrm{Ca}\frac{\partial \eta}{\partial z} + \frac{1}{6}\eta(z,t) + \frac{1}{6}\frac{\partial^2 \eta}{\partial z^2}$$

$$\eta(0,t) = \eta(l,t) = 0$$

Find eigenmodes $\eta(x,t) = \bar{\eta}(z)e^{\lambda t}$

$$\frac{\partial \eta}{\partial t} = -\mathrm{Ca}\frac{\partial \eta}{\partial z} + \frac{1}{6}\eta(z,t) + \frac{1}{6}\frac{\partial^2 \eta}{\partial z^2}$$

$$\eta(0,t) = \eta(l,t) = 0$$

$$\boxed{\begin{gathered}\left(\lambda - \frac{1}{6}\right)\bar{\eta}(z) = -\mathrm{Ca}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}z} + \frac{1}{6}\frac{\mathrm{d}^2\bar{\eta}}{\mathrm{d}z^2} \\ \bar{\eta}(0) = \bar{\eta}(l) = 0\end{gathered}}$$

Fundamental solutions $\bar{\eta}(z) = \hat{\eta}e^{\alpha z}$

$$\alpha^2 - 6\mathrm{Ca}\alpha + 1 - 6\lambda = 0$$

$$\alpha_{1,2} = 3\mathrm{Ca} \pm \frac{1}{2}\sqrt{36\mathrm{Ca}^2 + 24\lambda - 4}$$

Fundamental solutions $\bar{\eta}(z) = \hat{\eta} e^{\alpha z}$

$$\alpha^2 - 6\mathrm{Ca}\alpha + 1 - 6\lambda = 0$$

$$\alpha_{1,2} = 3\mathrm{Ca} \pm \frac{1}{2}\sqrt{36\mathrm{Ca}^2 + 24\lambda - 4}$$

Impose boundary conditions

$$\begin{bmatrix} 1 & 1 \\ e^{\alpha_1 l} & e^{\alpha_2 l} \end{bmatrix} \begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad e^{\alpha_2 l} - e^{\alpha_1 l} = 0$$

$$\Rightarrow \quad \alpha_1 - \alpha_2 = i\frac{2\pi m}{l}$$ (m integer)

$$\boxed{\sqrt{36\mathrm{Ca}^2 + 24\lambda_m - 4} = i\frac{2\pi m}{l}}$$

Find eigenmodes $\eta(x,t)=\bar{\eta}(z)e^{\lambda t}$

$$\boxed{\lambda_m = -\frac{3}{2}\mathrm{Ca}^2 + \frac{1}{6} - \frac{m^2\pi^2}{6l^2} = -i\omega_0 - \frac{m^2\pi^2}{6l^2}}$$

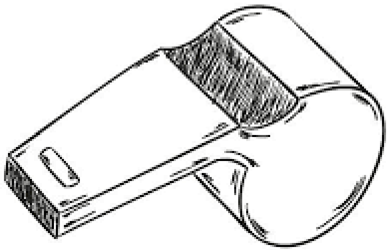
>0 if Im(ω_0)>0

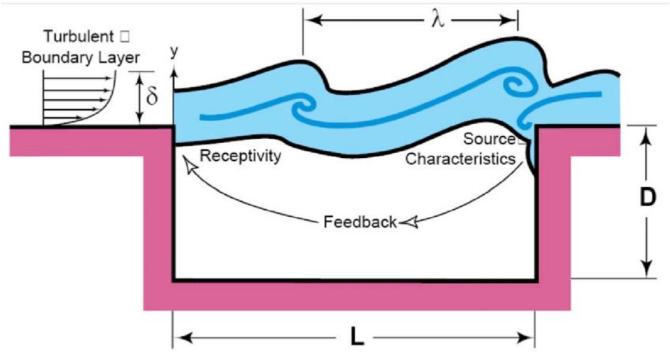
absolute frequency $\omega(k_0)$

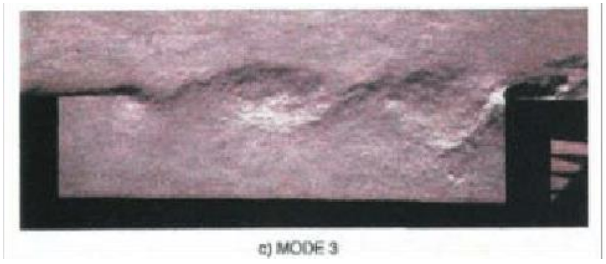
The flow including boundary conditions is **globally** unstable only if it is locally absolutely unstable

Global instability is needed when the streamwise invariance is broken

-by inlet/outlet boundary conditions

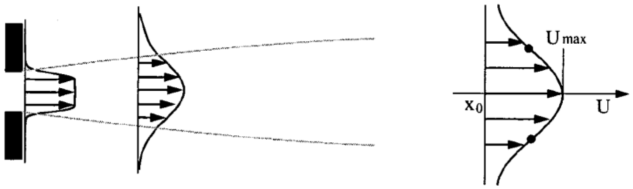






-by the nonparallelism of the flow

2D jet



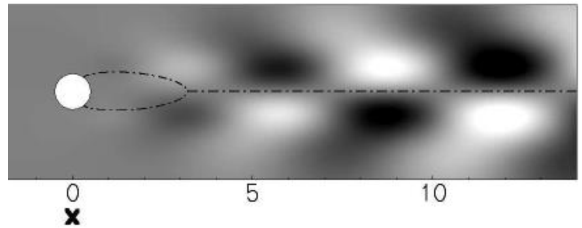
Global instability

The normal mode expansion $u(y)\exp(i(kx-\omega t))$
is replaced by
a global mode expansion $u(x,y)\exp(\sigma t)$

The dispersion relation $\omega(k)$
is replaced by
a set of global eigenvalues σ

Most often, a necessary condition for global inst. $\sigma_r>0$
is not temporal instability $\exists k\ \omega_i(k)>0$
But rather that of absolute instability $\omega_{0i}(k_0)>0$

Global linear instability of flows



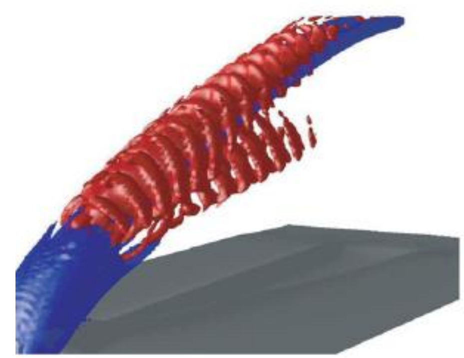
Stability of viscous flow past a circular cylinder
A. Zebib

1D **1987…..** **2005** **3 D**

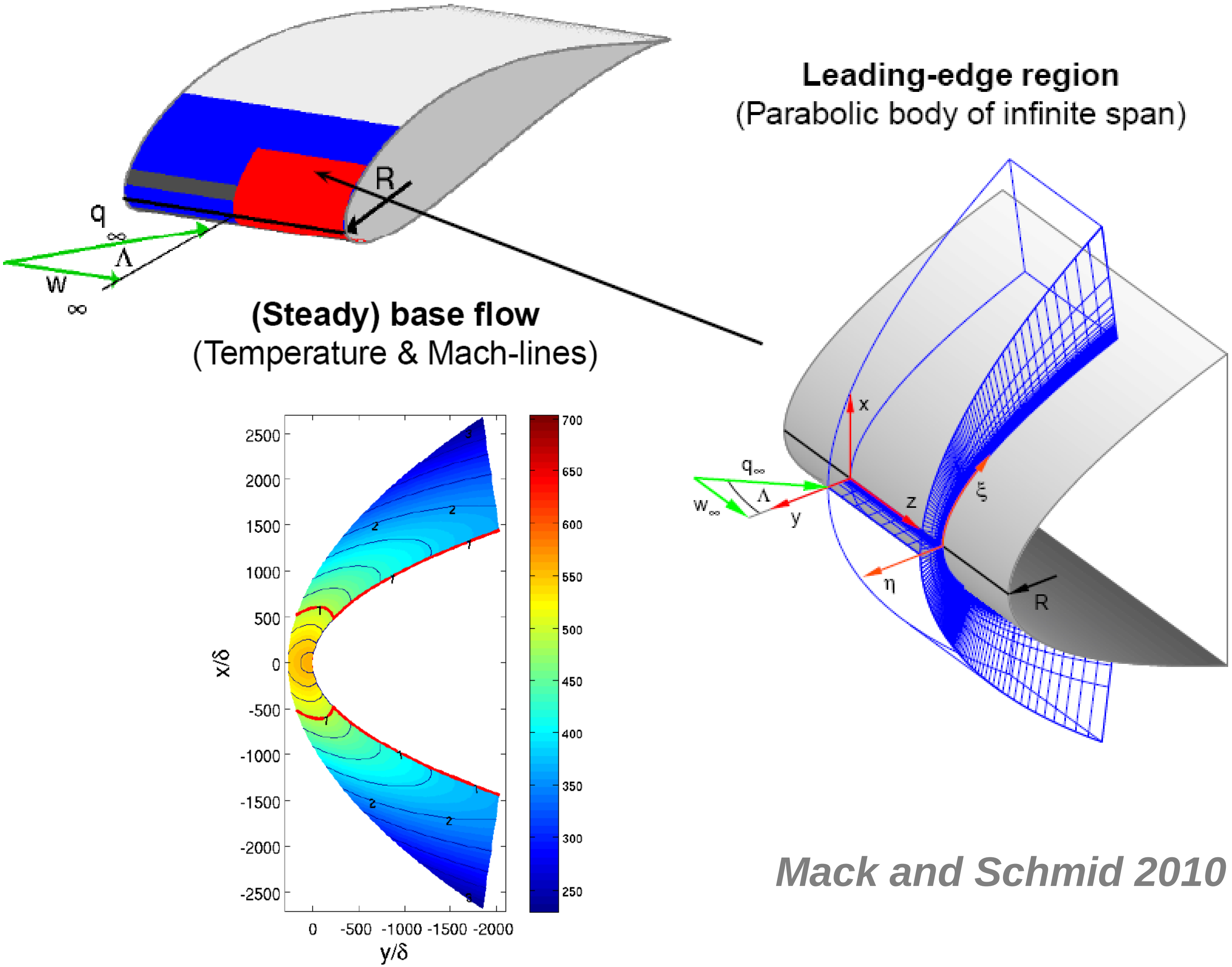
1971
Accurate solution of the Orr–Sommerfeld stability equation
S. A. Orszag

2 D

2009
Global stability of a jet in crossflow
S. Bagheri

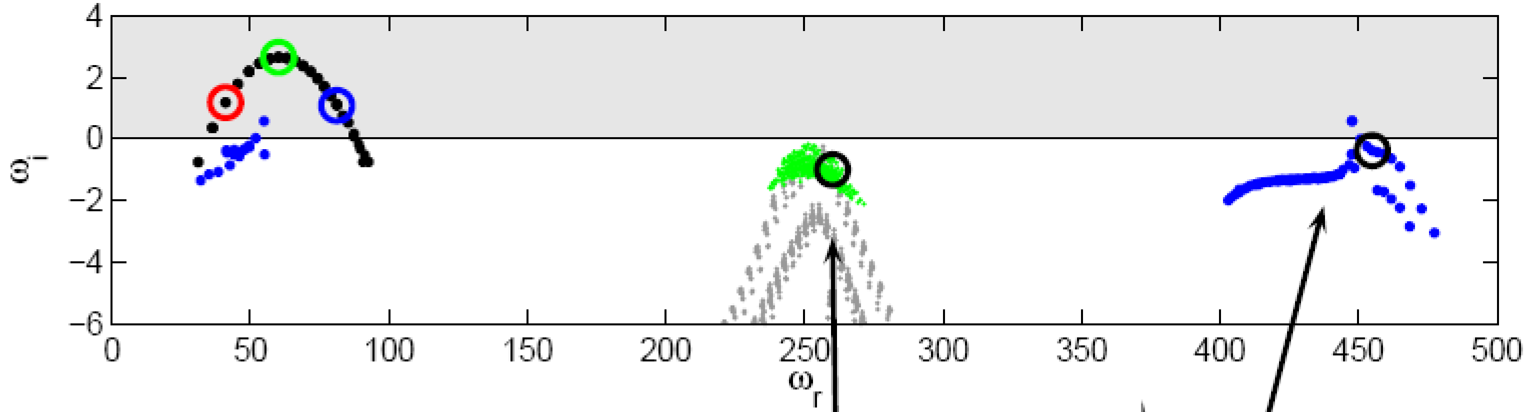


Global linear instability of flows in complex geometry

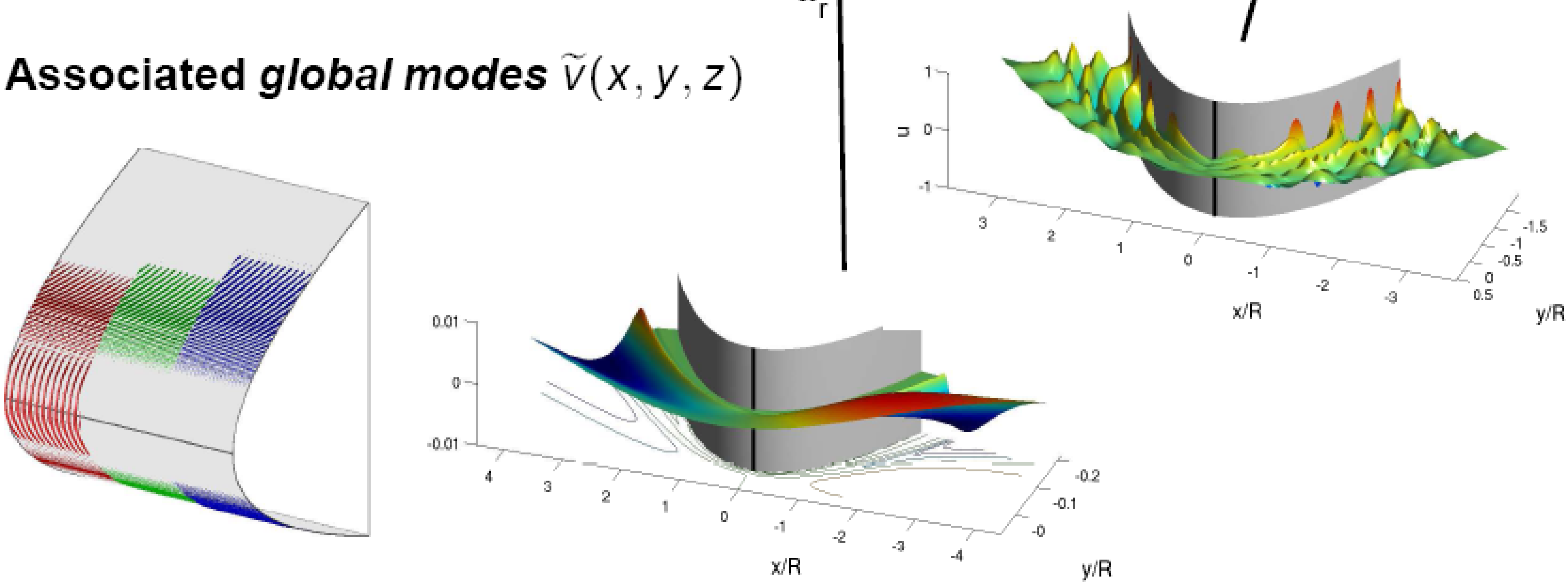


Mack and Schmid 2010

***Global* spectrum:** $\phi'(x,y,z,t) = \tilde{\phi}(x,y)e^{i(\beta z - \omega t)}$ with $\omega = \omega_r + i\omega_i$

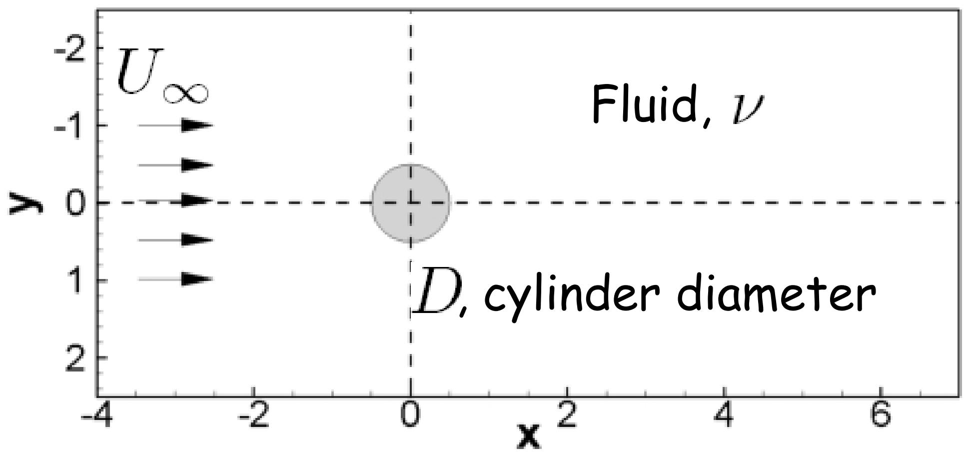


Associated *global modes* $\tilde{v}(x,y,z)$



Mack and Schmid 2010

Wake behind cylinder flow



$$Re = \frac{U_\infty D}{\nu}$$

Incompressible 2D flow described by

$$(\boldsymbol{u}, p)$$

Velocity $\qquad$ Pressure

Navier-Stokes equations

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u} \cdot \boldsymbol{u} = -\boldsymbol{\nabla} p + Re^{-1} \boldsymbol{\nabla}^2 \boldsymbol{u},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

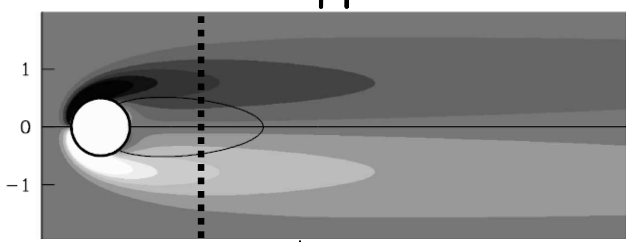
Stability analysis

Perturbation expansion

$$(\boldsymbol{u}, p) = (\boldsymbol{U}, P) + (\boldsymbol{u}^{'}, p^{'})$$ Perturbations

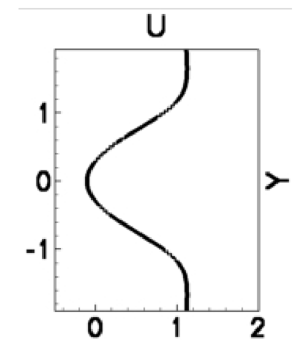
Stationary base flow

Global approach $\longleftarrow$ $\boldsymbol{U}(x, y)$ $\qquad$ $U(y)$ $\longrightarrow$



Local approach

Huerre & Monkewitz (1990)



Zebib, Jackson (1987)

Base flow equations

$$\nabla \boldsymbol{U} \cdot \boldsymbol{U} = -\nabla P + Re^{-1} \nabla^2 \boldsymbol{U},$$
$$\nabla \cdot \boldsymbol{U} = 0$$

Linearized **perturbation equations**

$$\partial_t \boldsymbol{u}^{'} + \nabla \boldsymbol{U} \cdot \boldsymbol{u}^{'} + \nabla \boldsymbol{u}^{'} \cdot \boldsymbol{U} + \cancel{\nabla \boldsymbol{u}^{'} \cdot \boldsymbol{u}^{'}} = -\nabla p^{'} + Re^{-1} \nabla^2 \boldsymbol{u}^{'},$$
$$\nabla \cdot \boldsymbol{u}^{'} = 0$$

Analyse de stabilité globale

$$\boldsymbol{U}(x,y)$$

$$(\boldsymbol{u}^{'},p^{'})(x,y,t) = (\hat{\boldsymbol{u}},\hat{p})(x,y)\exp[\sigma t]$$

Global mode(complex)

$$\sigma = \lambda + i\omega$$

Growth-rate

frequency

$$St = \frac{\omega}{2\pi}$$

Global stability equations

$$\sigma\hat{\boldsymbol{u}} + \boldsymbol{\nabla}\hat{\boldsymbol{u}}\cdot\boldsymbol{U} + \boldsymbol{\nabla}\boldsymbol{U}\cdot\hat{\boldsymbol{u}} = -\boldsymbol{\nabla}\hat{p} + Re^{-1}\boldsymbol{\nabla}^2\hat{\boldsymbol{u}},$$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}} = 0,$$

Global stability analysis solvers

For a given value of Re, numerically solve

- non linear equations,

$$\boldsymbol{\nabla U} \cdot \boldsymbol{U} = -\boldsymbol{\nabla} P + Re^{-1} \boldsymbol{\nabla}^2 \boldsymbol{U},$$

(Newton method)

- Eigenvalue problem

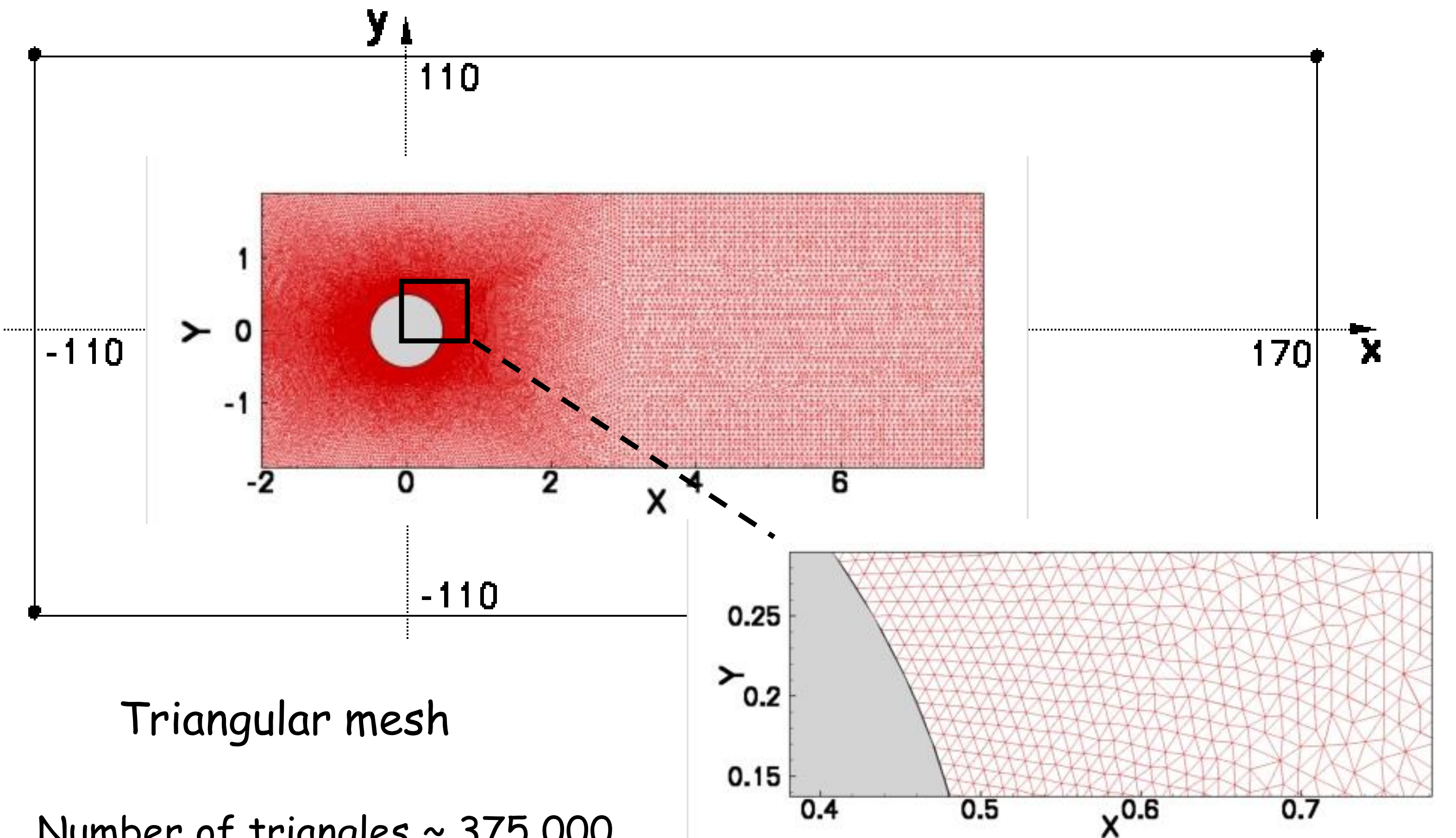
$$\sigma \hat{\boldsymbol{u}} + \boldsymbol{\nabla} \hat{\boldsymbol{u}} \cdot \boldsymbol{U} + \boldsymbol{\nabla U} \cdot \hat{\boldsymbol{u}} = -\boldsymbol{\nabla} \hat{p} + Re^{-1} \boldsymbol{\nabla}^2 \hat{\boldsymbol{u}},$$

(Krylov-Arnoldi method)

Spatial discretization = finite element methods

(FreeFem++ freeware)

Computational model and mesh



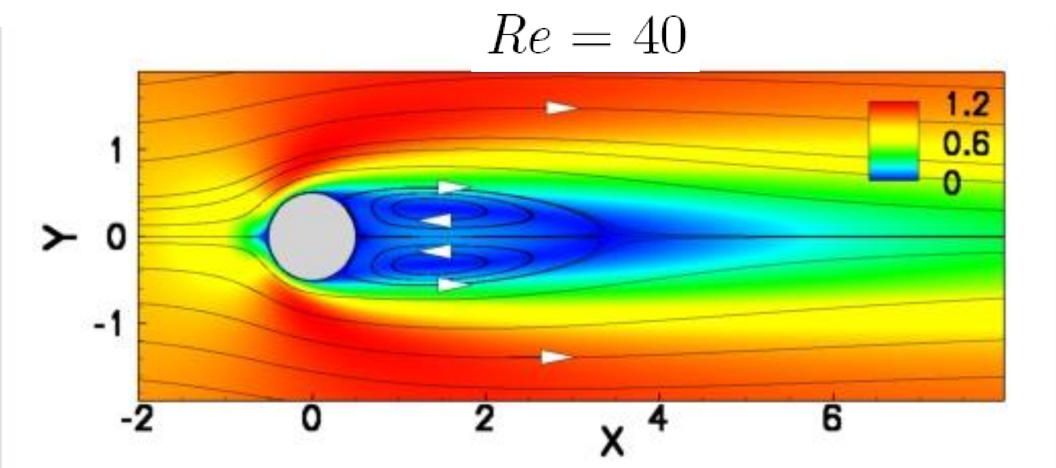
Triangular mesh

Number of triangles ~ 375 000

Number of nodes ~ 190 000

Taylor-Hood finite elements (P2,P2,P1) $\rightarrow$ number of degrees of freedom~ 1.6 10^6

Base flow



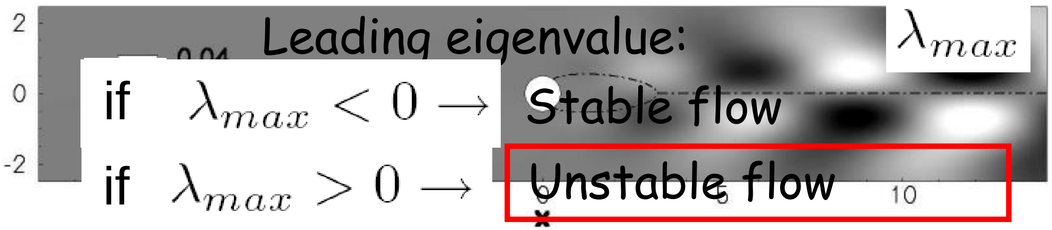
Drag coefficient $C_D = \dfrac{F_D}{0.5 D\, U_\infty^2}$

C_D	$Re = 20$	$Re = 40$
Dennis & Chang (1970)	2.05	1.52
Fornberg (1980)	2.00	1.50
Ye *et al.* (1999)	2.03	1.52
Giannetti & Luchini (2007)	2.05	1.54
Marquet et al. (2008)	2.00	1.50

Valeur propre et mode global dominants

$\Re\{\hat{u}\}$ $\quad (\hat{\boldsymbol{u}}, \hat{p}) \exp[\sigma t] \quad \sigma = \lambda + i\omega$

Spectrum at Re = 50

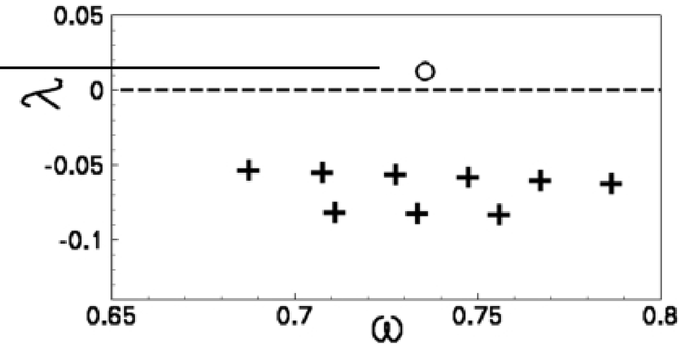


Leading eigenvalue: λ_{max}

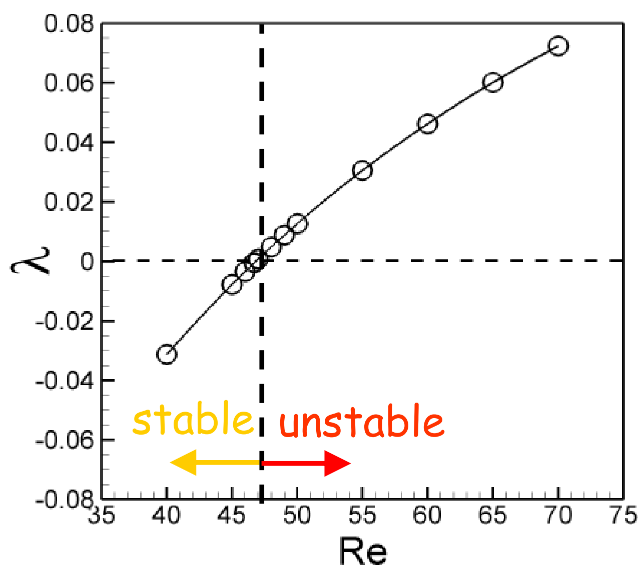
if $\lambda_{max} < 0 \rightarrow$ Stable flow

if $\lambda_{max} > 0 \rightarrow$ Unstable flow

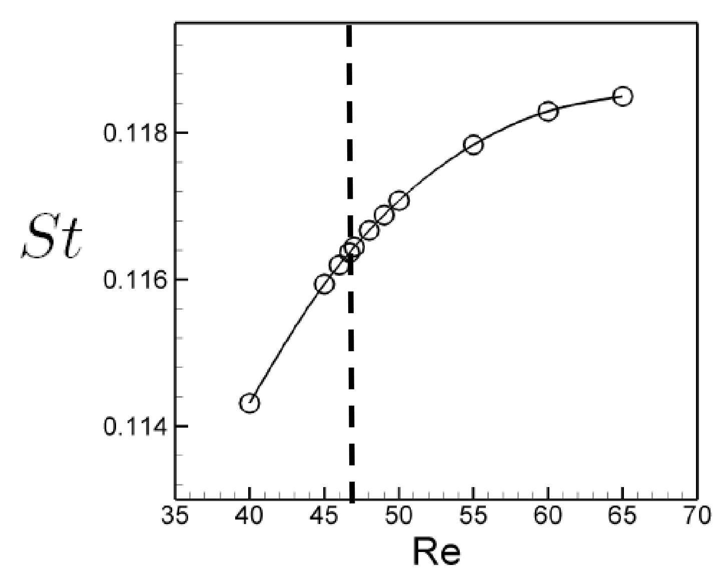
Leading global mode



Evolution as a function of the Reynolds number



$Re_c = 46.7(\pm 0.1)$



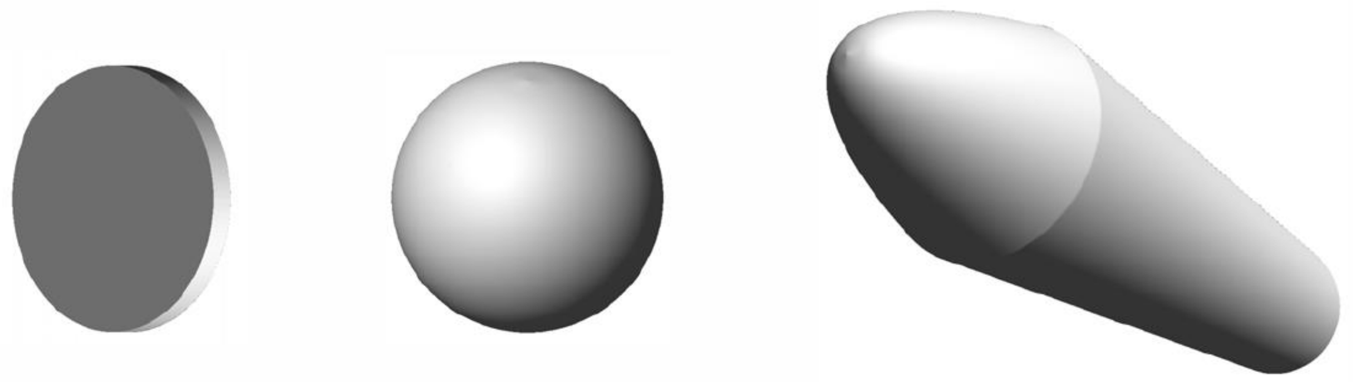
$St_c = 0.116$

To keep im mind

Global stability analysis=
Nonlinear problem (base flow) +
Eigenvalue problem

Prototype flows

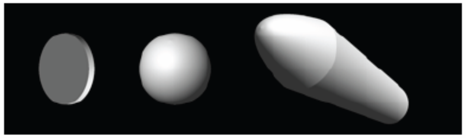
Axisymmetric wakes



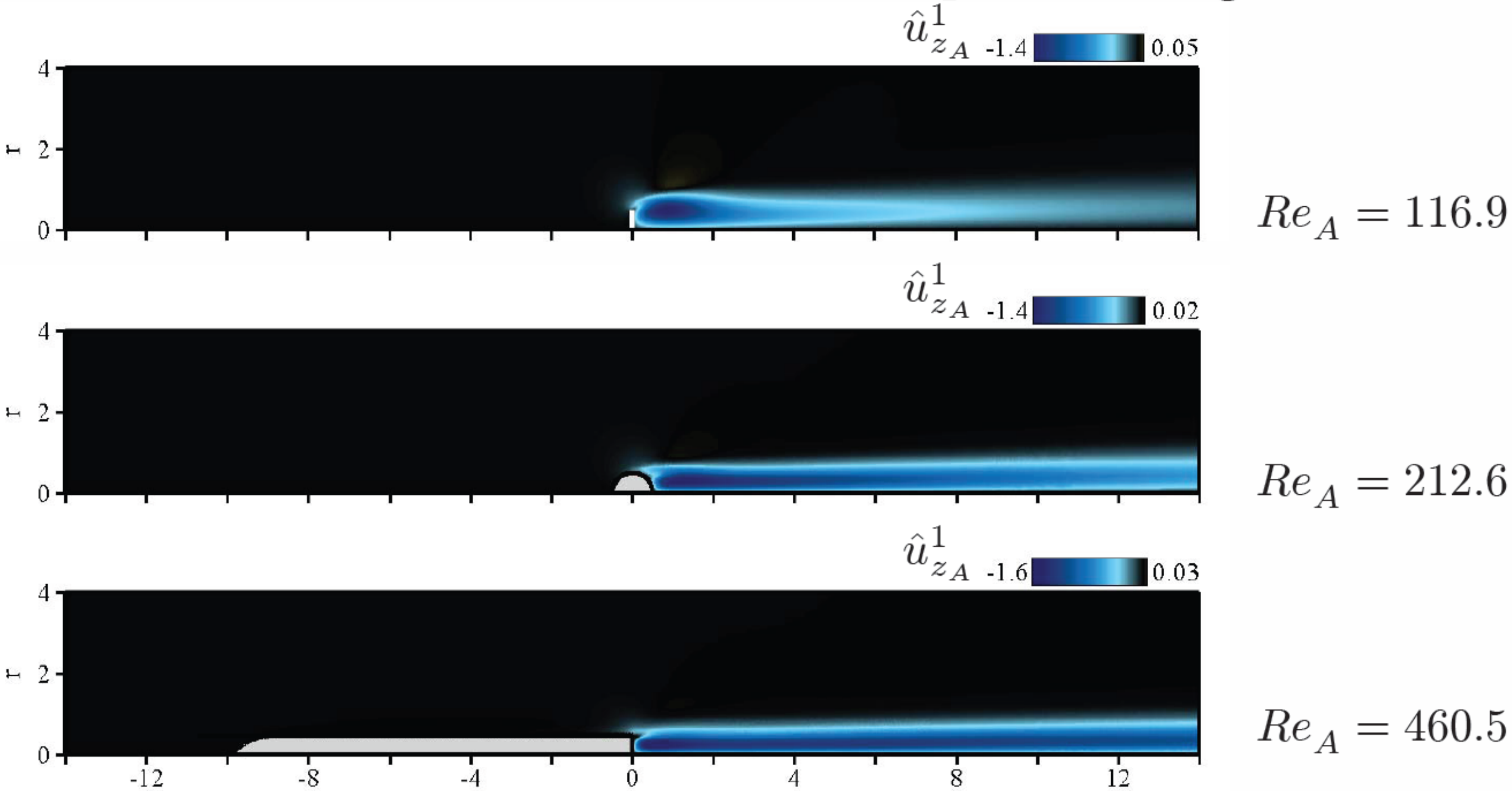
Disk Sphere Blunt base

Meliga et al. 2008,2009,2010

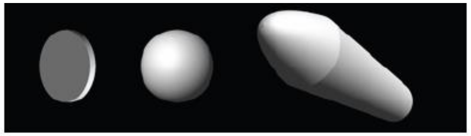
Leading global modes



Same bifurcation sequence for all prototype flows, in the INCOMPRESSIBLE & compressible regimes



Leading global modes



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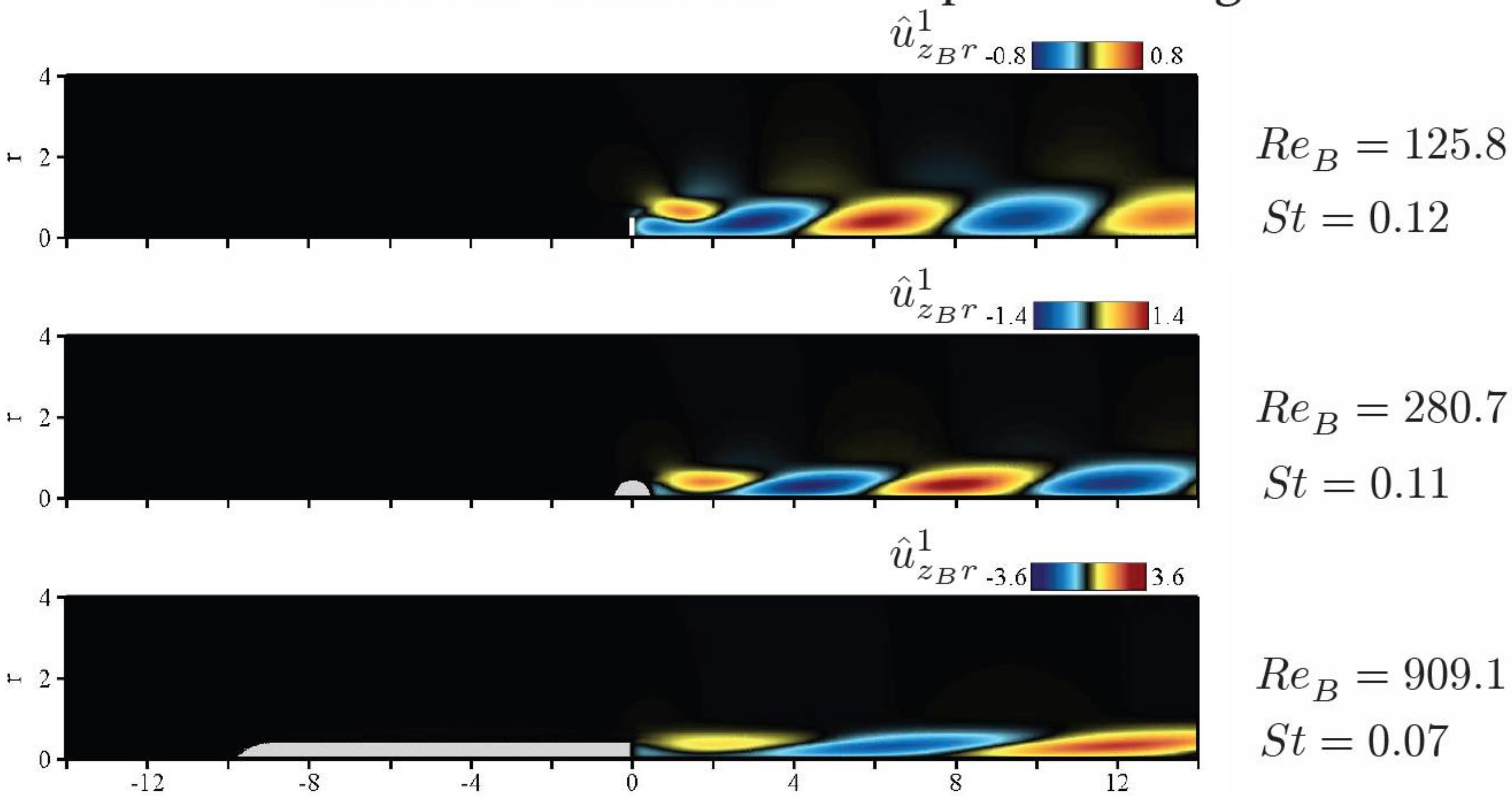
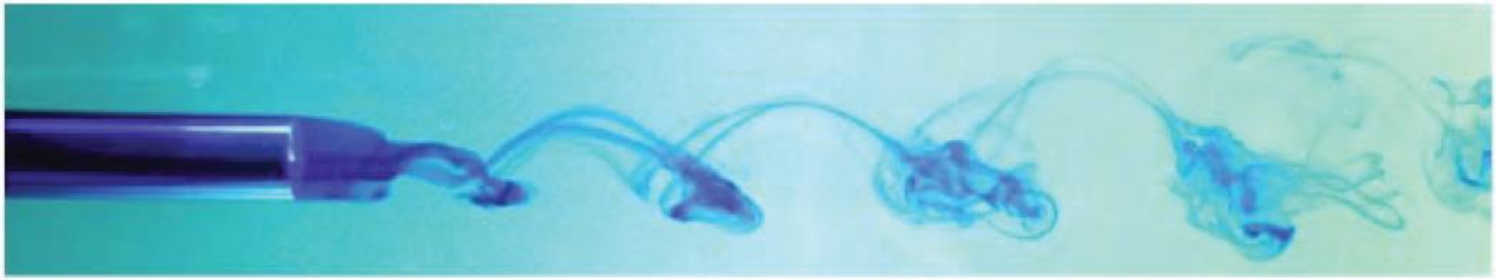


How do these theoretical considerations compare with experiments

Sphere – Johnson & Patel (1999)

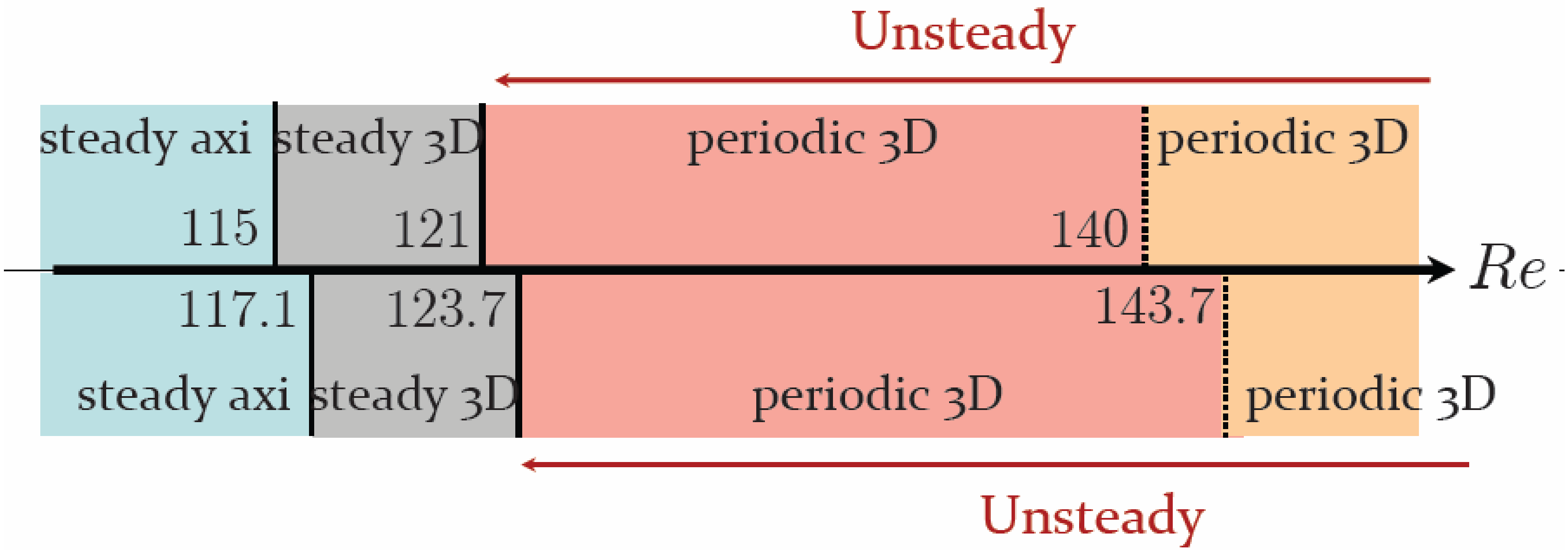


Blunt base – Siegel et al. (2008)

Remarkable predictions of successive symmetry breakings and thresholds

Direct numerical simulation of the 3D flow

Fabre et al. (2008)

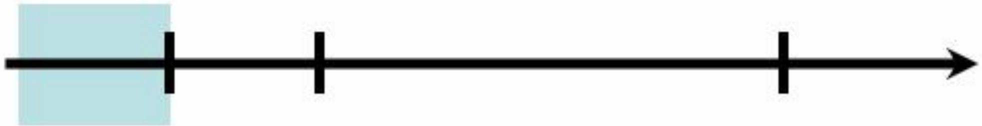


Weakly nonlinear global stability

Meliga et al. 2009

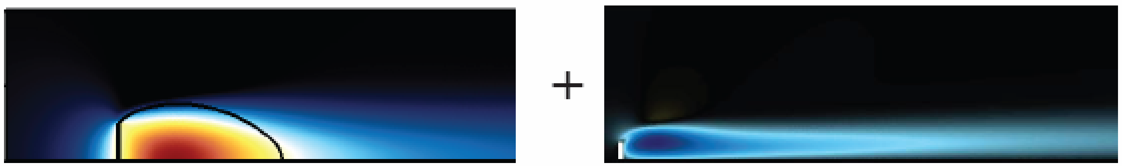
Stable solutions

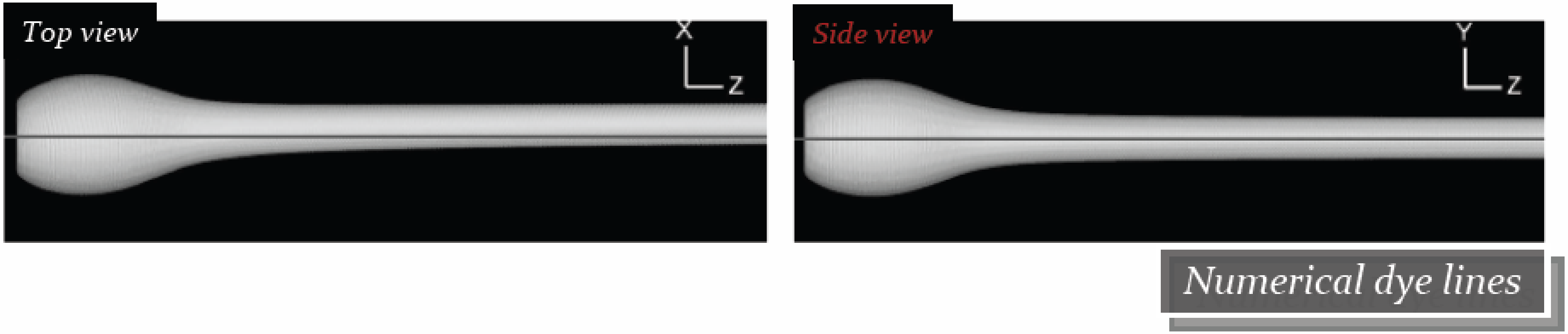
- Steady axi



Stable solutions

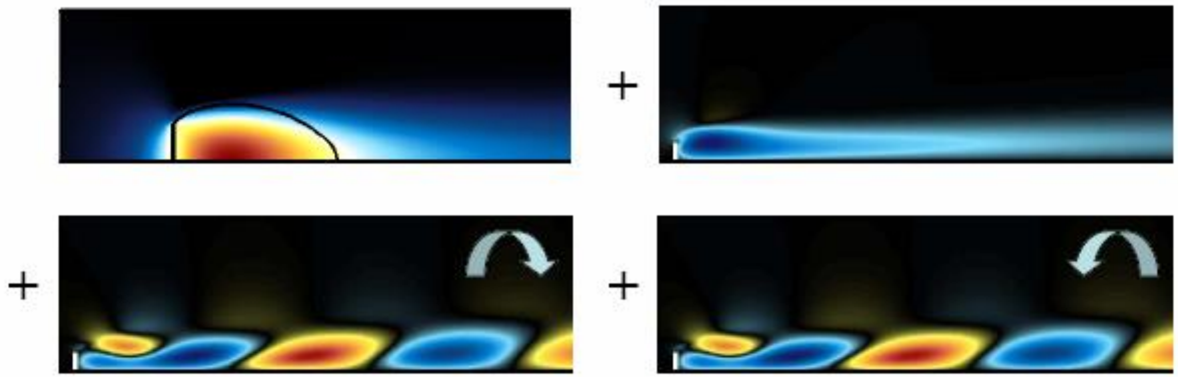
- Steady 3D + planar sym.

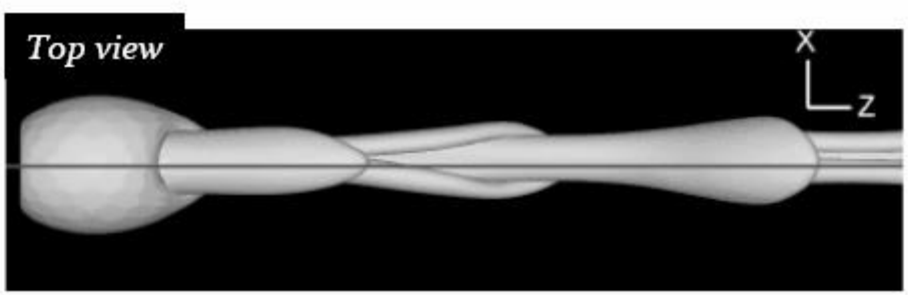


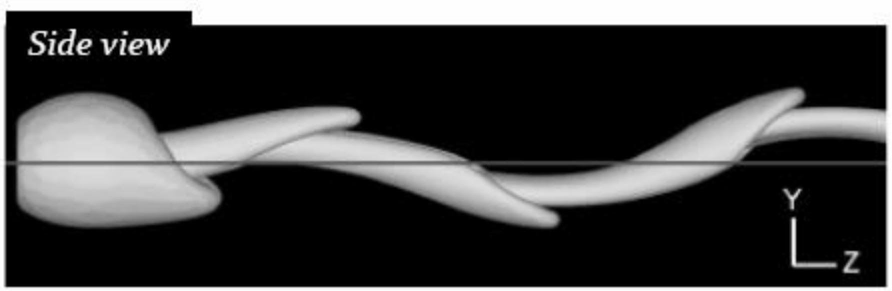


Stable solutions

- Periodic 3D, no sym.

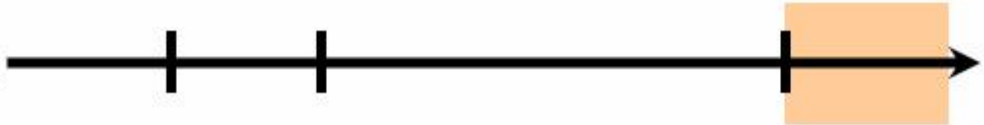


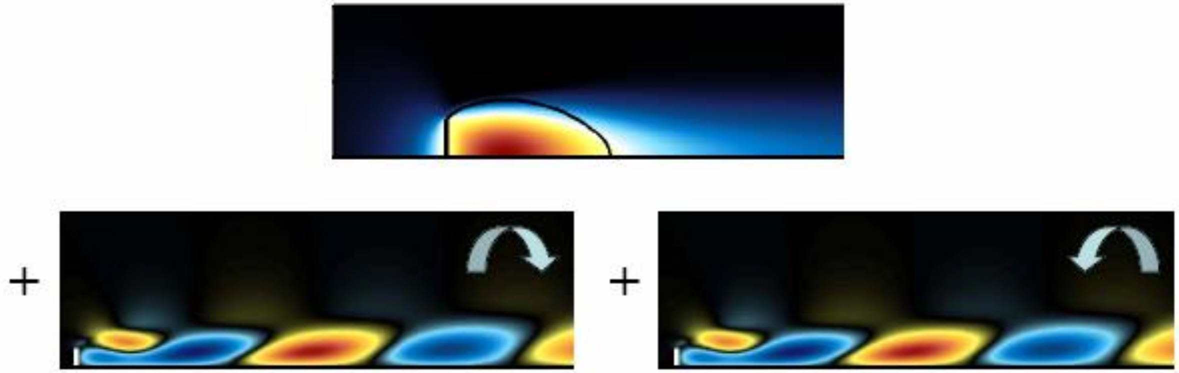


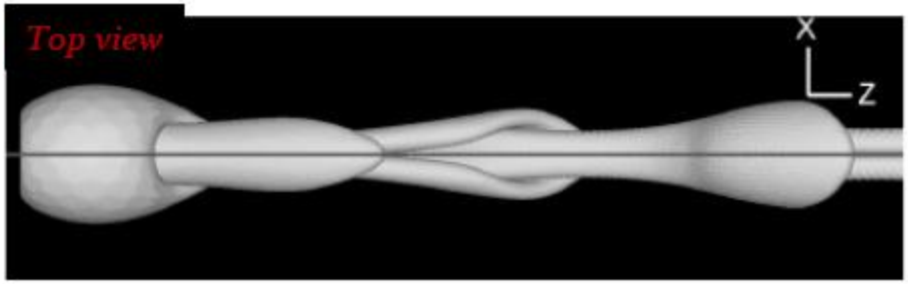


Stable solutions

- Periodic 3D + planar sym.

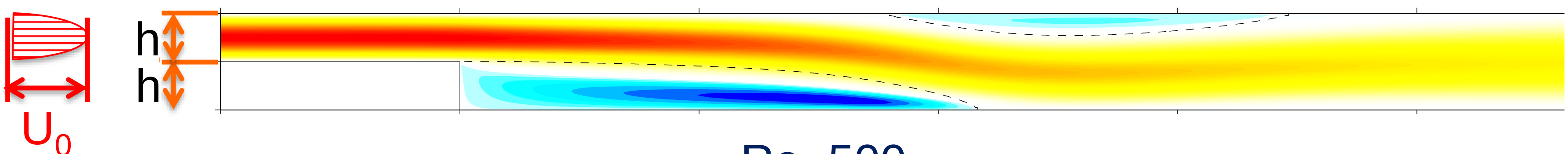




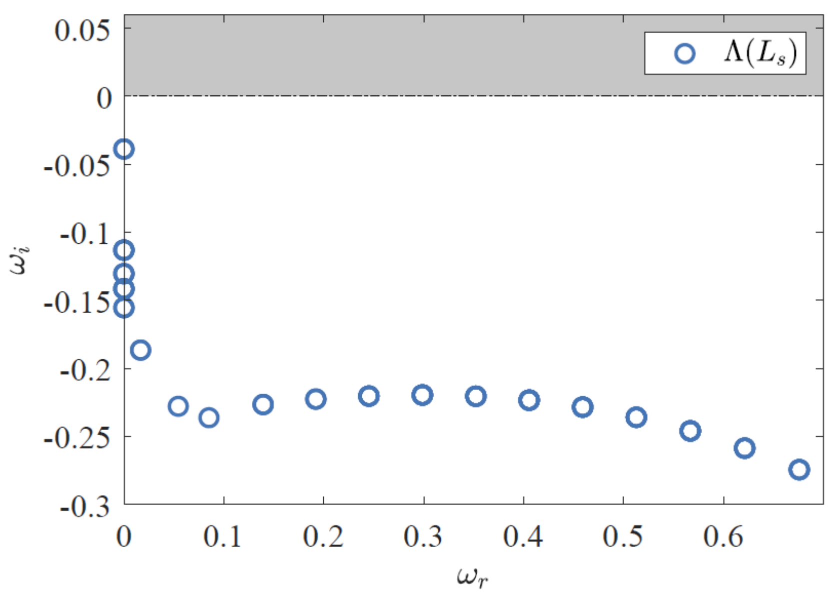


What about globally stable flows but locally unstable flows?

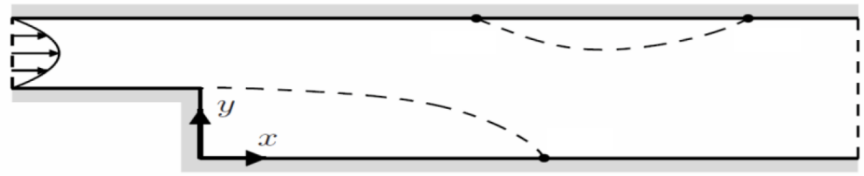
Amplifier flows
Linearly **globally stable**
Amplification of external disturbances, irrespective of the eigenfrequency spectrum
Selective but broad band response
Examples: Jets, Separated boundary layers,…



Re=500



Amplifier flows



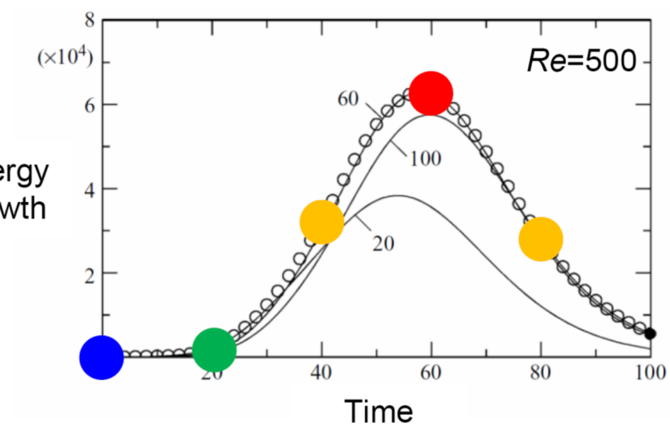
Response to initial disturbance

Response to sustained harmonic forcing

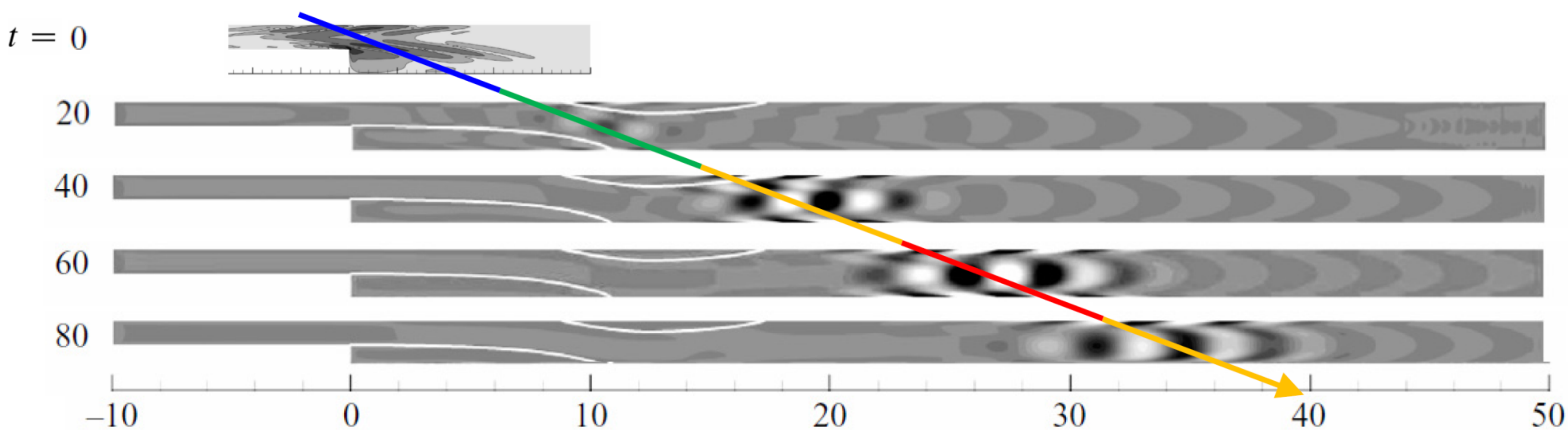
Response to stochastic forcing

Large transient growth

[Blackburn, Barkley & Sherwin, 2008]



Growth in space & time: optimal perturbations amplified while convected downstream

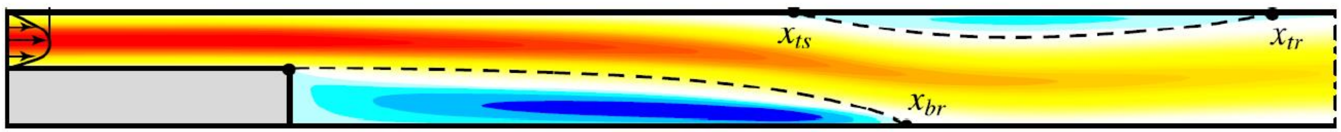


Backward facing step flow

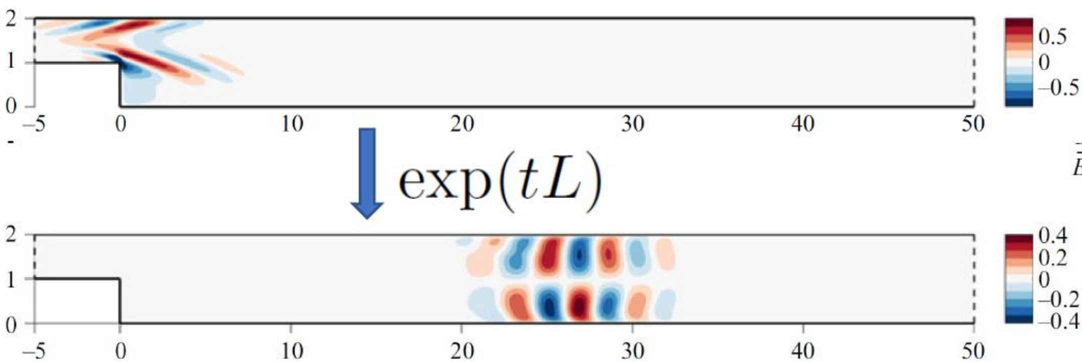
$$\frac{\partial \boldsymbol{u}}{\partial t} = L\boldsymbol{u} + \cancel{\boldsymbol{f}}, \qquad \underline{\boldsymbol{u}(0) \neq \boldsymbol{0}} \qquad \text{and} \qquad LL^{\dagger} \neq L^{\dagger}L$$

Transient growth:

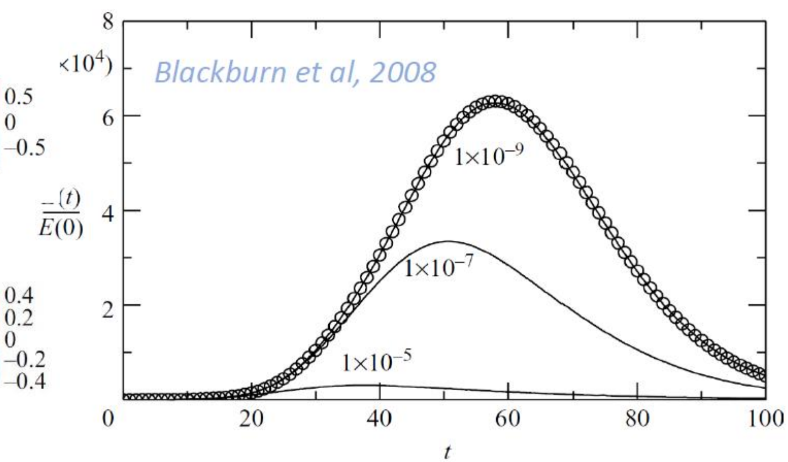
Re=500



initial condition

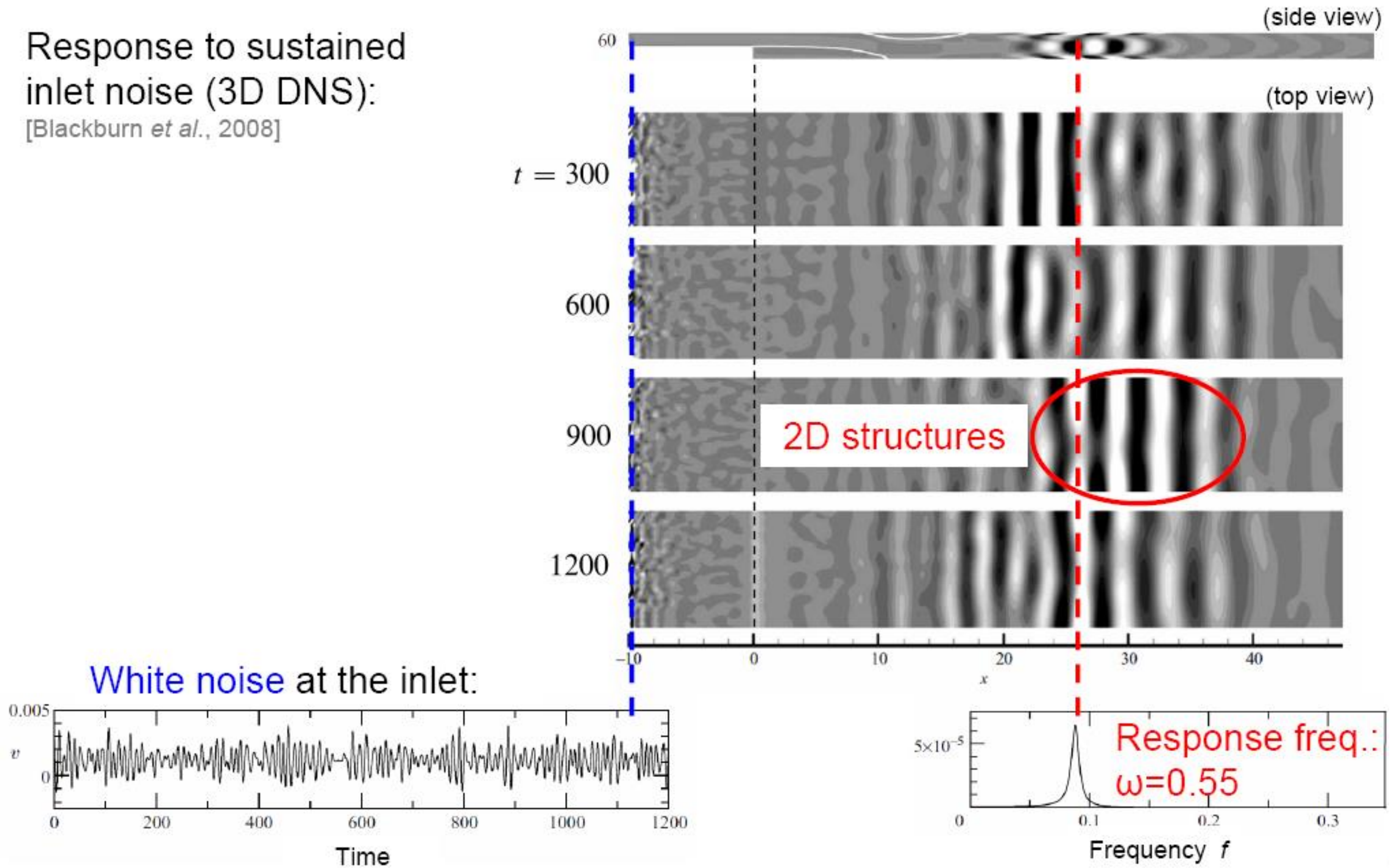


amplified solution at t=58



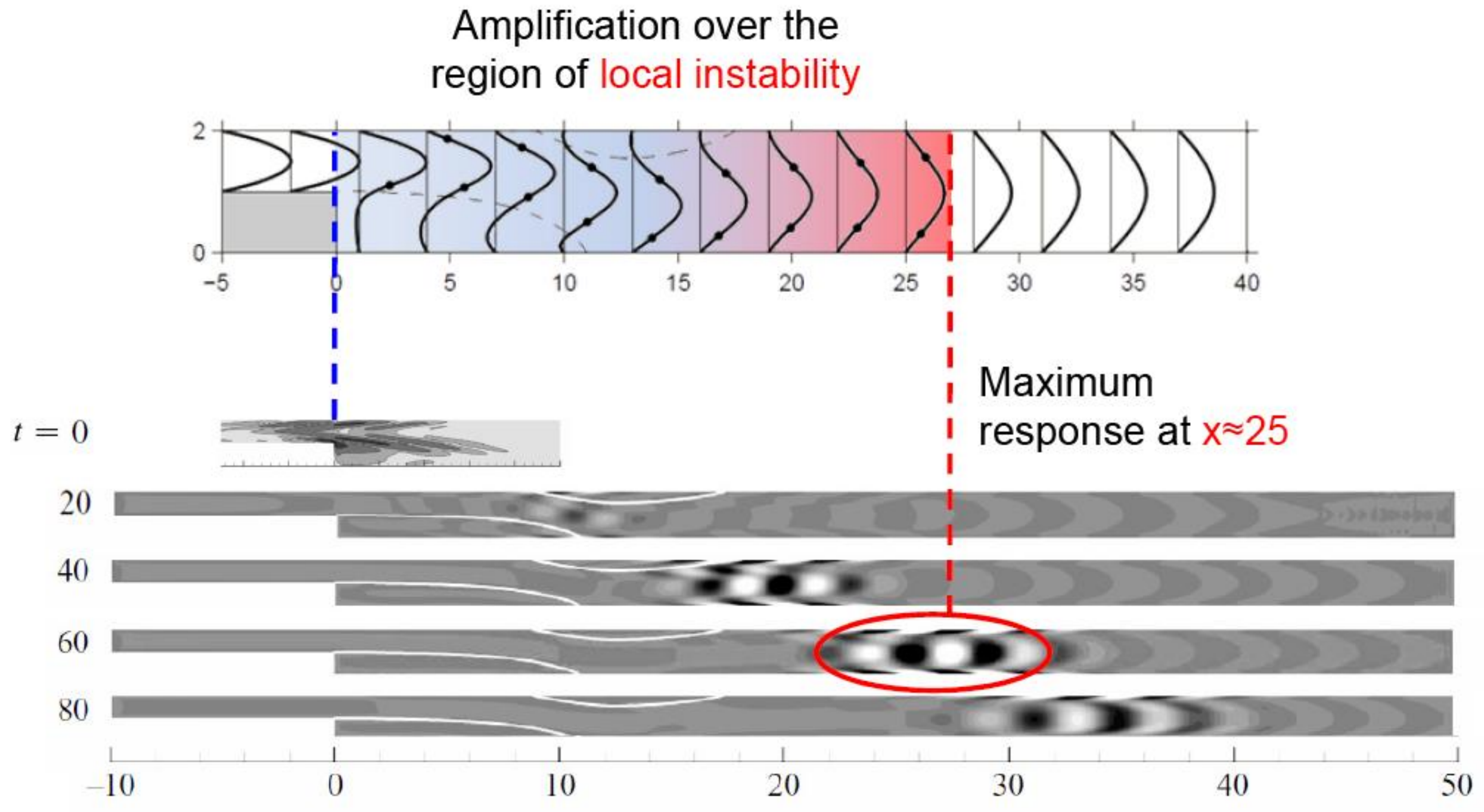
Response to white noise

Response to sustained inlet noise (3D DNS):
[Blackburn *et al.*, 2008]



⇨Very selective amplifier

The global transient growth is due to the local spatial growth



The global transient growth is due to the local spatial growth

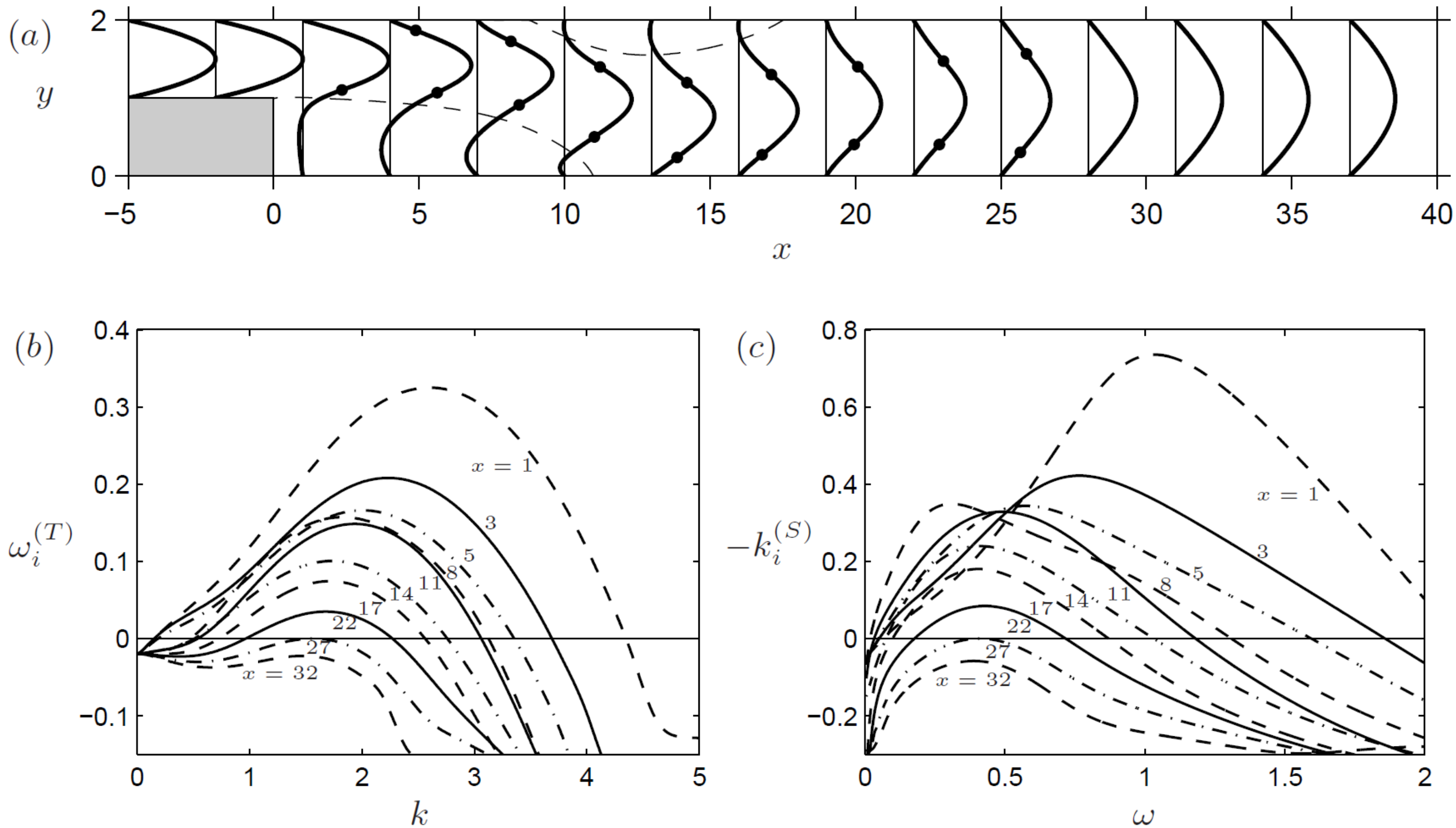


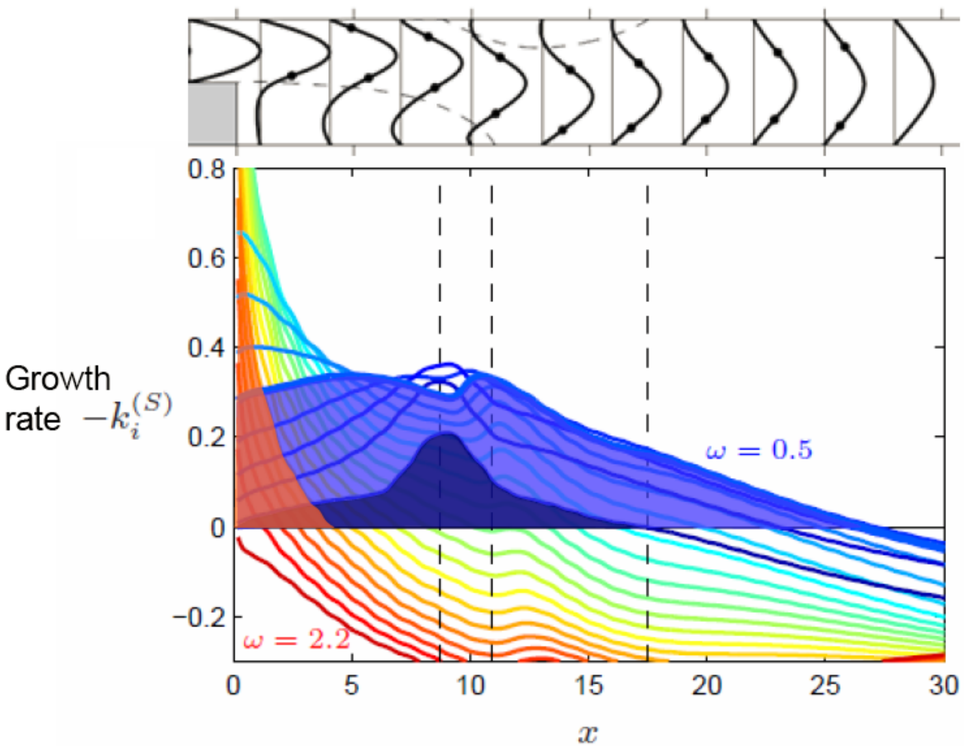
FIGURE 7. (a) Profiles of streamwise velocity for $\Gamma = 0.5$, $Re = 500$, with inflexion points shown as dots. (b) Temporal and (c) spatial growth rates obtained from local stability analysis.

Boujo and Gallaire 2015

Method 1: weakly non parallel spatial stability analysis

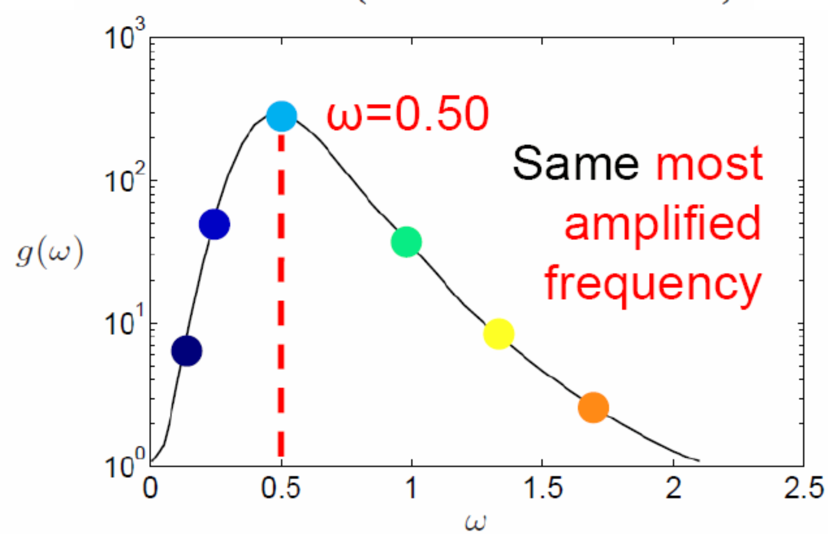
Parallel flow: $\mathbf{u}'(x,y,t) = \mathbf{u}(y)\,\mathrm{e}^{i(kx-\omega t)}$

Spatial analysis $\omega \in \mathbb{R}, k \in \mathbb{C}$



Overall amplification: integral gain

$$g(\omega) = \exp\left(\int -k_i^{(S)}(\omega, x)\,\mathrm{d}x\right)$$



Method 2: transfer function

Nonlinear problem (base flow)

$$\boldsymbol{\nabla} \cdot \mathbf{U}_b = 0,$$

$$\mathbf{U}_b \cdot \boldsymbol{\nabla} \mathbf{U}_b + \boldsymbol{\nabla} P_b - Re^{-1} \boldsymbol{\nabla}^2 \mathbf{U}_b = \mathbf{0},$$

+ linear perturbation

$$\boldsymbol{\nabla} \cdot \mathbf{u}' = 0,$$

$$\partial_t \mathbf{u}' + \boldsymbol{\nabla} \mathbf{u}' \cdot \mathbf{U}_b + \boldsymbol{\nabla} \mathbf{U}_b \cdot \mathbf{u}' + \boldsymbol{\nabla} p' - Re^{-1} \boldsymbol{\nabla}^2 \mathbf{u}' = \mathbf{F}$$

Method 2: transfer function

Nonlinear problem (base flow)

$$\boldsymbol{\nabla} \cdot \mathbf{U}_b = 0,$$

$$\mathbf{U}_b \cdot \boldsymbol{\nabla} \mathbf{U}_b + \boldsymbol{\nabla} P_b - Re^{-1} \boldsymbol{\nabla}^2 \mathbf{U}_b = \mathbf{0}$$

+ linear perturbation

$$\boldsymbol{\nabla} \cdot \mathbf{u}' = 0,$$

$$\partial_t \mathbf{u}' + \boldsymbol{\nabla} \mathbf{u}' \cdot \mathbf{U}_b + \boldsymbol{\nabla} \mathbf{U}_b \cdot \mathbf{u}' + \boldsymbol{\nabla} p' - Re^{-1} \boldsymbol{\nabla}^2 \mathbf{u}' = \mathbf{F}$$

for harmonic forcing $\mathbf{F}(t) = \mathbf{f} e^{i\omega t}$

look for solution $(\mathbf{u}', p')(\mathbf{x}, t) = (\mathbf{u}, p)(\mathbf{x})\, \mathrm{e}^{i\omega t}$

Method 2: transfer function

harmonic forcing

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0,$$

$$i\omega \mathbf{u} + \boldsymbol{\nabla}\mathbf{u} \cdot \mathbf{U}_b + \boldsymbol{\nabla}\mathbf{U}_b \cdot \mathbf{u} + \boldsymbol{\nabla} p - Re^{-1} \boldsymbol{\nabla}^2 \mathbf{u} = \mathbf{f}$$

Resolvent operator (transfer function)

$$\mathbf{u} = \mathcal{R}(\omega)\mathbf{f}$$

Method 2: transfer function

harmonic forcing

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0,$$

$$i\omega \mathbf{u} + \boldsymbol{\nabla} \mathbf{u} \cdot \mathbf{U}_b + \boldsymbol{\nabla} \mathbf{U}_b \cdot \mathbf{u} + \boldsymbol{\nabla} p - Re^{-1} \boldsymbol{\nabla}^2 \mathbf{u} = \mathbf{f}$$

Resolvent operator (transfer function)

$$\mathbf{u} = \mathcal{R}(\omega)\mathbf{f}$$

Harmonic gain

$$G^2(\omega) = \frac{||\mathbf{u}||^2}{||\mathbf{f}||^2} = \frac{(\mathcal{R}\mathbf{f} \mid \mathcal{R}\mathbf{f})}{(\mathbf{f} \mid \mathbf{f})} = \frac{(\mathcal{R}^\dagger \mathcal{R}\mathbf{f} \mid \mathbf{f})}{(\mathbf{f} \mid \mathbf{f})}$$

Optimal response to harmonic forcing

$$\frac{\partial \boldsymbol{u}}{\partial t} = L\boldsymbol{u} + \boldsymbol{f}$$

insert:

$$\boldsymbol{f}(\boldsymbol{x},t) = \hat{\boldsymbol{f}}(\boldsymbol{x})e^{i\omega_o t} + \boldsymbol{c.c}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \hat{\boldsymbol{u}}(\boldsymbol{x})e^{i\omega_o t} + \boldsymbol{c.c}$$

to obtain: $\hat{\boldsymbol{u}} = (i\omega_o I - L)^{-1}\,\hat{\boldsymbol{f}} = R(i\omega_o)\,\hat{\boldsymbol{f}}$

optimize: $G(i\omega_o) = \max_{\hat{f}} \dfrac{\|\hat{\boldsymbol{u}}\|}{\|\hat{\boldsymbol{f}}\|} = \|R(i\omega_o)\| = \dfrac{1}{\epsilon_o}$ under the scalar product $\langle \hat{\boldsymbol{u}}_a | \hat{\boldsymbol{u}}_b \rangle = \int_\Omega \hat{\boldsymbol{u}}_a^H \hat{\boldsymbol{u}}_b \, d\Omega$

Singular value decomposition:

$$R(i\omega_o)^{-1}\,\hat{\boldsymbol{u}}_o = \epsilon_o \hat{\boldsymbol{f}}_o$$

With normalization

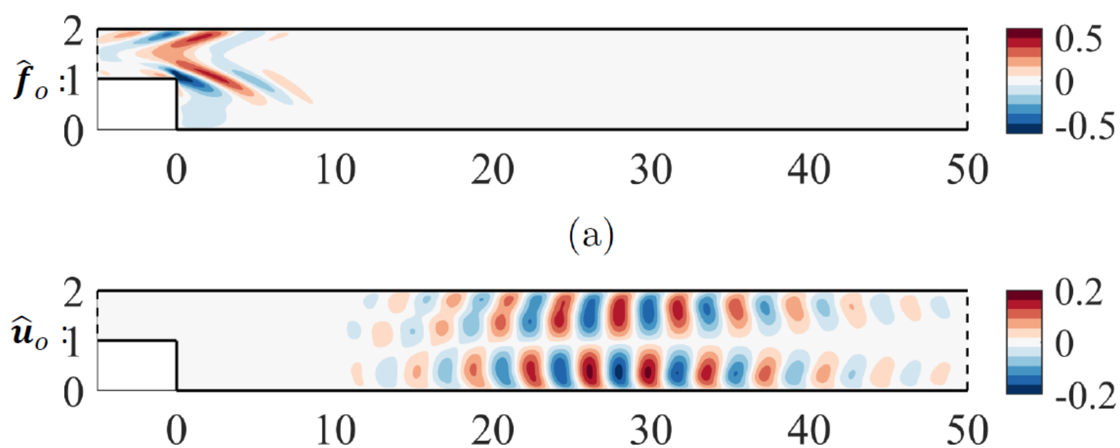
$$\langle \hat{\boldsymbol{u}}_o | \hat{\boldsymbol{u}}_o \rangle = \|\hat{\boldsymbol{u}}_o\|^2 = 1$$

$$\|\hat{\boldsymbol{f}}_o\| = 1$$

Strong nonnormality: $\boxed{\epsilon_o \ll 1}$

apply

$R(i\omega_o)^{-1}$



Optimal response to harmonic forcing

Bounds of resolvent norm

- Diagonalize the system matrix *L*

$$\boxed{L = S\Lambda S^{-1}}$$

Eigenvalue decomposition

S: Column eigenvector

Λ: Diagonal eigenvalues

$$\frac{1}{dist\{i\omega, \Lambda\}} \leq ||(i\omega - L)^{-1}|| = \boxed{||S(i\omega - \Lambda)^{-1}S^{-1}|| \leq \kappa(S)\frac{1}{dist\{i\omega, \Lambda\}}}$$

$$\kappa(S) \gg 1$$

Non-Normal system: upper and lower bound differ

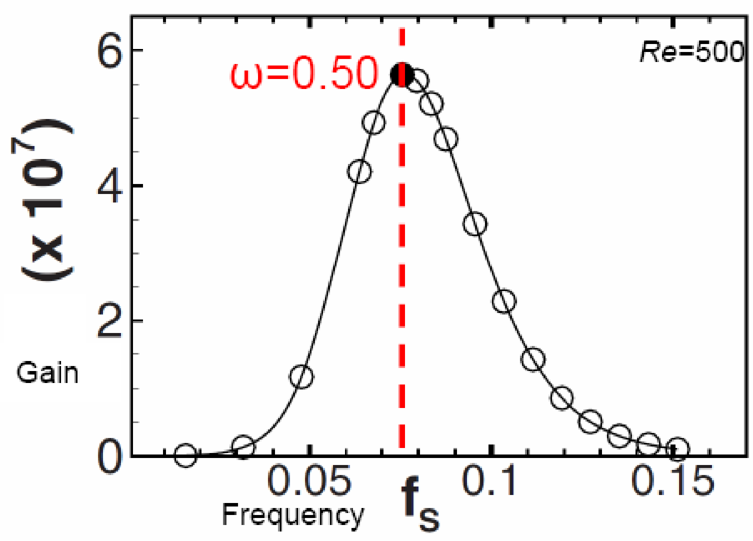
we can have a pseudo-resonance

Strong amplification also far from system eigenfrequency

Method 2: transfer function

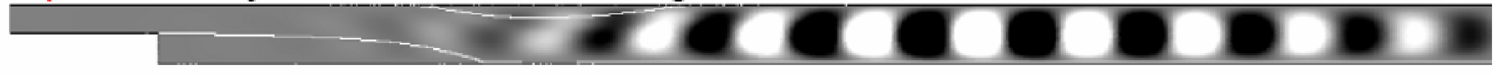
Optimal harmonic gain [Marquet &Sipp, 2010; Marquet *et al.*, 2010, priv. comm.]:

same preferred frequency,
large gain

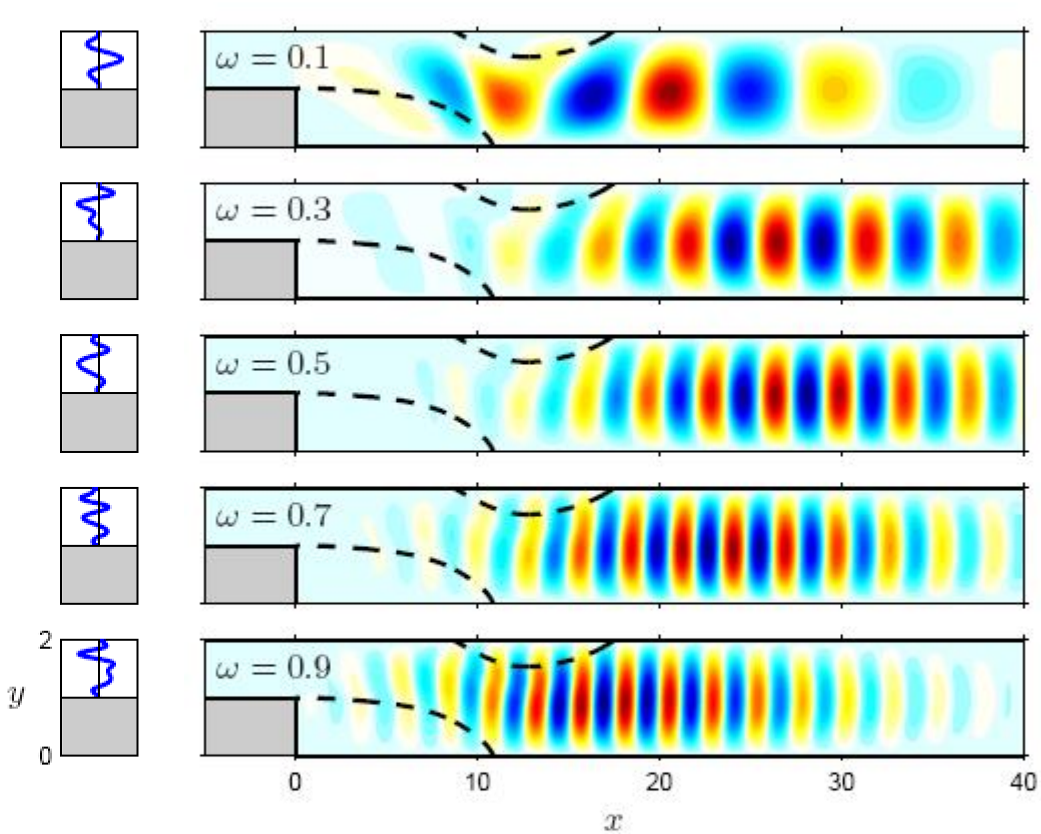


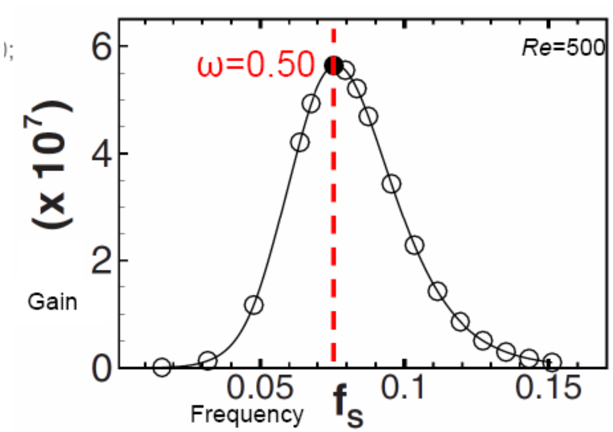
Optimal forcing

and optimal response very similar to transient growth:



Method 2: transfer function





Local weakly non parallel spatial stability analysis v.s. global resolvent analysis