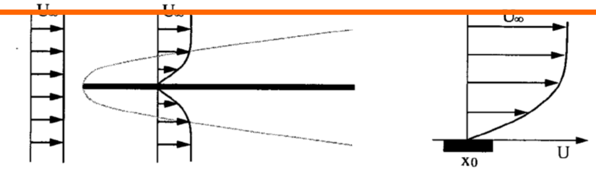
Most flows are unstable...

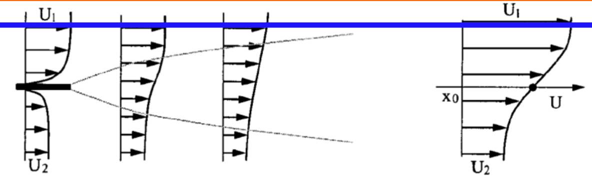
1. Intro-definitions
2. Rayleigh-Taylor
3. Rayleigh Plateau (destabilization through surface tension)
4. Rayleigh-Benard (convection)
5. Benard-Marangoni
6. Taylor Couette-Centrifugal instability
7. Kelvin-Helmholtz
8. Inflection point theorem Rayleigh, Orr sommerfeld
9. Orr sommerfeld, transient growth
10. Spatial growth

SPATIALLY DEVELOPING SHEAR FLOWS

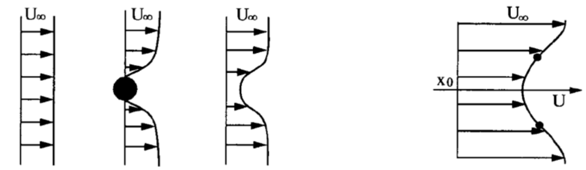
Flat plate boundary layer



Mixing layer

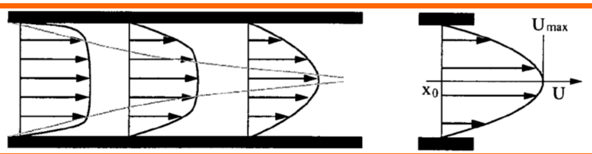


Cylinder wake



Inviscid **in**stability
(inflection point criterion)
Stabilized by viscosity
Transient growth on top

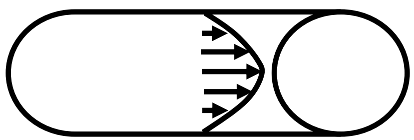
Plane channel flow



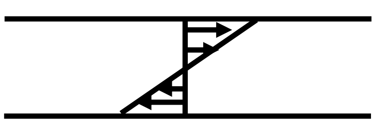
Inviscid stability
(inflection point criterion)
Destabilized by viscosity
Transient growth important

2D jet

Hagen Poiseuille

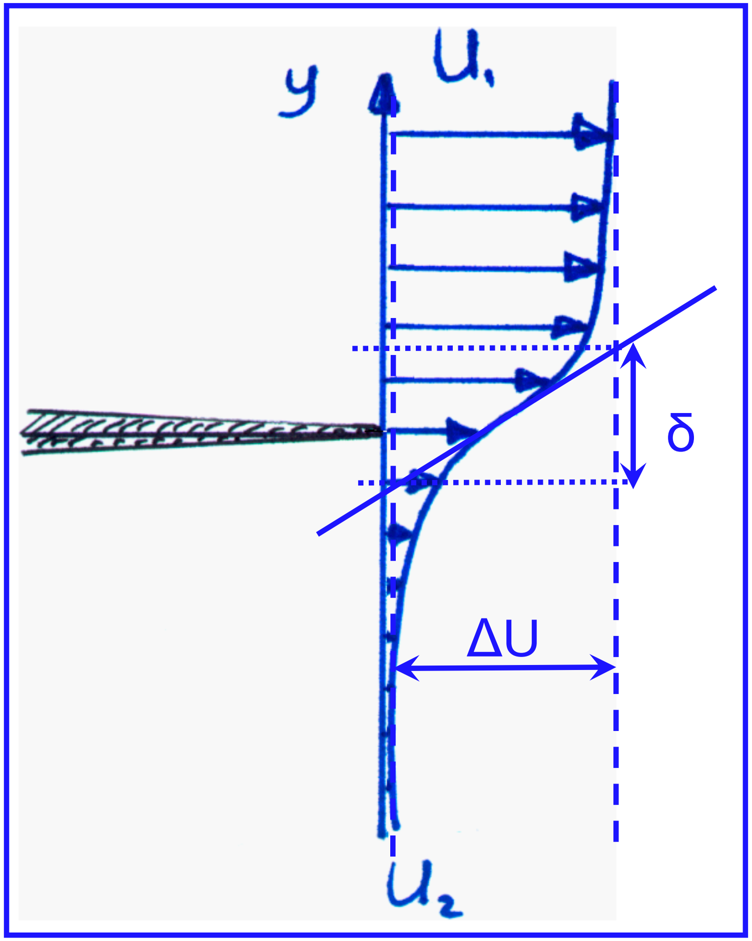


Couette



Linear stability
Transient growth essential

Viscosity has limited stabilizing influence on K-H instability



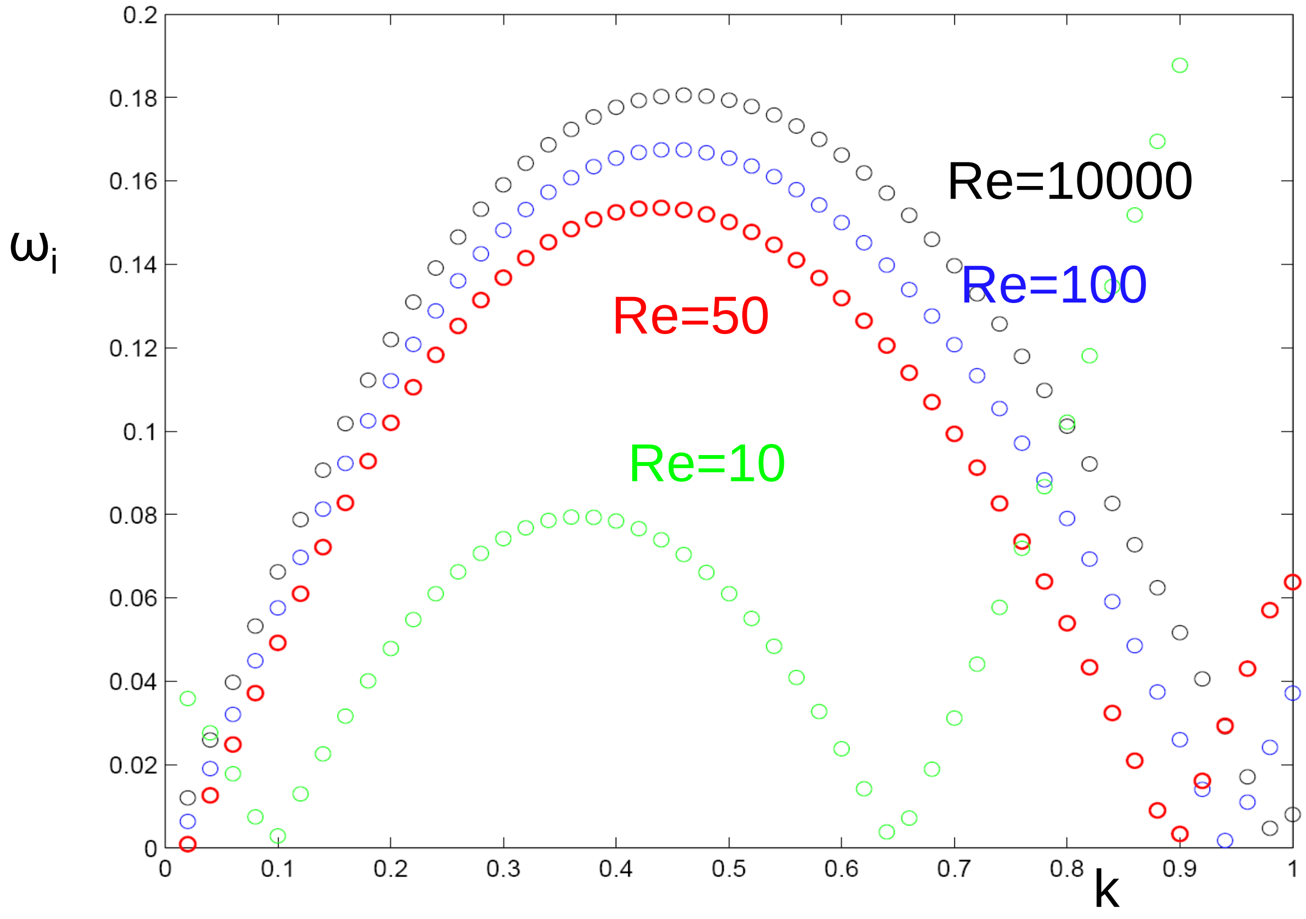
$$U_B(y) = \tanh\, y$$

$$Re := \Delta U\, \delta / \nu$$

$$\omega_{i,max} \frac{\delta}{\Delta U} \approx \sqrt{\frac{0,2}{1 + @/0,2 Re}}$$

constant

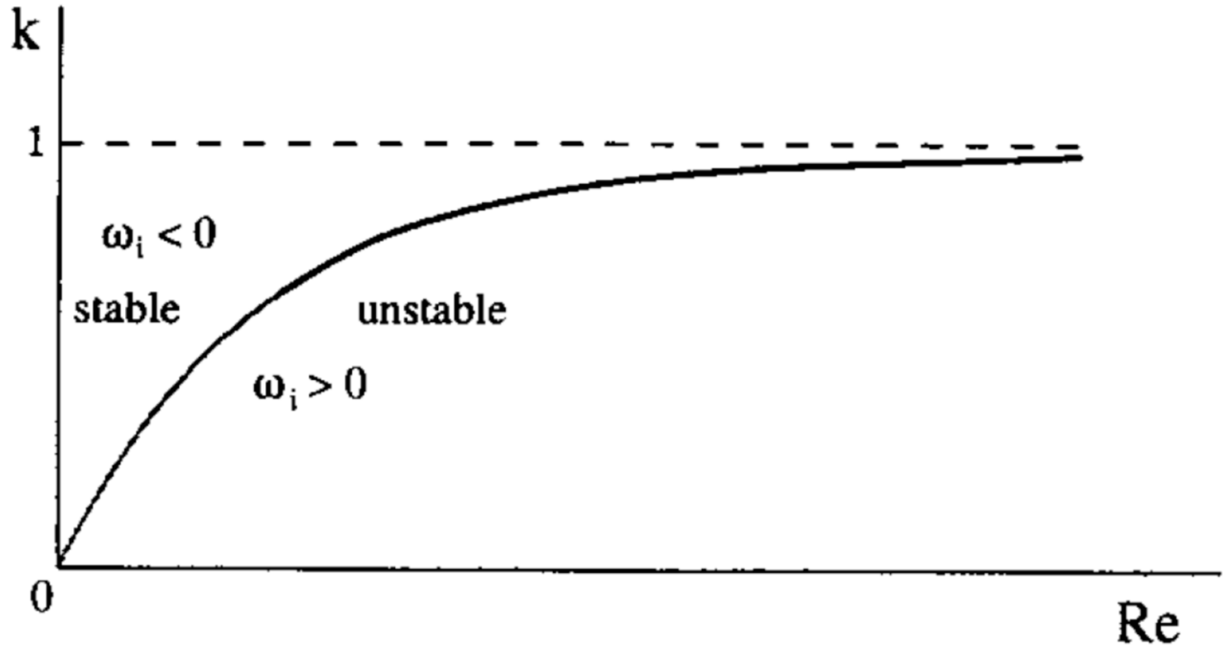
Viscosity has stabilizing influence on K-H instability



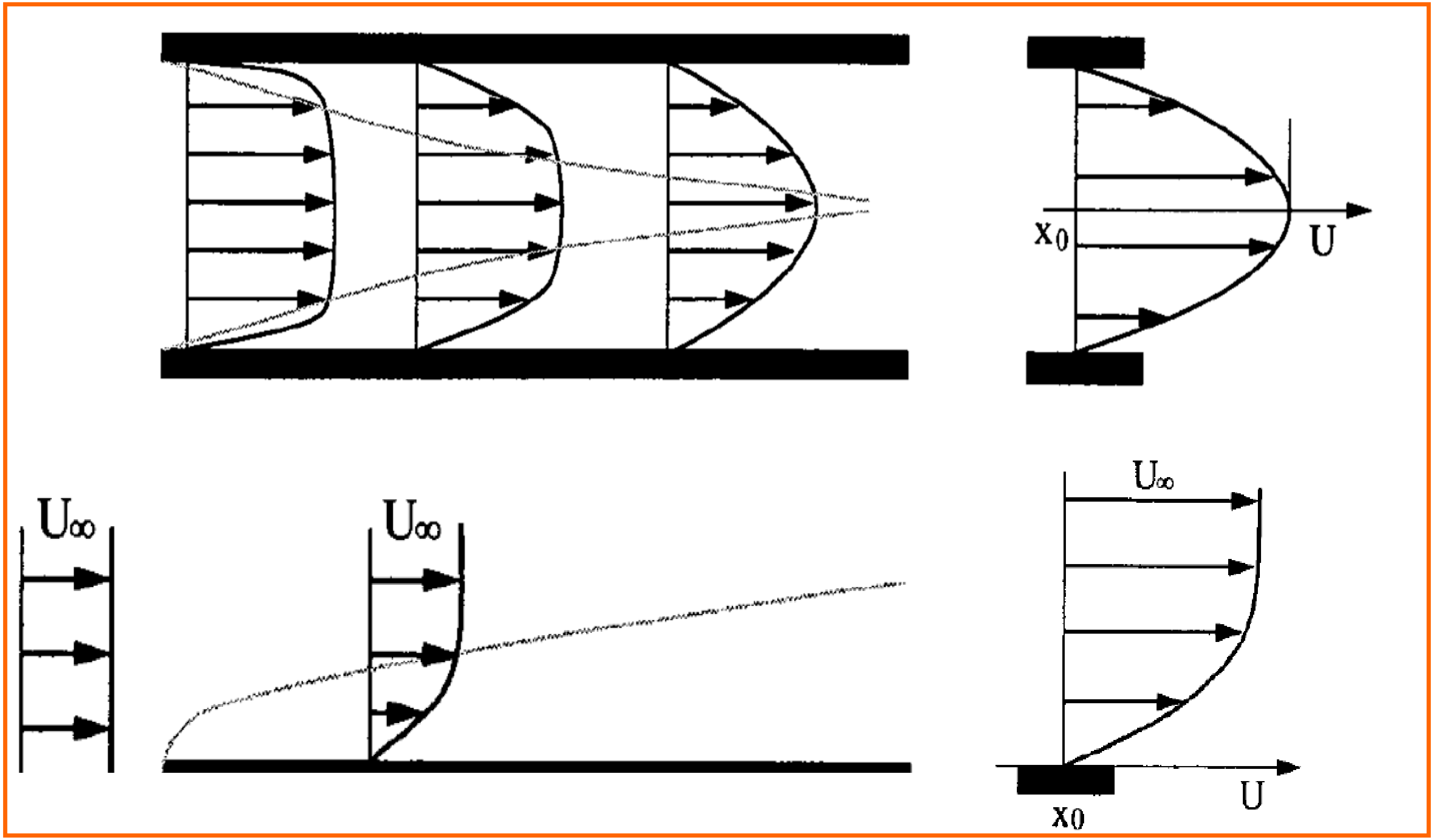
PARALLEL FLOW CONCEPTS

Viscous instabilities

Hyperbolic tangent mixing layer

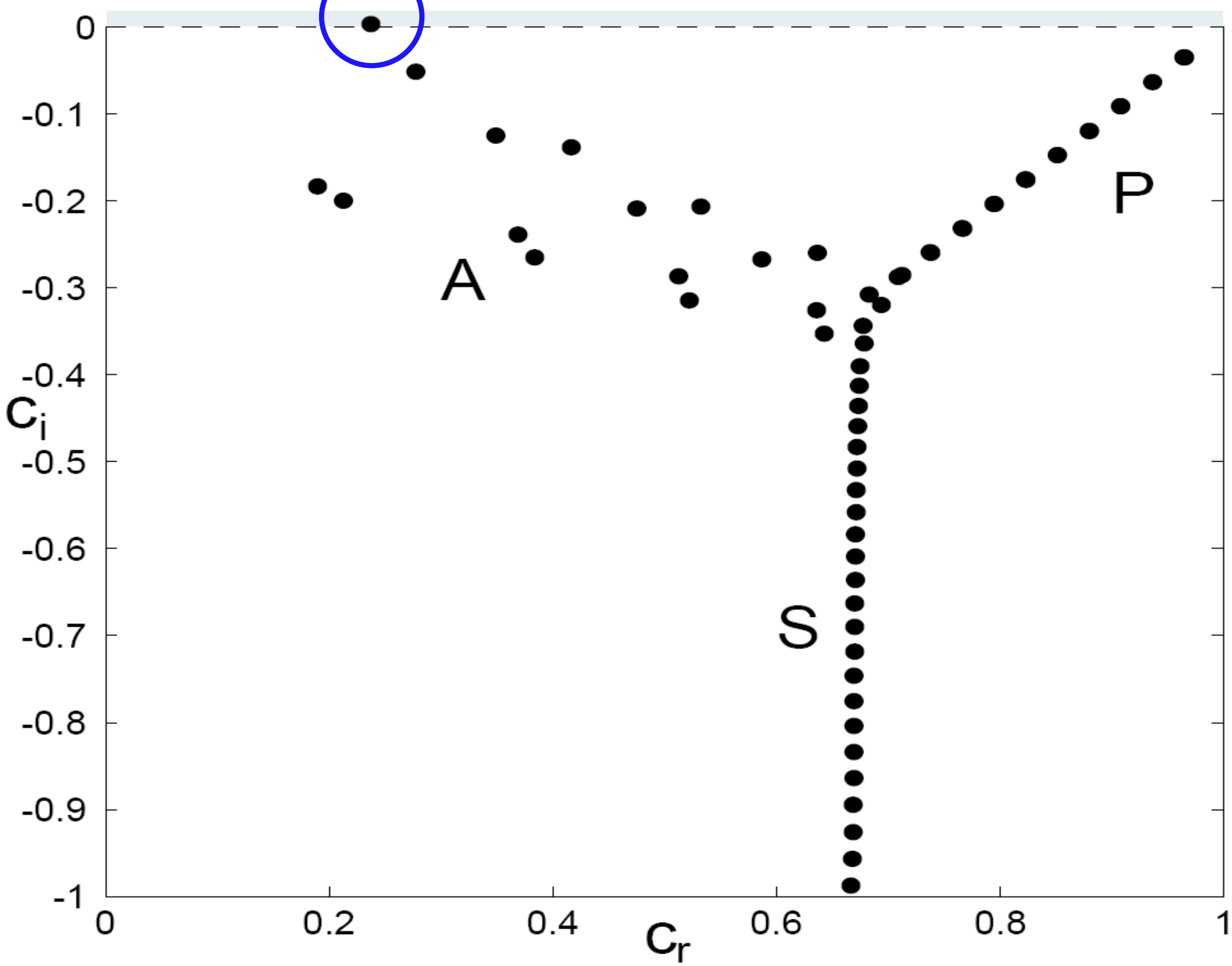


What about inviscidly stable flows (no inflexion point)?



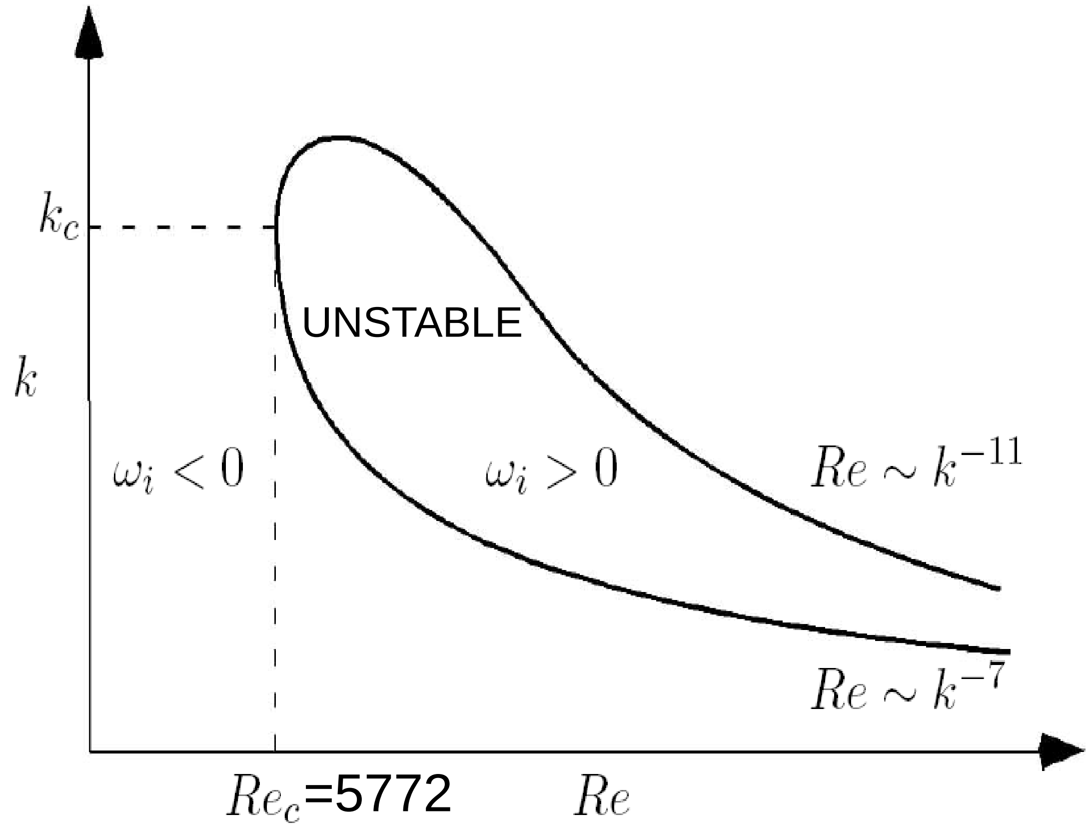
Plane poiseuille flow; Re=10000; α=1, β=0

No inflection point but unstable!



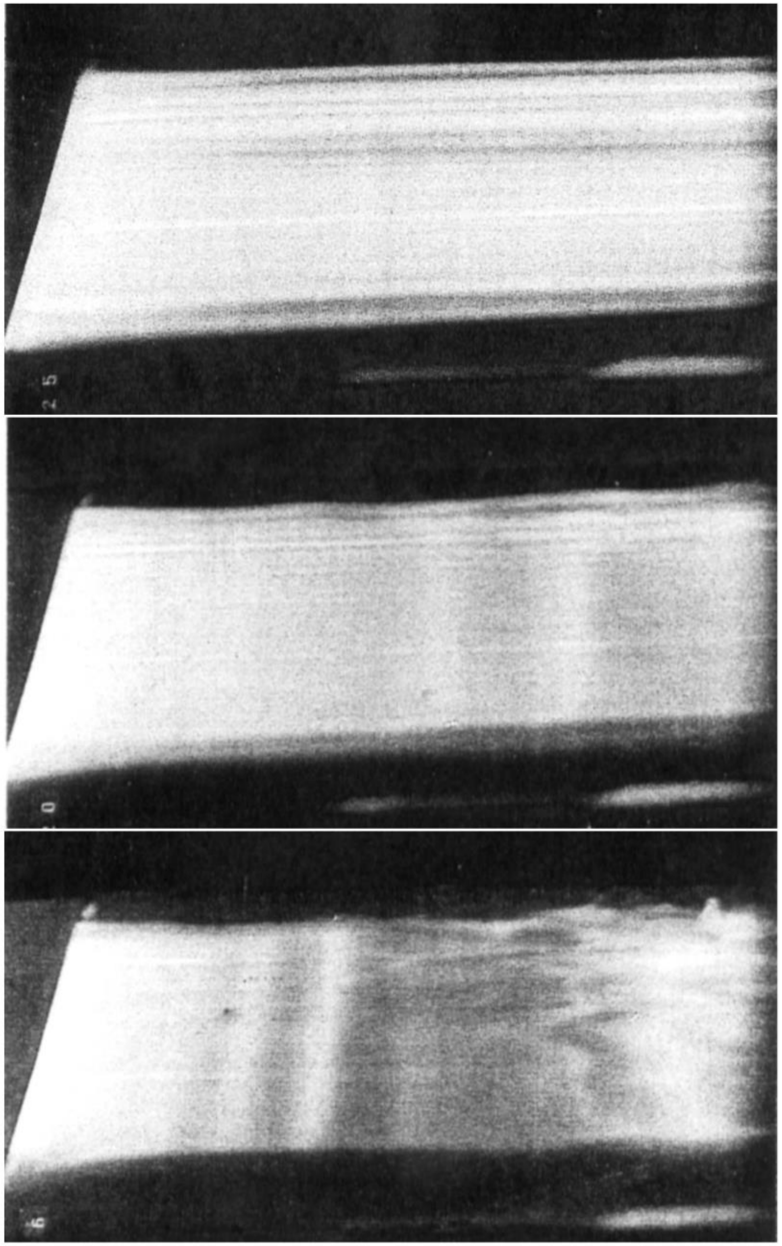
$c=\omega/k$

Neutral curve for plane poiseuille flow



– *Allure de la courbe de stabilité marginale dans le plan* $Re - k$ *pour l'écoulement de Poiseuille plan.*

Boundary layer at increasing Reynolds number



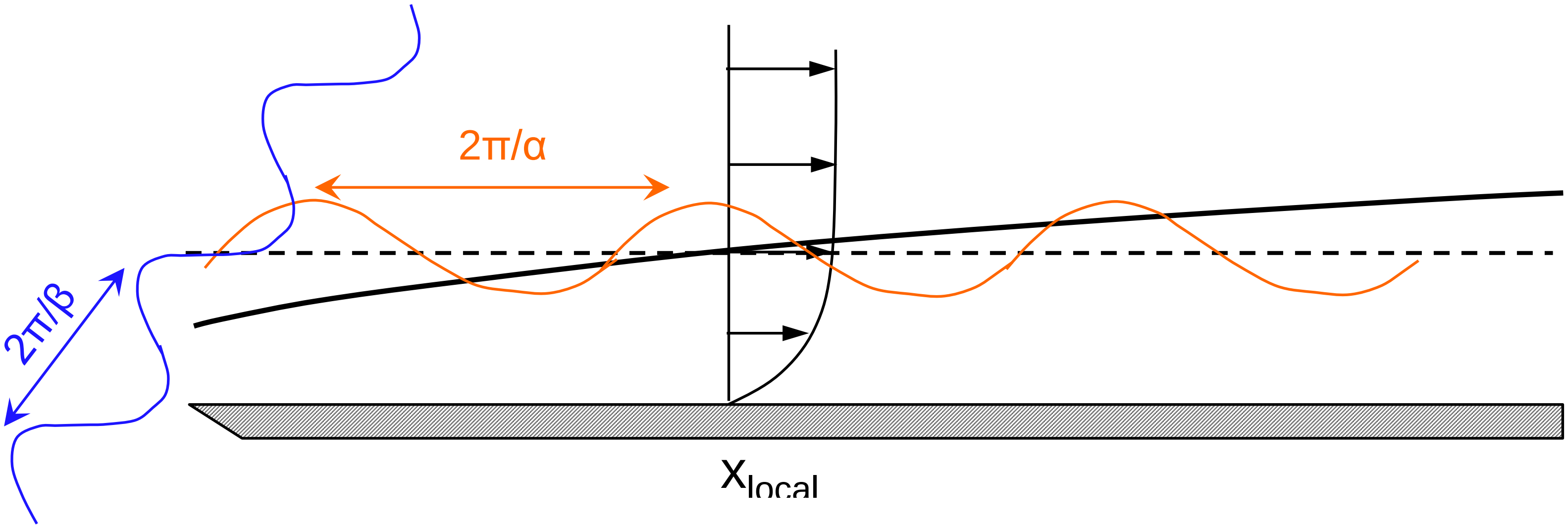
Boundary layers are destabilized by viscosity
Reynolds Orr equation

Perturbation kinetic energy: $e_c = \frac{1}{2}(u^2 + v^2 + w^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \langle e_c \rangle \mathrm{d}y = \int_{y_1}^{y_2} \partial_y \overline{U} \tau_{xy} \mathrm{d}y - \frac{1}{Re}\int_{y_1}^{y_2} \langle \boldsymbol{\omega} . \boldsymbol{\omega} \rangle \mathrm{d}y$$

Production term   Dissipation term

Local parallel flow approximation



$$(\mathbf{u}, p) = (\mathbf{u}(y), p(y))\, \mathrm{e}^{\sigma t + i(\alpha x + \beta z)}$$

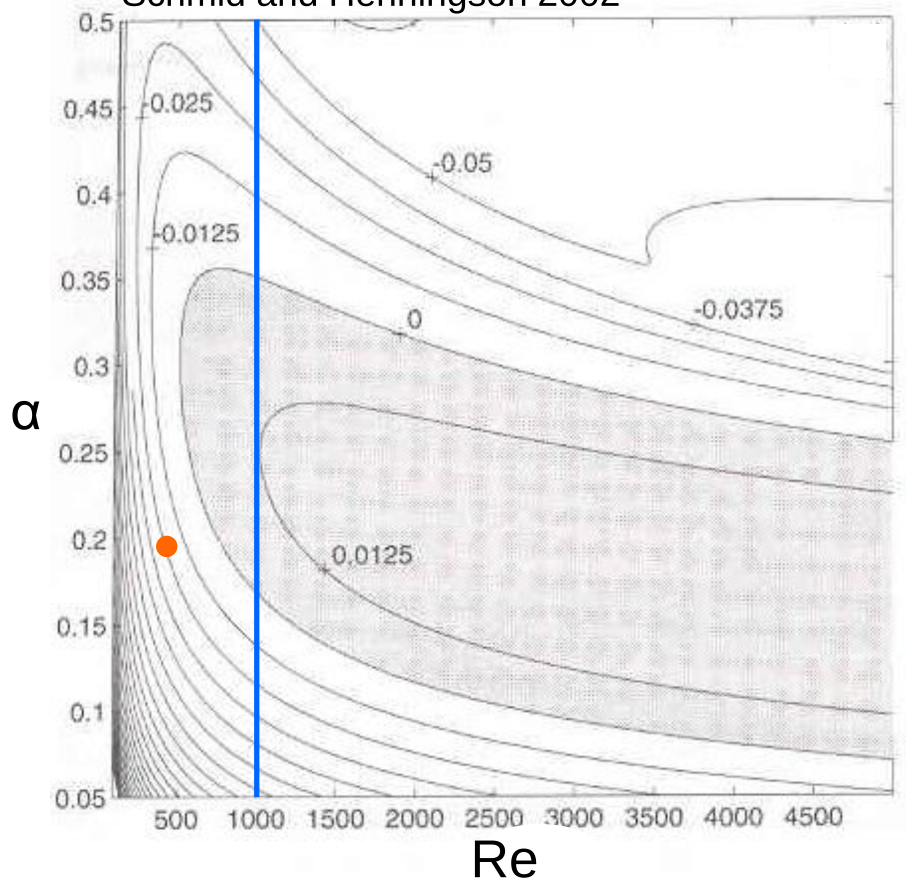
⇨Orr-Sommerfeld-Squire equation

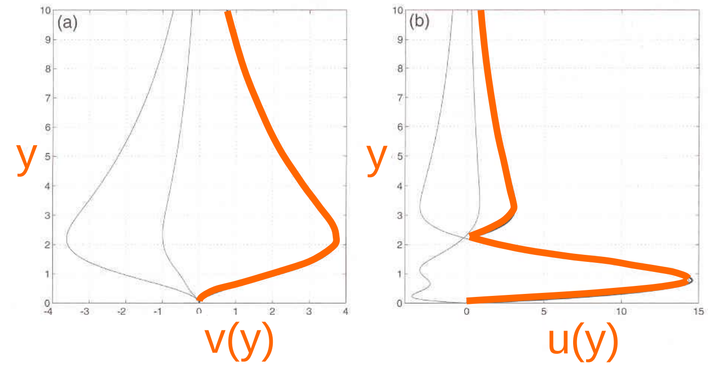
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}(y)\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{U}(y) = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$

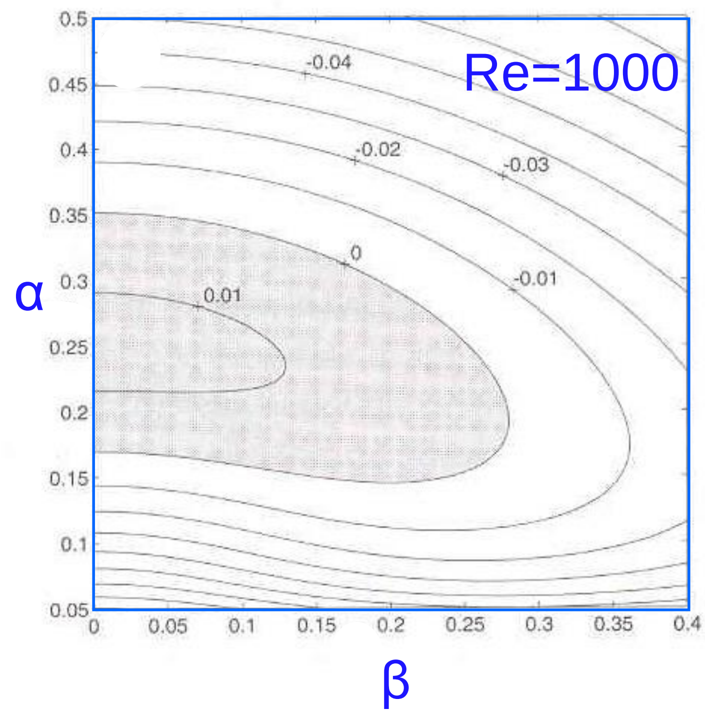
$$\nabla.\mathbf{u} = 0$$

Tollmien Schlichting waves

Schmid and Henningson 2002



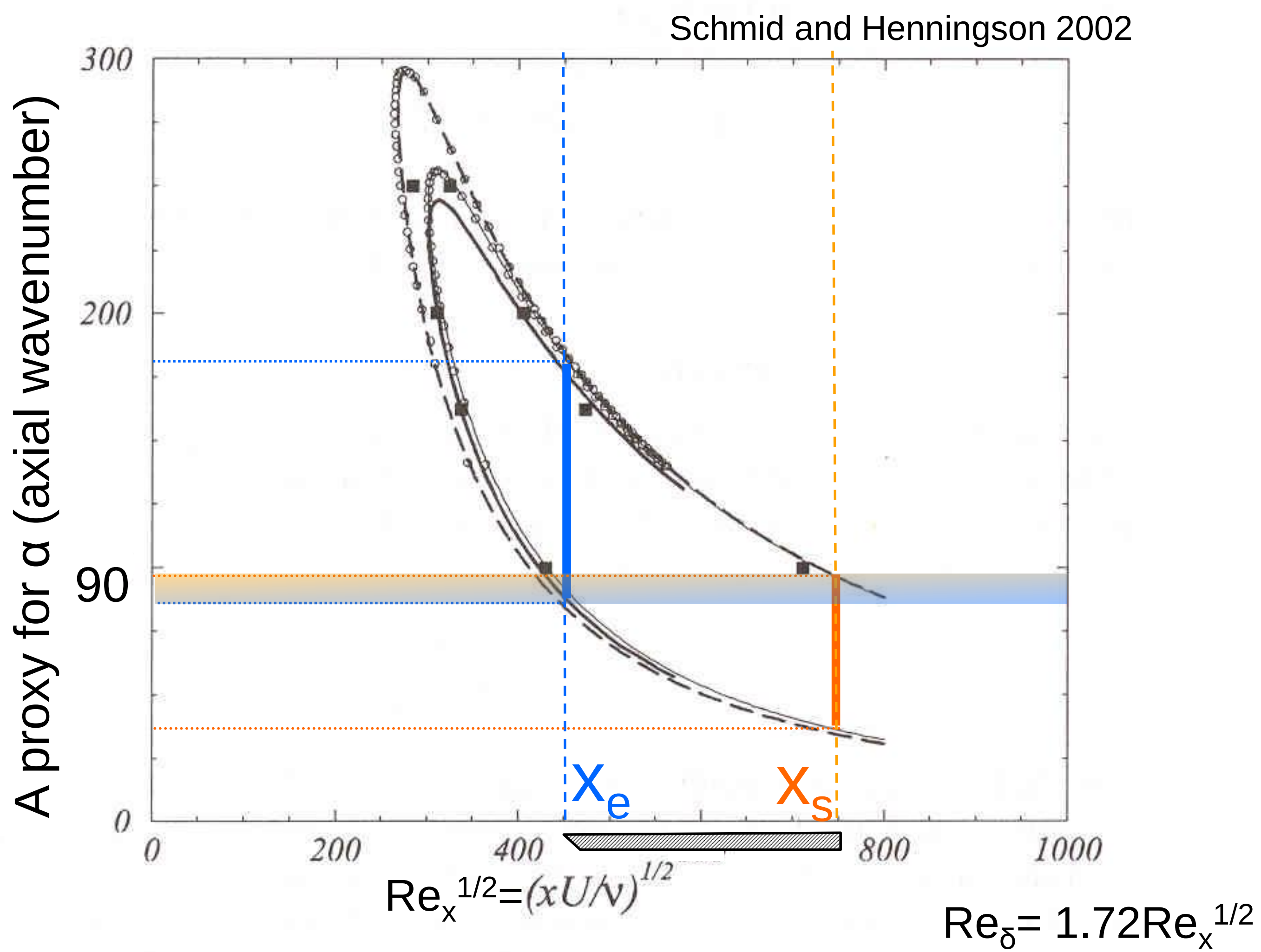




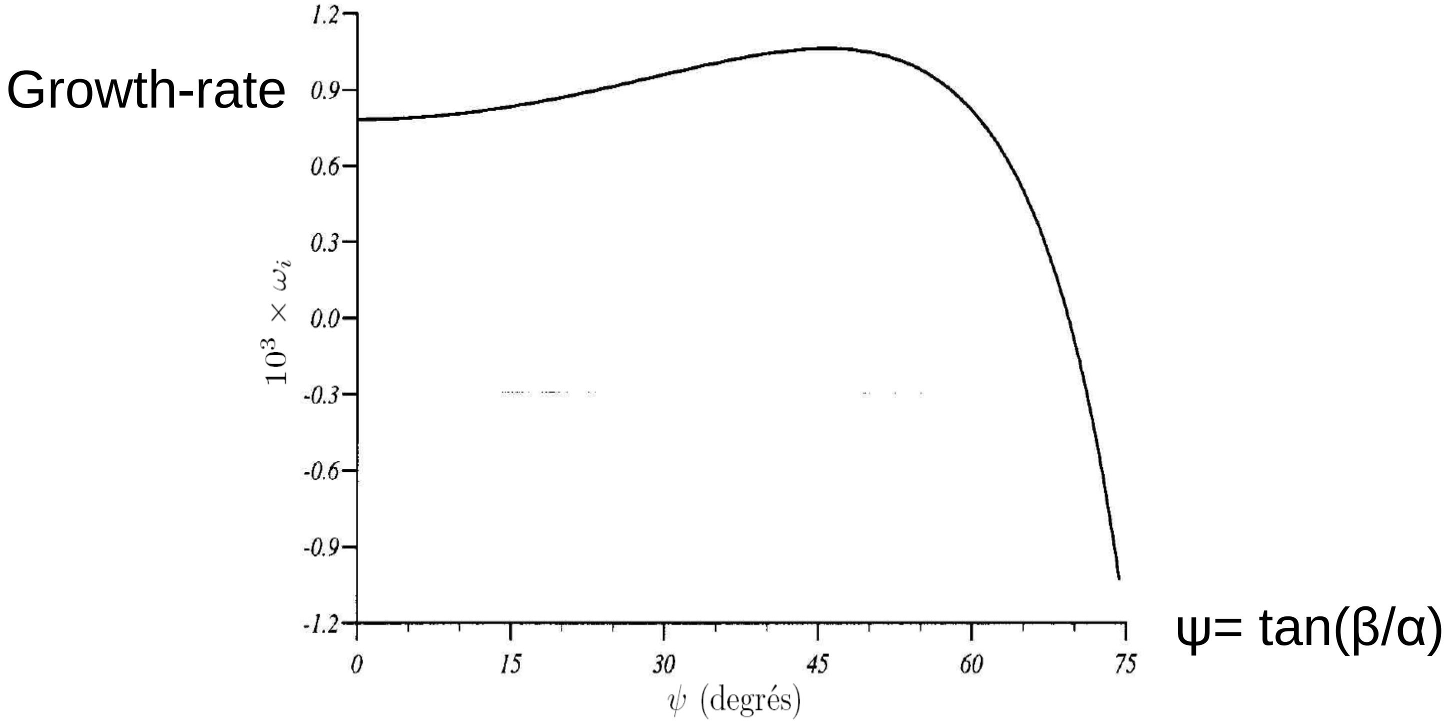
α: axial wavenumber

β: transverse wavenumber

Neutral curve



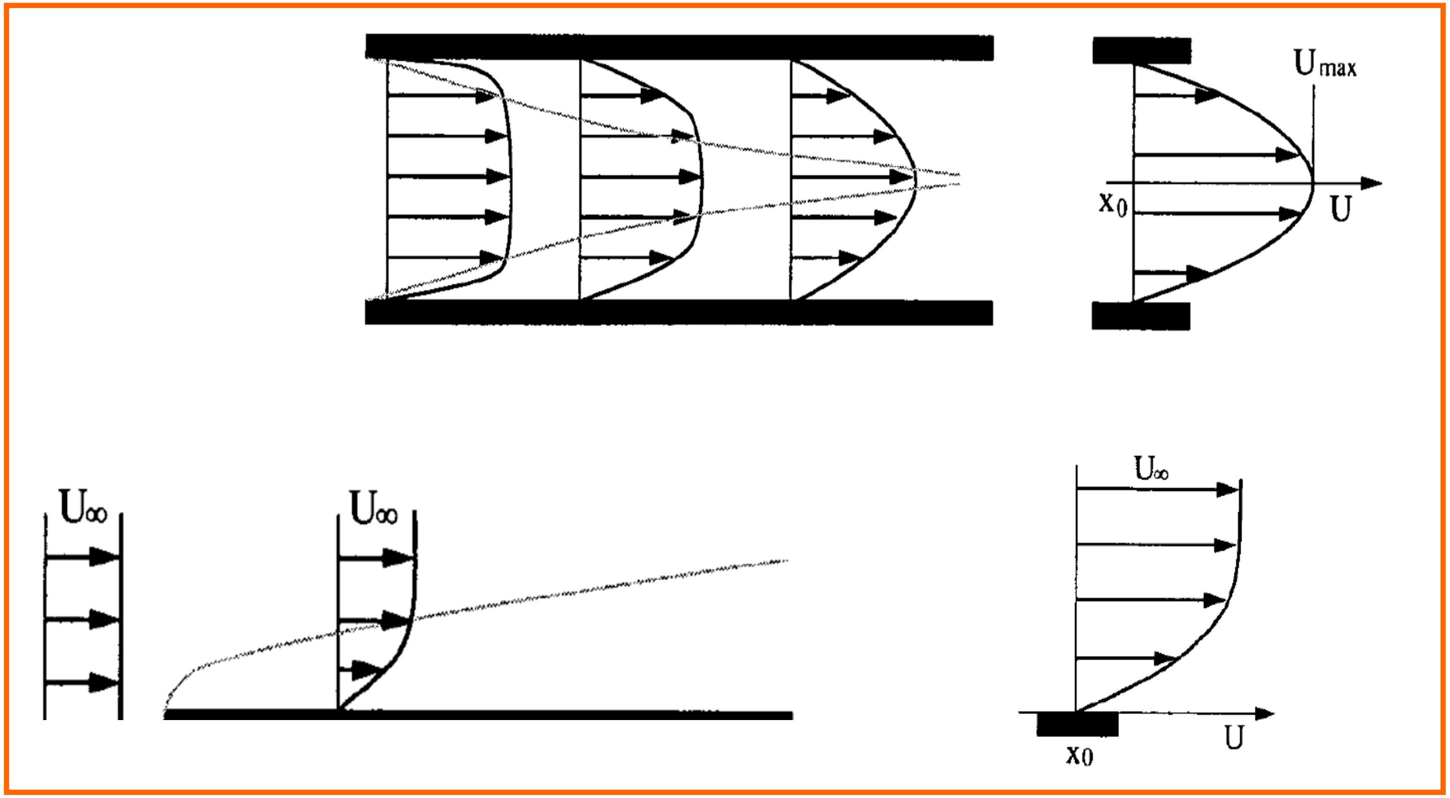
Boundary layer at Re_δ =1500



Taux de croissance temporel (s^{-1}) de l'instabilité d'une couche limite, en fonction de l'angle ψ de la perturbation avec la direction de l'écoulement de base. $R_{\delta 1} = 1500$, $\omega\nu/U_\infty^2 = 0,3 \times 10^{-4}$ (calcul G. Casalis, ONERA).

The most unstable perturbation is oblique!

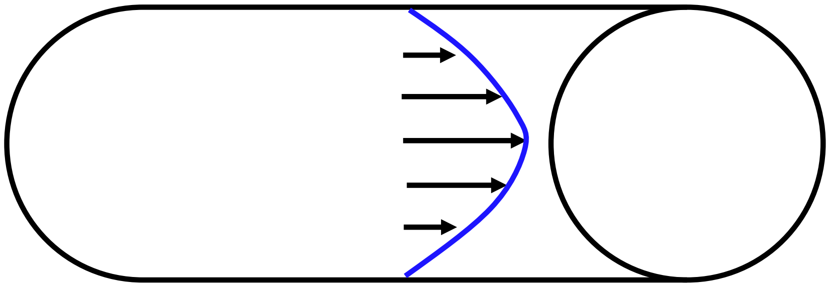
Transient flow is very important in these linearly unstable flows



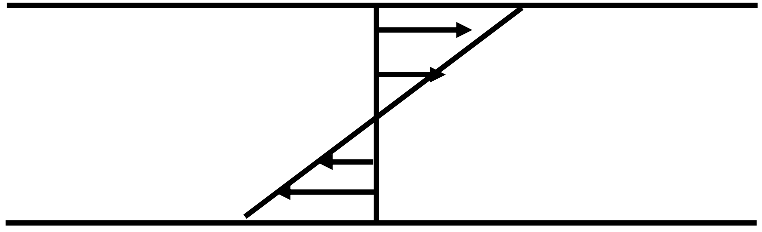
Flow type	Re_{exp}	Re_{lin}
Plane Poiseuille flow	≈ 1000	5772

Linearly stable flows

Hagen Poiseuille



Couette



Flow type	Re_{exp}	Re_{lin}
Pipe flow	≈ 2000	∞
Plane Couette flow	≈ 360	∞

Transient growth mechanisms are essential to explain experimental observations

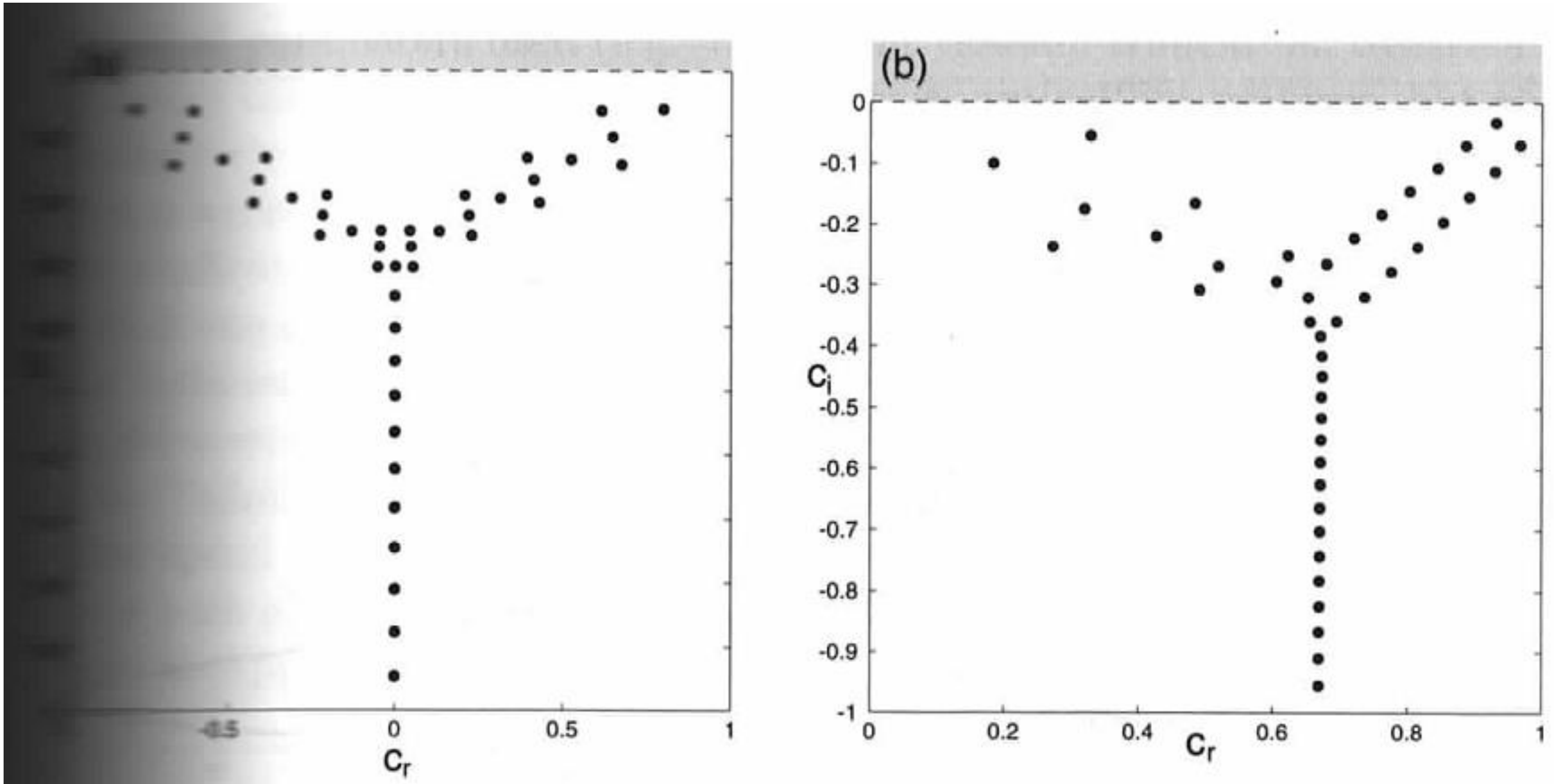


FIGURE 3.3. Spectrum of plane Couette and pipe flow. (a) Plane Couette flow for $\alpha = 1, \beta = 1, \mathrm{Re} = 1000$; (b) Pipe flow for $\alpha = 1, n = 1, \mathrm{Re} = 5000$.

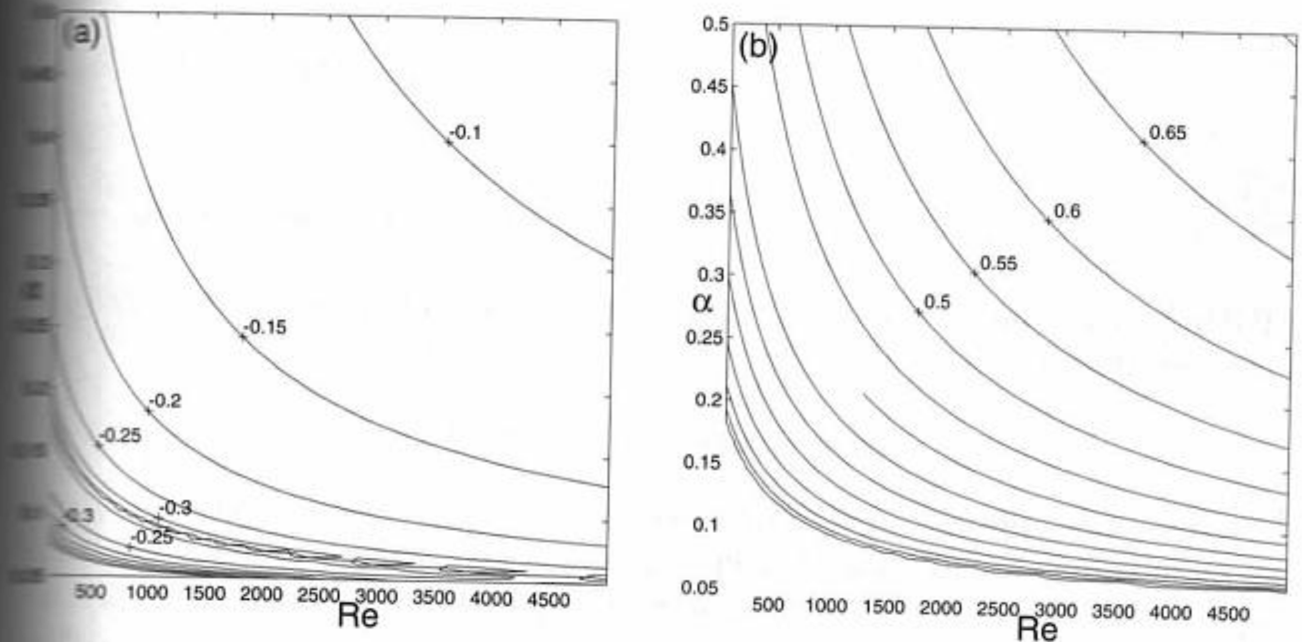


FIGURE 3.11. Contours of constant growth rate (a) and phase speed (b) for plane Couette flow.

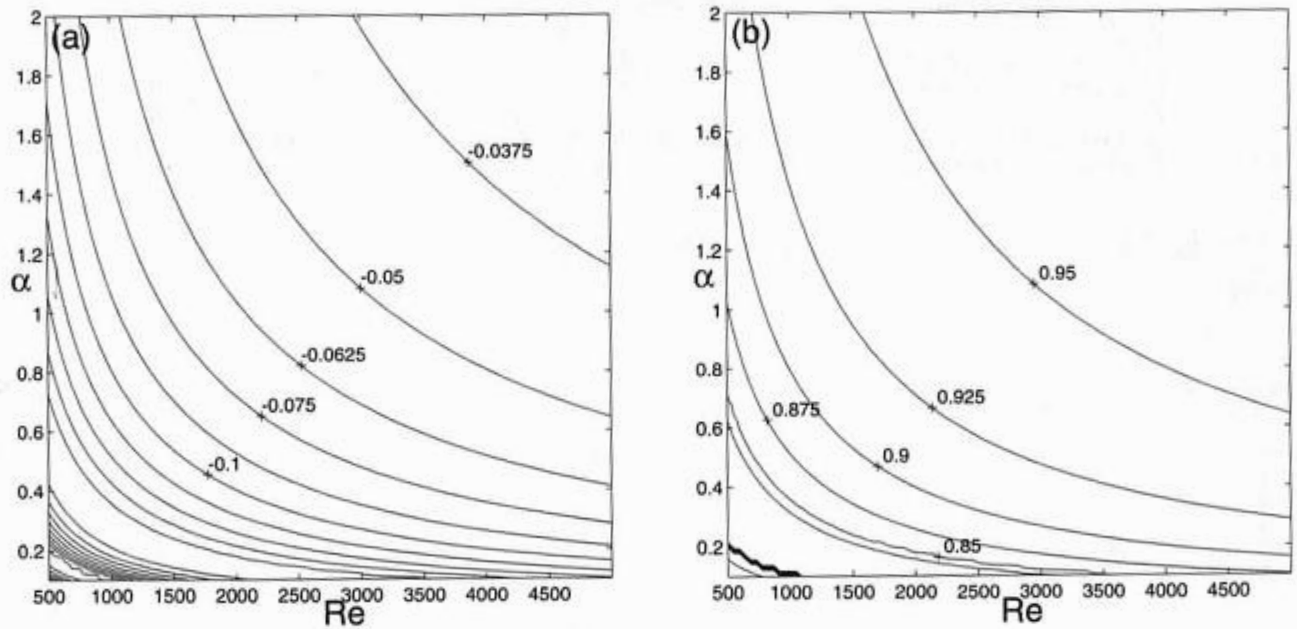
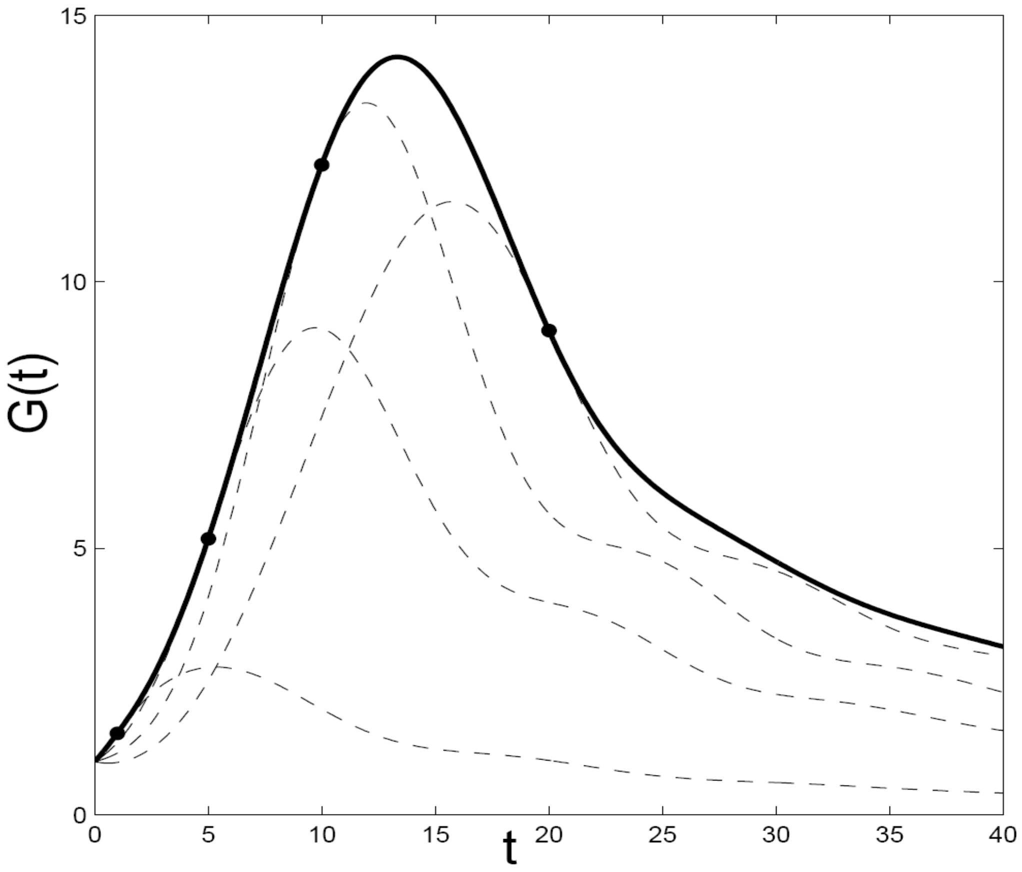


FIGURE 3.12. Contours of constant growth rate (a) and phase speed (b) for pipe Poiseuille flow.

Orr mechanism (2D)

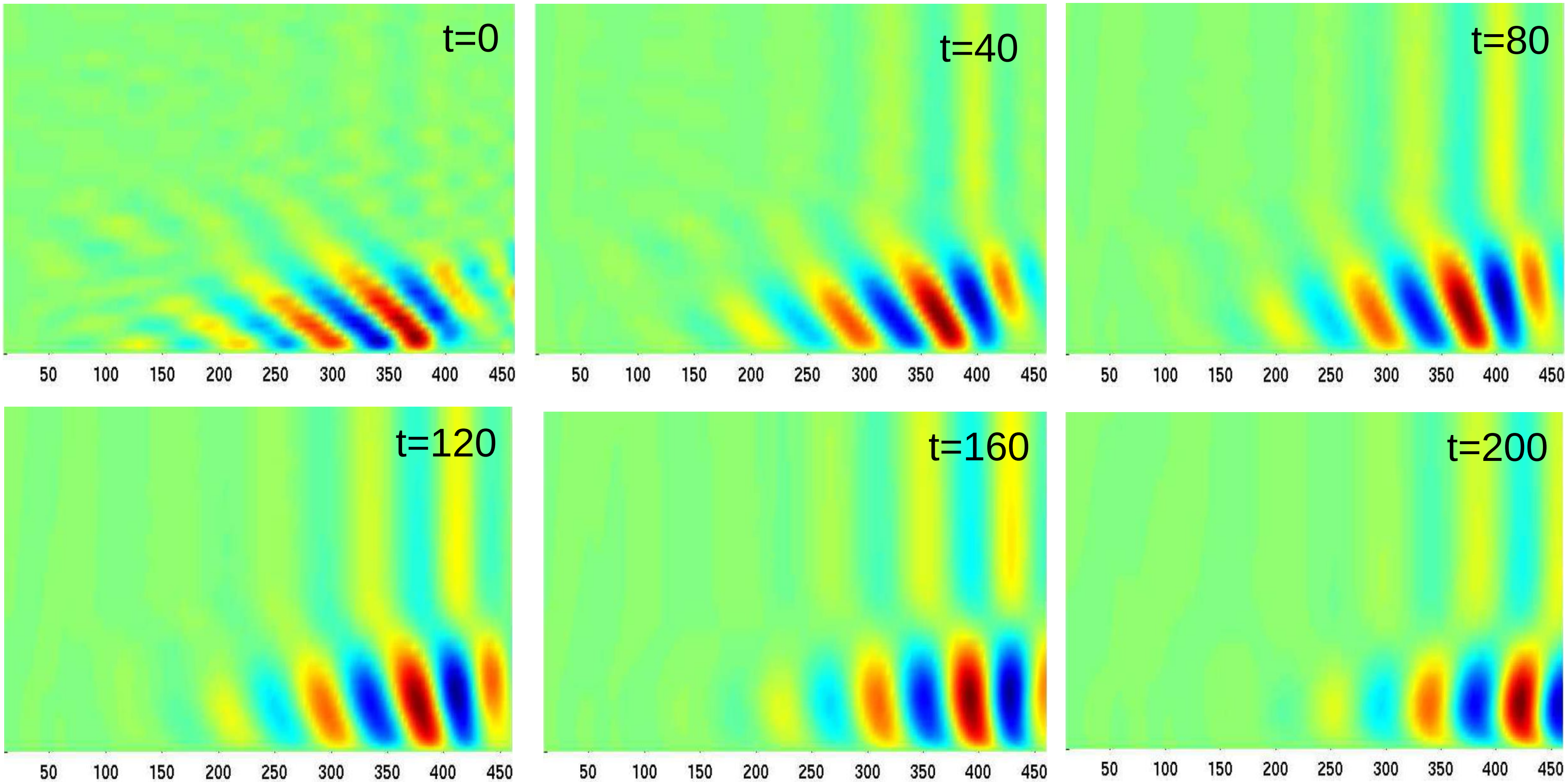


Amplification $G(t)$ for Poiseuille flow with $\mathrm{Re} = 1000, \alpha = 1$ (solid line)

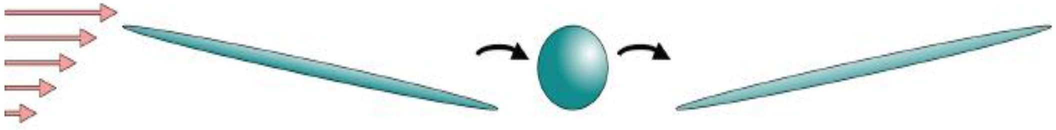
« By-pass transition »

Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

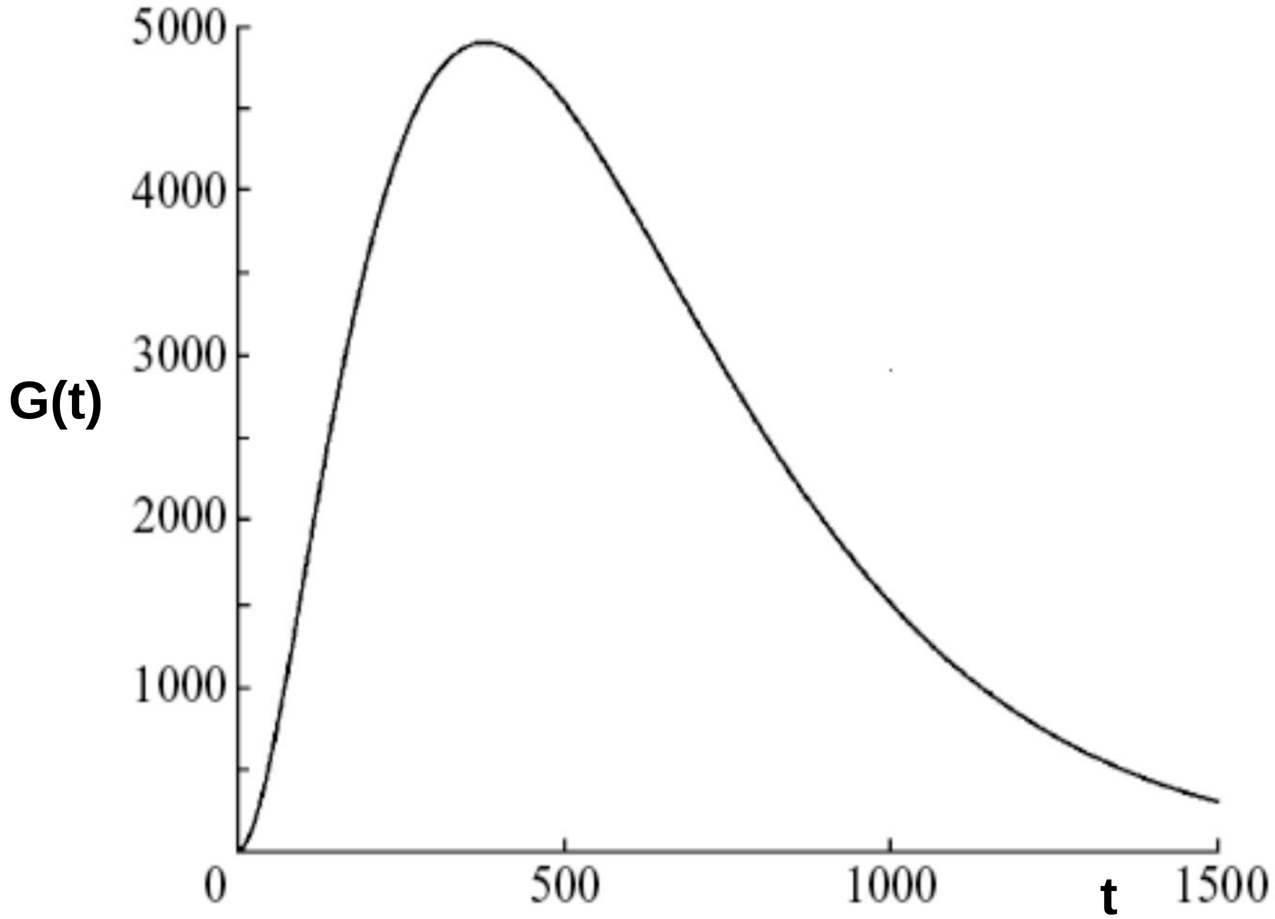
Optimal perturbation (here in boundary layer)



Orr mechanism



Plane Poiseuille flow
Lift-up mechanism (3D)



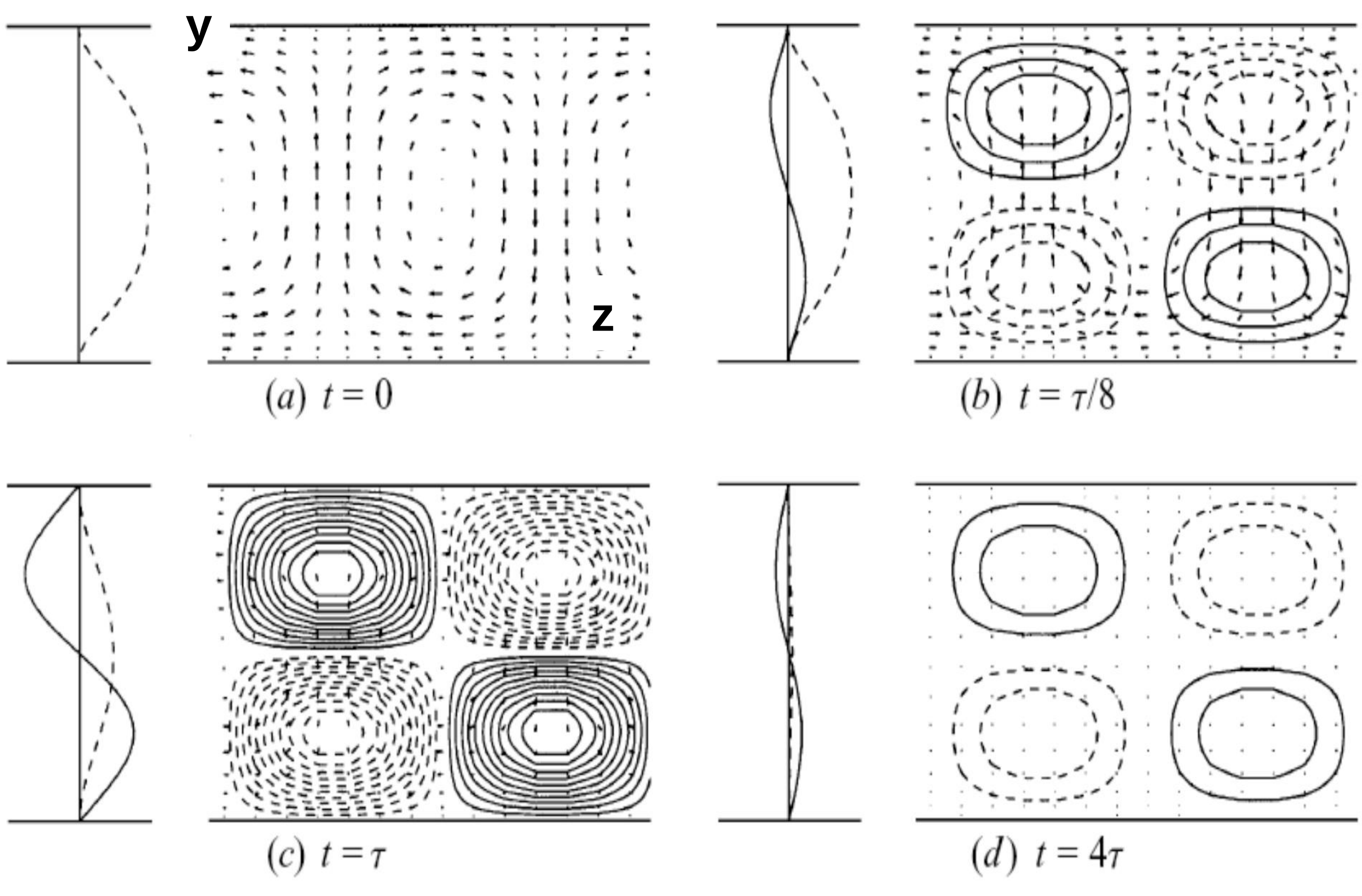
Re=5000, α=0, β=1

« By-pass transition »

Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

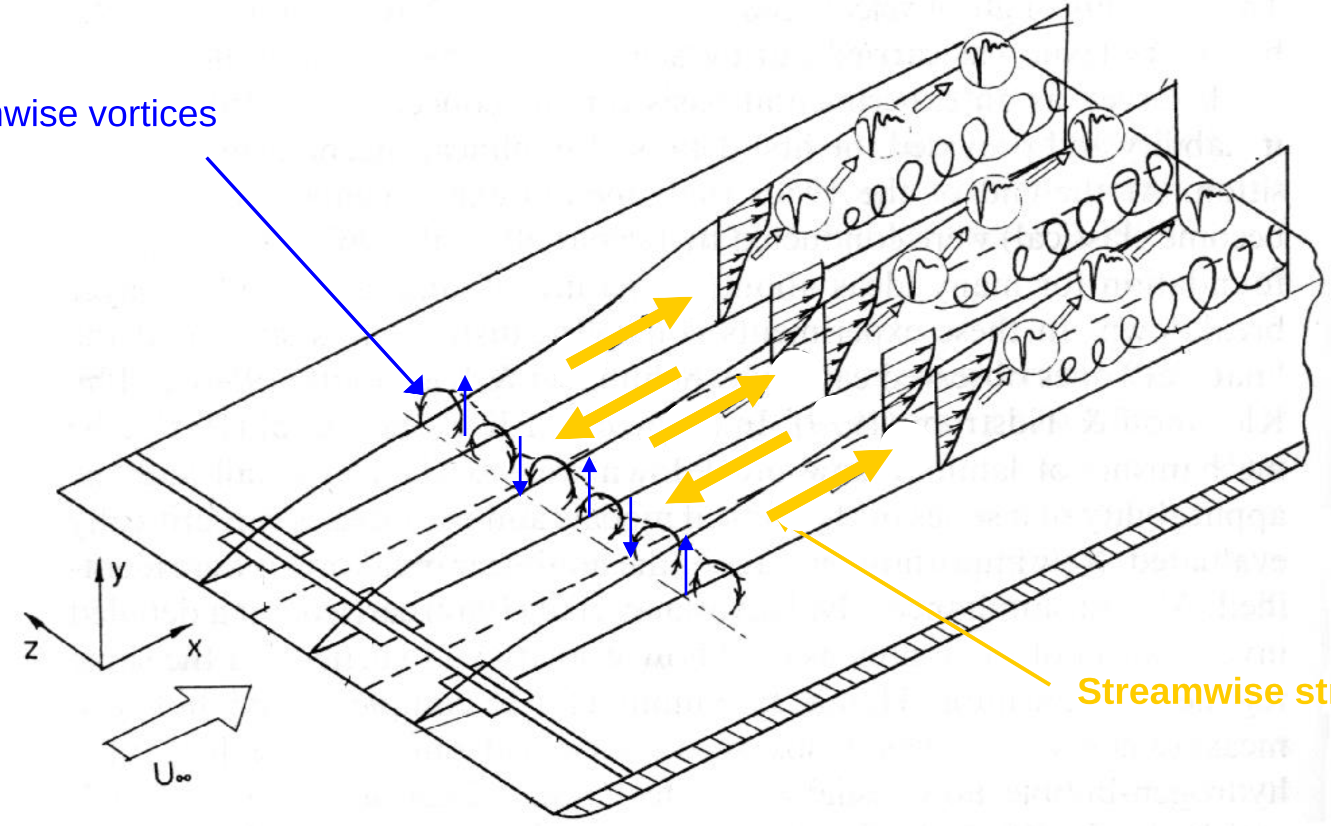
Lift-up mechanisms

Optimal transformation of vortices into streaks



Lift-up mechanism

Streamwise vortices

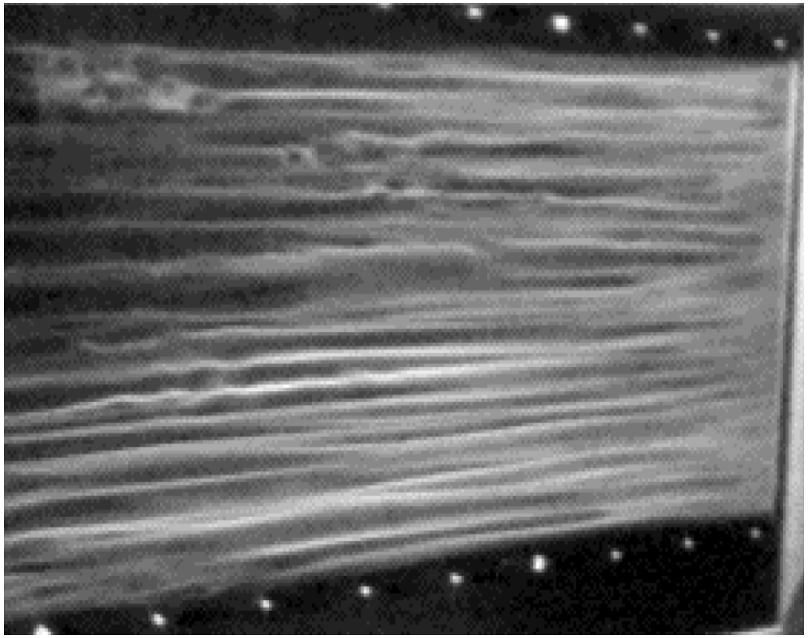


Streamwise streaks

Schmid & Henningson.(2001)

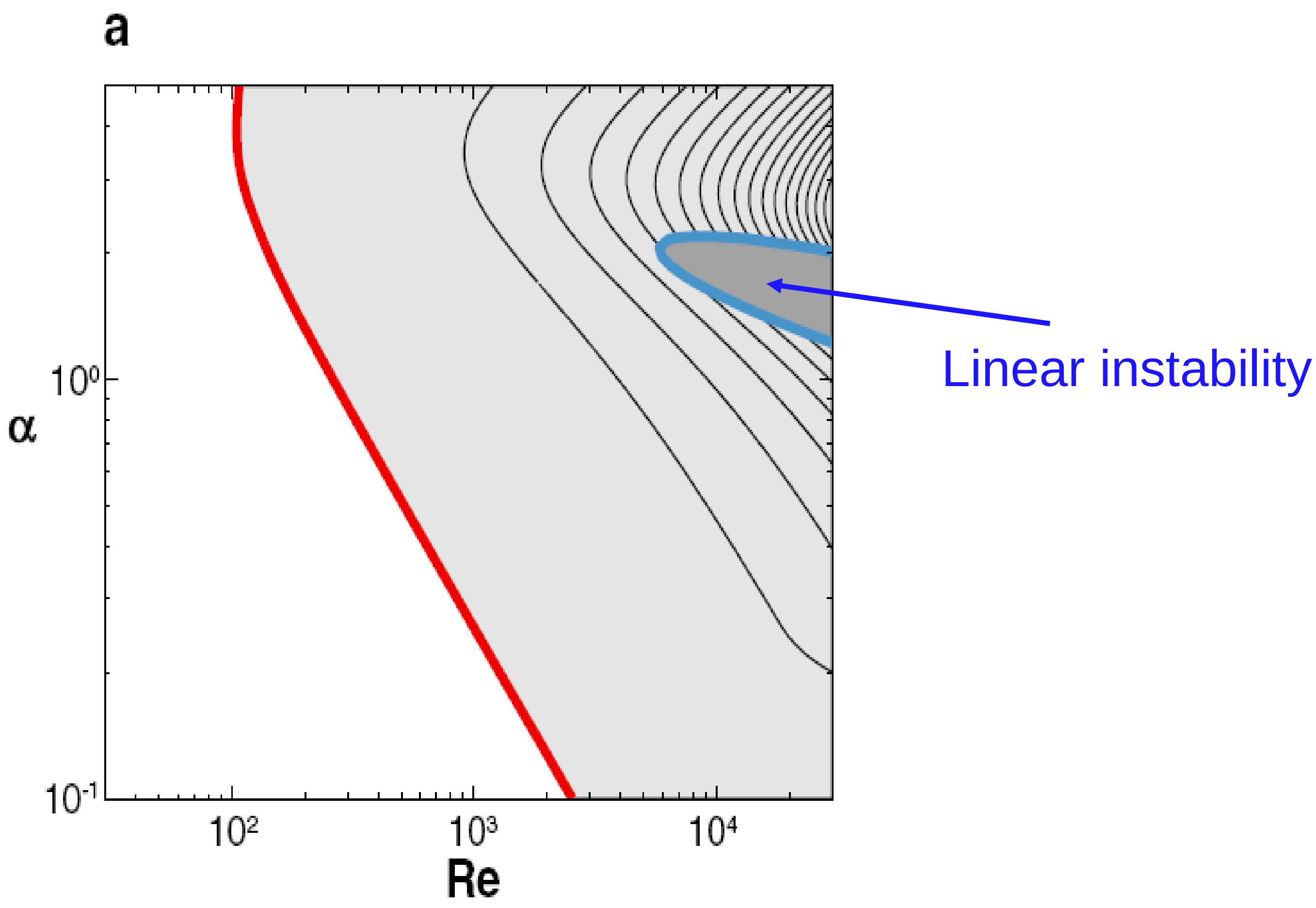
Lift-up mechanism

Optimal transformation of vortices into streaks



Alfredson & Matsubara (1996), streaky structures in the boudary layer

Isolines of maximum transient amplification
β=0



« By-pass transition »

Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

	$G_{\max}$ (10^{-3})	t_{max}	α	β
Plane Poiseuille	0.20 Re^2	0.076 Re	0	2.04
Couette	1.18 Re^2	0.117 Re	35/Re	1.6
Pipe	0.07 Re^2	0.048 Re	0	1
Boundary layer	1.50 Re^2	0.778 Re	0	0.65

Most flows are unstable...

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5. Benard-Marangoni
6. Taylor Couette-Centrifugal instability
7. Kelvin-Helmholtz
8. Inflection point theorem Rayleigh, Orr sommerfeld
9. Orr sommerfeld, transient growth
10. Spatial growth

2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi=\frac{1}{Re}\nabla^4\Psi$$

Basic flow + perturbation

$$\Psi(x,t)=\int U(y)dy+\psi(x,y,t)$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial\psi}{\partial x}=\frac{1}{Re}\nabla^4\psi$$

Dispersion relation

$$D(k, \omega) = 0$$

Temporal approach:
k is real; ω is complex
Perturbation grow and
decay in time!

Spatial approach:
ω is real; k is complex
Perturbations grow and
decay in space!

Shear layer is inviscidly unstable!

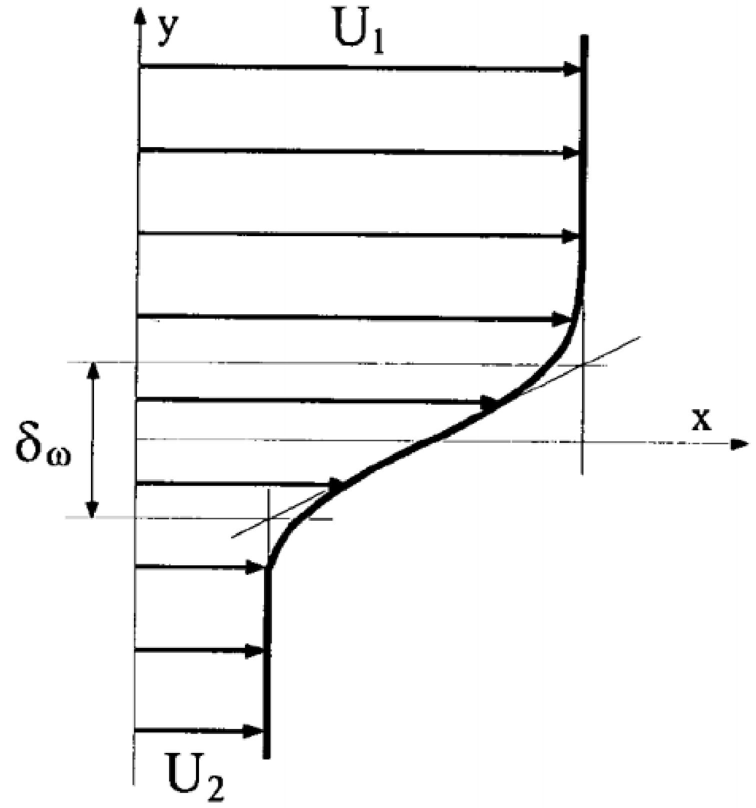
Hyperbolic tangent mixing layer

$$U(y) = \bar{U} + \frac{\Delta U}{2}\tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

Velocity ratio

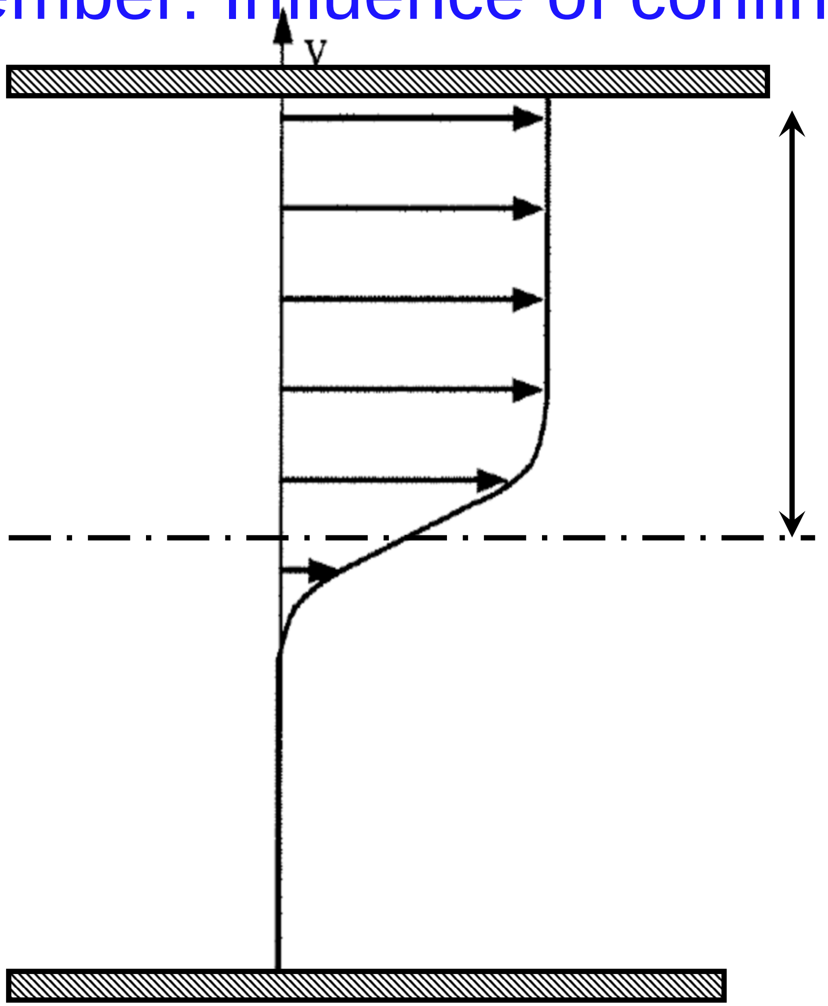
$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$



Why?

Only a necessary condition for instability!

Remember: Influence of confinement



R = 1

2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

$$U(y;R) = 1 + R\tanh y$$

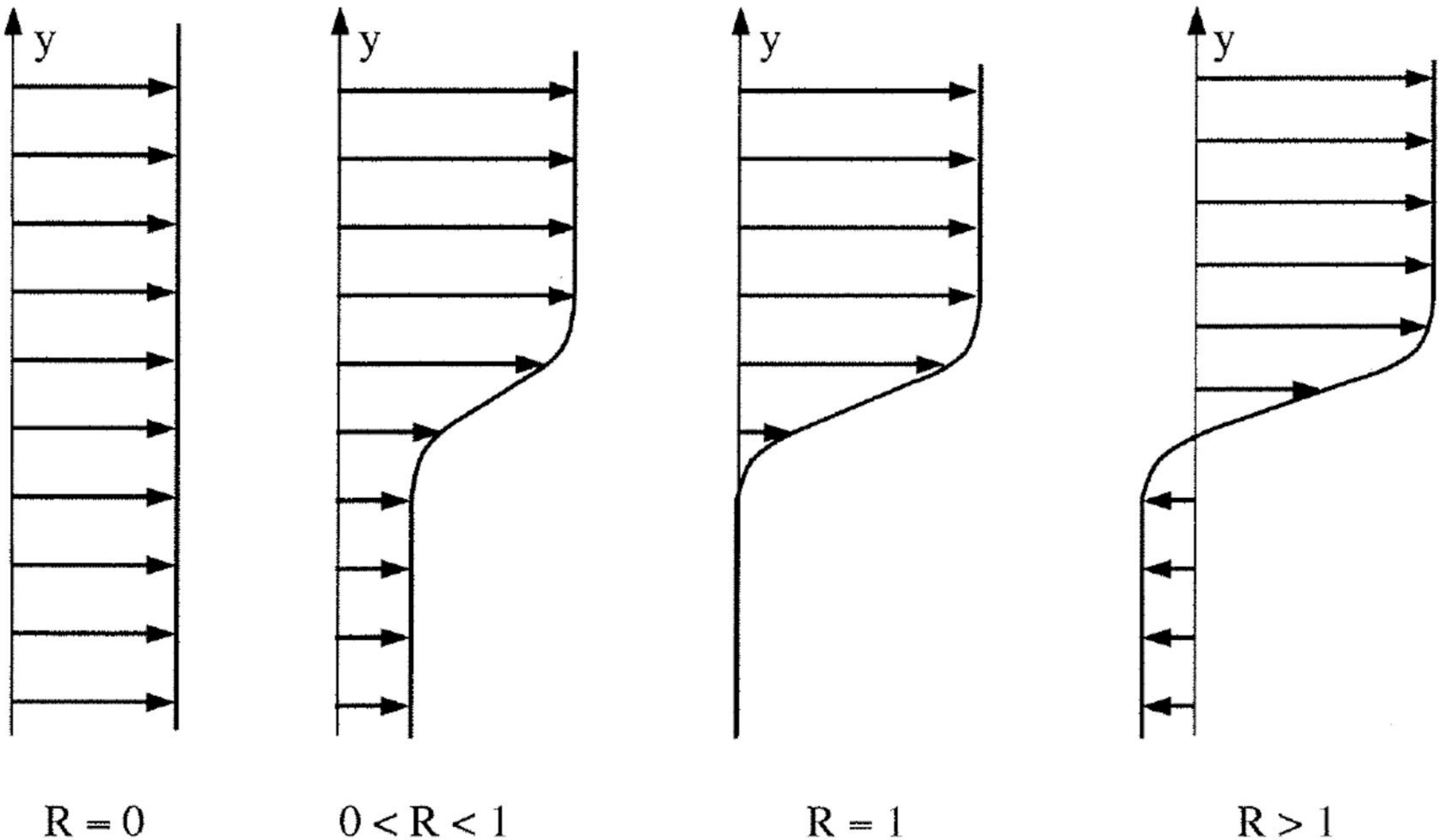
$$U_1(y) = \tanh\, y$$

Dispersion relation

$$\omega(k;R) = k + R\,\omega_1(k)$$

PARALLEL FLOW CONCEPTS

Effect of velocity ratio



2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

Temporal approach

$$\omega_1(k) = i\,\omega_{1,i}(k)$$

$$\omega_i(k;R) = R\,\omega_{1,i}(k)$$

$$c_r = \omega_r/k = 1$$

Temporal approach: k is real; ω is complex

2D PARALLEL FLOW CONCEPTS
Hyperbolic tangent mixing layer

Spatial approach

$$k + R\,\omega_1(k) = \omega$$

$$R \ll 1$$

$$\boxed{-k_i(\omega, R) \sim R\,\omega_{1,i}(\omega)}$$

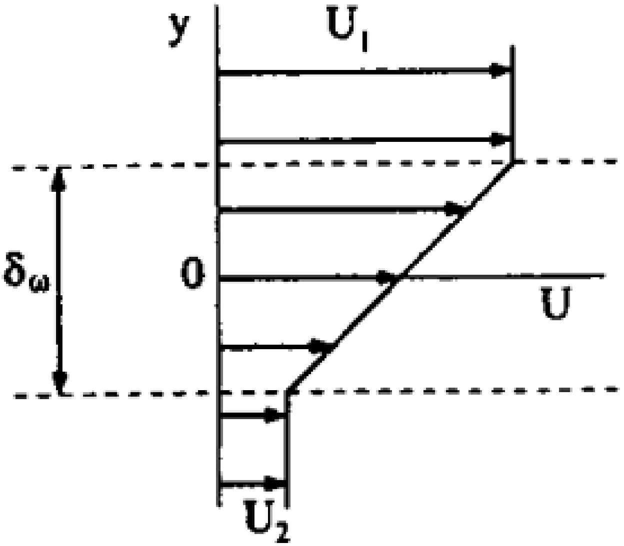
Gaster transformation

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$U(y) = \begin{cases} U_1 , & y > \delta_\omega/2 \\ (U_1+U_2)/2 + (U_1-U_2)y/\delta_\omega , & |y| < \delta_\omega/2 \\ U_2 , & y < -\delta_\omega/2 \end{cases}$$

$$\phi'' - k^2\phi = 0$$



$$\phi_1(y) = A_1\, e^{-ky}, \quad y > \delta_\omega/2 ,$$
$$\phi_2(y) = B_2\, e^{ky}, \quad y < -\delta_\omega/2 ,$$
$$\phi_0(y) = A_0\, e^{-ky} + B_0\, e^{ky}, \quad |y| < \delta_\omega/2$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$
\begin{aligned}
A_1\,\mathrm{e}^{-k\delta_\omega/2} &= A_0\,\mathrm{e}^{-k\delta_\omega/2} + B_0\,\mathrm{e}^{k\delta_\omega/2}\,,\\
B_2\,\mathrm{e}^{-k\delta_\omega/2} &= A_0\,\mathrm{e}^{k\delta_\omega/2} + B_0\,\mathrm{e}^{-k\delta_\omega/2}\,,\\
-k(U_1 - c)A_1\,\mathrm{e}^{-k\delta_\omega/2} &= k(U_1 - c)(-A_0\,\mathrm{e}^{-k\delta_\omega/2} + B_0\,\mathrm{e}^{k\delta_\omega/2})\\
&\quad - \frac{\Delta U}{\delta_\omega}(A_0\,\mathrm{e}^{-k\delta_\omega/2} + B_0\,\mathrm{e}^{k\delta_\omega/2})\,,\\
k(U_2 - c)B_2\,\mathrm{e}^{-k\delta_\omega/2} &= k(U_2 - c)(-A_0\,\mathrm{e}^{k\delta_\omega/2} + B_0\,\mathrm{e}^{-k\delta_\omega/2})\\
&\quad - \frac{\Delta U}{\delta_\omega}(A_0\,\mathrm{e}^{k\delta_\omega/2} + B_0\,\mathrm{e}^{-k\delta_\omega/2})\,.
\end{aligned}
$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$-\frac{\Delta U}{\delta_\omega} A_0 \, e^{-k\delta_\omega/2} + \left[2k(U_1 - c) - \frac{\Delta U}{\delta_\omega}\right] B_0 \, e^{k\delta_\omega/2} = 0$$

$$\left[2k(U_2 - c) + \frac{\Delta U}{\delta_\omega}\right] A_0 \, e^{k\delta_\omega/2} + \frac{\Delta U}{\delta_\omega} B_0 \, e^{-k\delta_\omega/2} = 0$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$4(k\delta_\omega)^2(c-\bar{U})^2-\left[(k\delta_\omega-1)^2-\mathrm{e}^{-2k\delta_\omega}\right]\Delta U^2=0$$

$$k\delta_\omega \mapsto 2k,\ c/\bar{U}\mapsto c$$

$$4k^2(c-1)^2-R^2\left[(2k-1)^2-\mathrm{e}^{-4k}\right]=0$$

$$\boxed{c\equiv\frac{\omega}{k}=1\pm\frac{R}{2k}\left[(2k-1)^2-\mathrm{e}^{-4k}\right]^{1/2}}$$

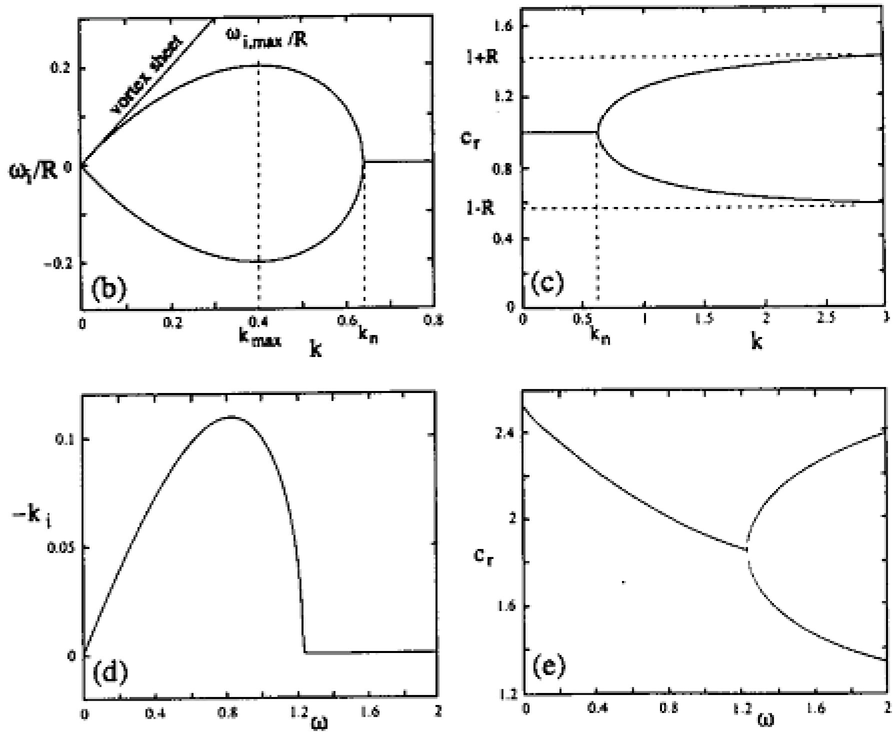
2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

Temporal approach

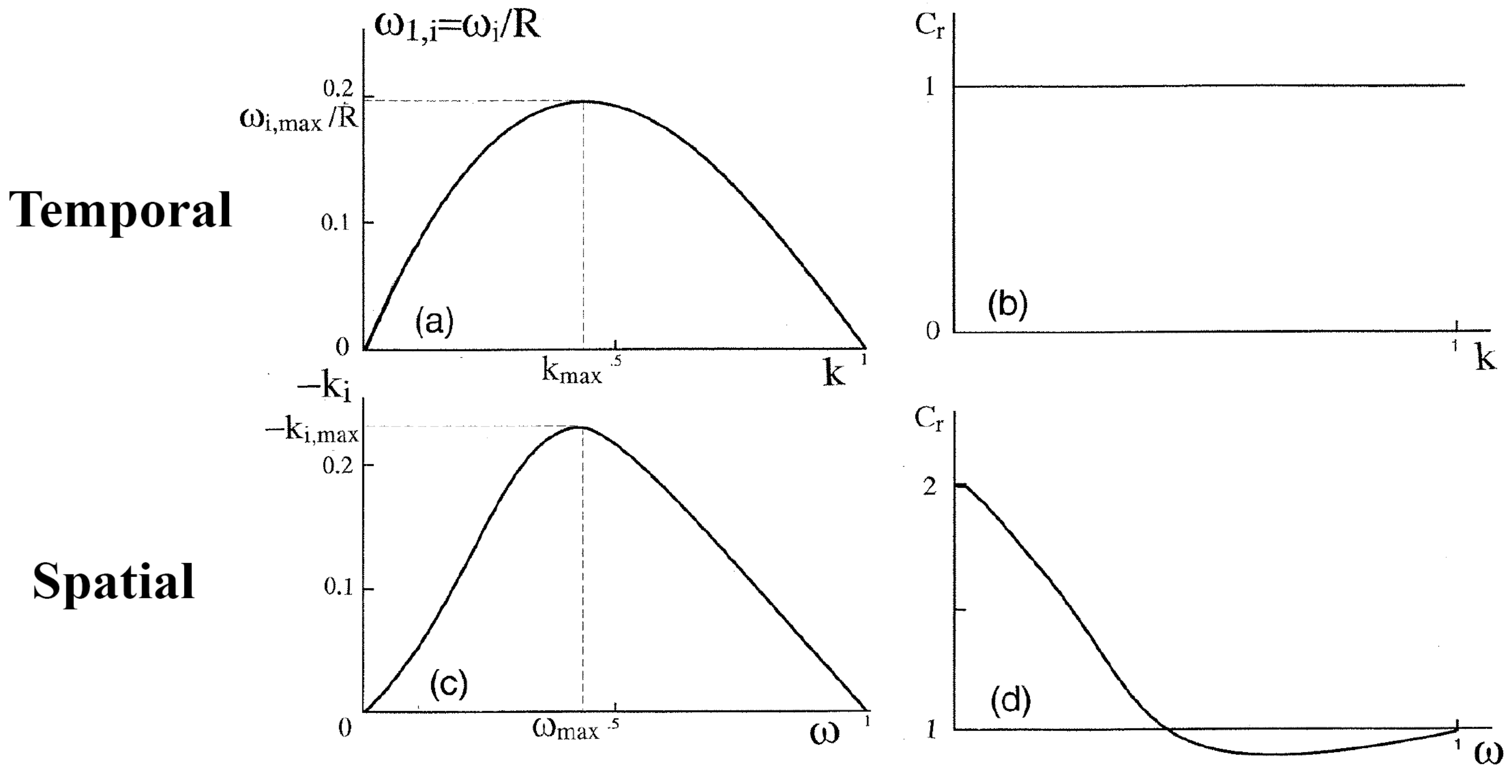
$$2k_n - 1 = e^{-2k_n}$$

Spatial approach
R = 0.5



2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer



Michalke (1964, 65)

Solving a spatial instability problem
ex: Rayleigh equation

Back to temporal stability analysis!
How to solve Rayleigh equation for real k and complex ω?

We fix k, we need to find all ω and ψ such that

$$k\left(U\left(\frac{d^2}{dy^2}-k^2\right)-U''(y)\right)\psi = \omega\left(\frac{d^2}{dy^2}-k^2\right)\psi$$

$$\psi(-L)=\psi(L)=0$$

Formally,

$$\mathcal{A}\psi = c\,\mathcal{E}\psi$$

$c=\omega/k$

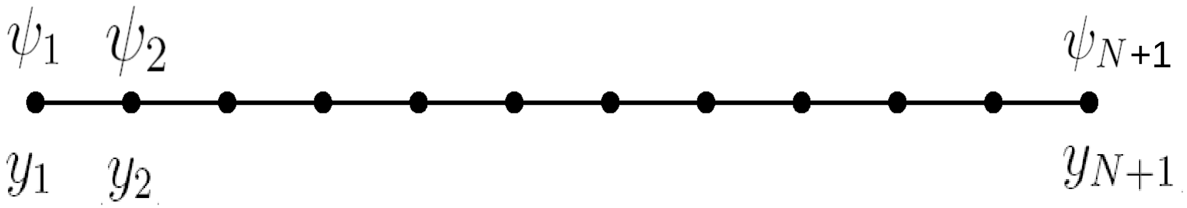
Discretize

$$\mathbf{A}\Psi = c\mathbf{E}\Psi$$

Generalized eigenvalue problem

How to solve Rayleigh equation for real k and complex ω?

Finite differences of order 1



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \\ \psi(y_{N+1}) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \psi_{N+1} \end{pmatrix} \qquad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \\ \psi''(y_{N+1}) \end{pmatrix}$$

How to solve Rayleigh equation for real k and complex ω?

Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

How to solve Rayleigh equation for complex k and real ω?

We fix ω, we need to find all k and ψ such that

$$\boxed{\begin{gathered} k\left(U\left(\frac{d^2}{dy^2}-k^2\right)-U''(y)\right)\psi = \omega\left(\frac{d^2}{dy^2}-k^2\right)\psi \\ \psi(-L)=\psi(L)=0 \end{gathered}}$$

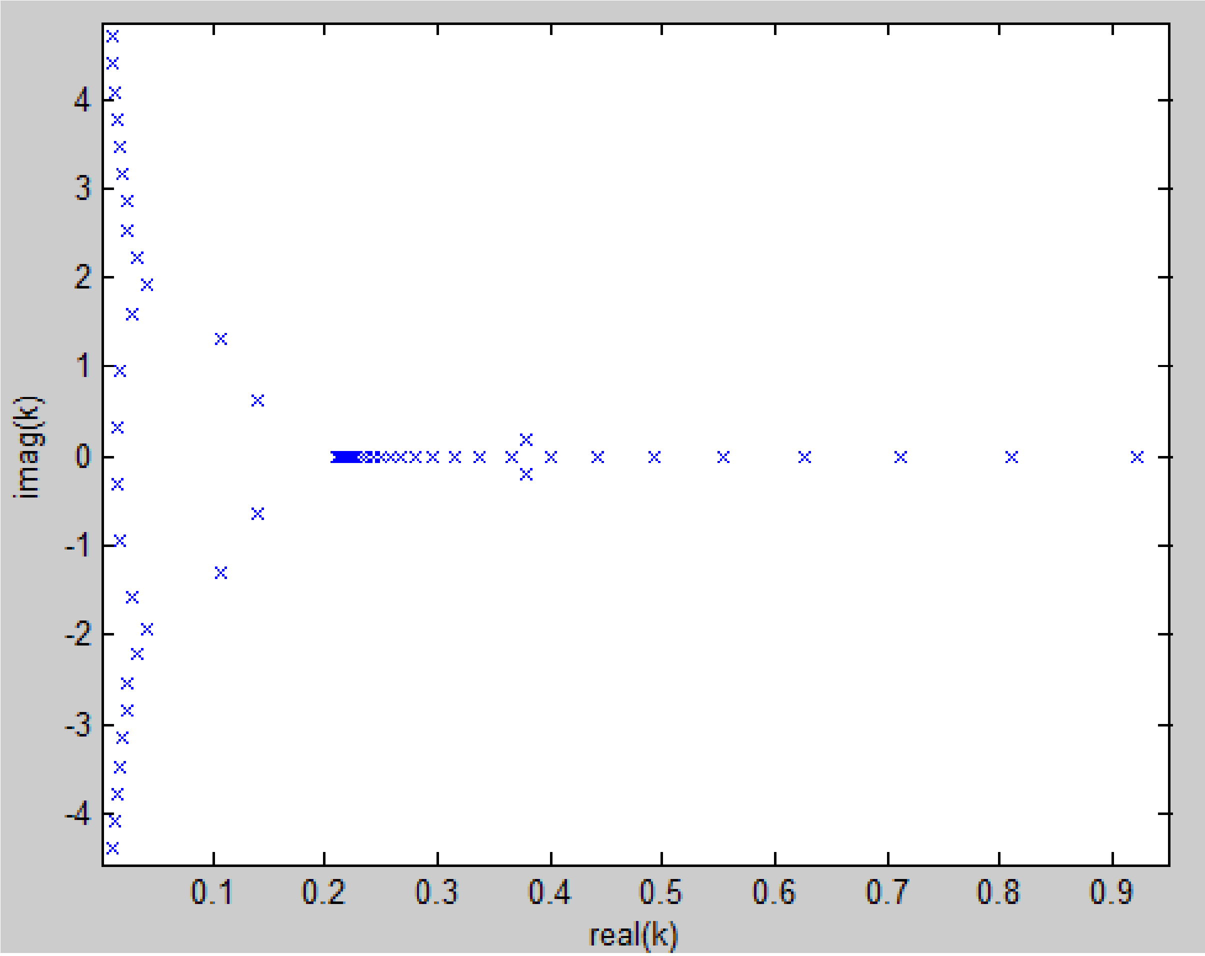
Formally,

$$\boxed{\left(A_0(\omega,y)+kA_1(\omega,y)+k^2A_2(\omega,y)+k^3A_3(\omega,y)\right)\ \psi = 0}$$

Polynomial eigenvalue problem

Many more eigenvalues
(for Rayleigh equation: 3 x more!)

U=1+0.9*tanh(y); ω=0.4; L=5



Which of these waves are unstable?

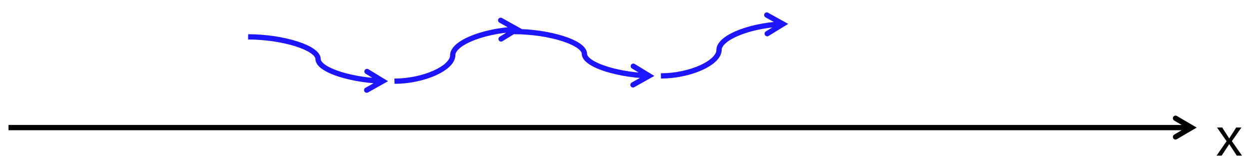
$\mathrm{Im}(k)<0$?

$\mathrm{Im}(k)>0$?

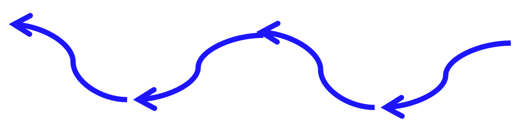
Recall : $\exp(i(kx-\omega t))$

The stability of a spatial wave can be only determined if one knows in which direction it propagates!

k^+ waves propagate towards positive x



k^- waves propagate towards negative x

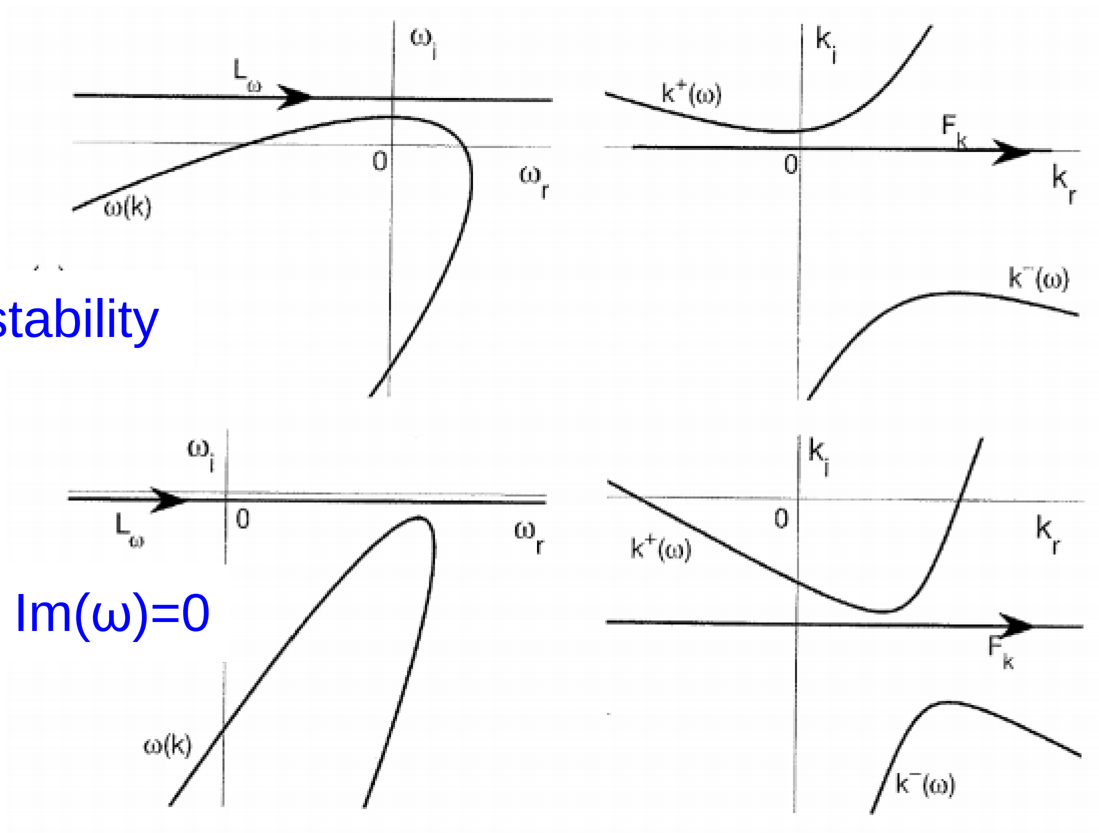


However, determining this direction of propagation is particularly difficult, except in the case of a temporally stable flow.

The addition of a positive imaginary offset to the frequency makes the temporal problem stable!
This separates the spatial waves into k+ and k- waves.

offset spatial stability

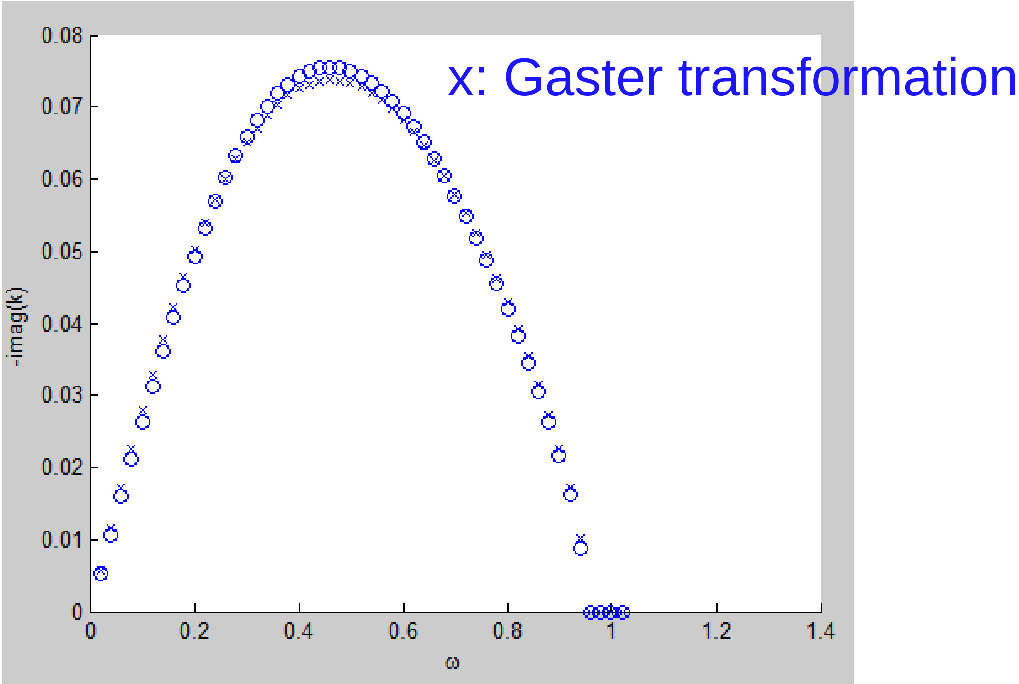
spatial stability: Im(ω)=0



The branches are then followed by continuity

Validity of Gaster transformation?

R=0.4

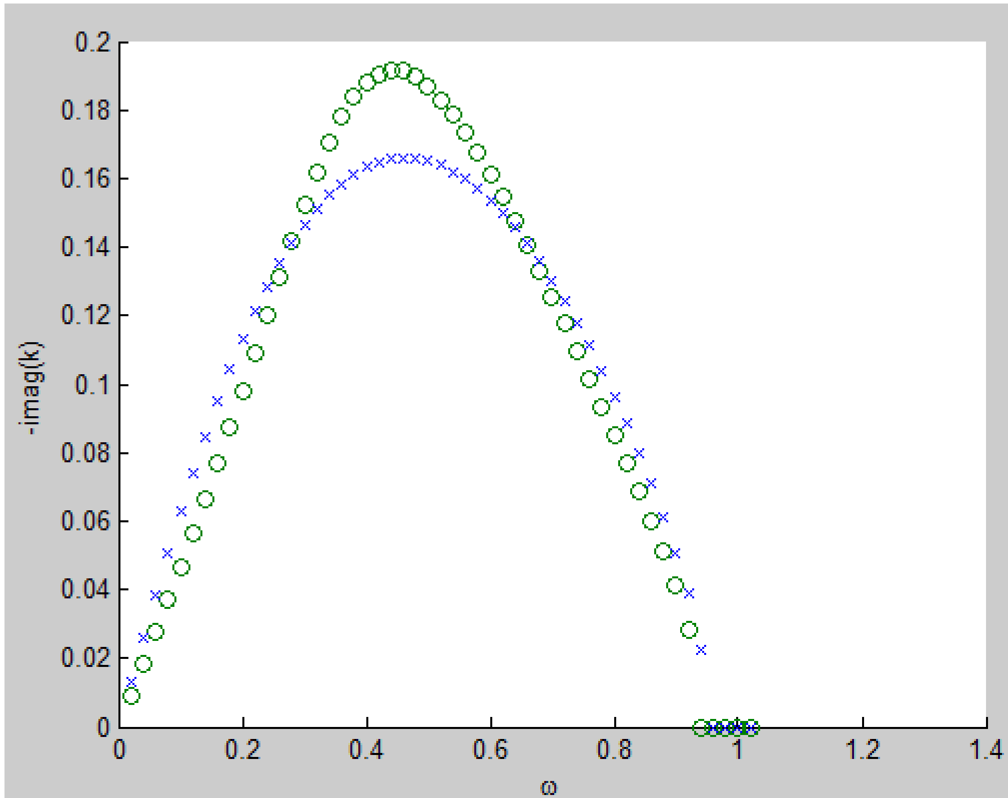


$$R \ll 1$$

$$\boxed{-k_i(\omega, R) \sim R\omega_{1,i}(\omega)}$$

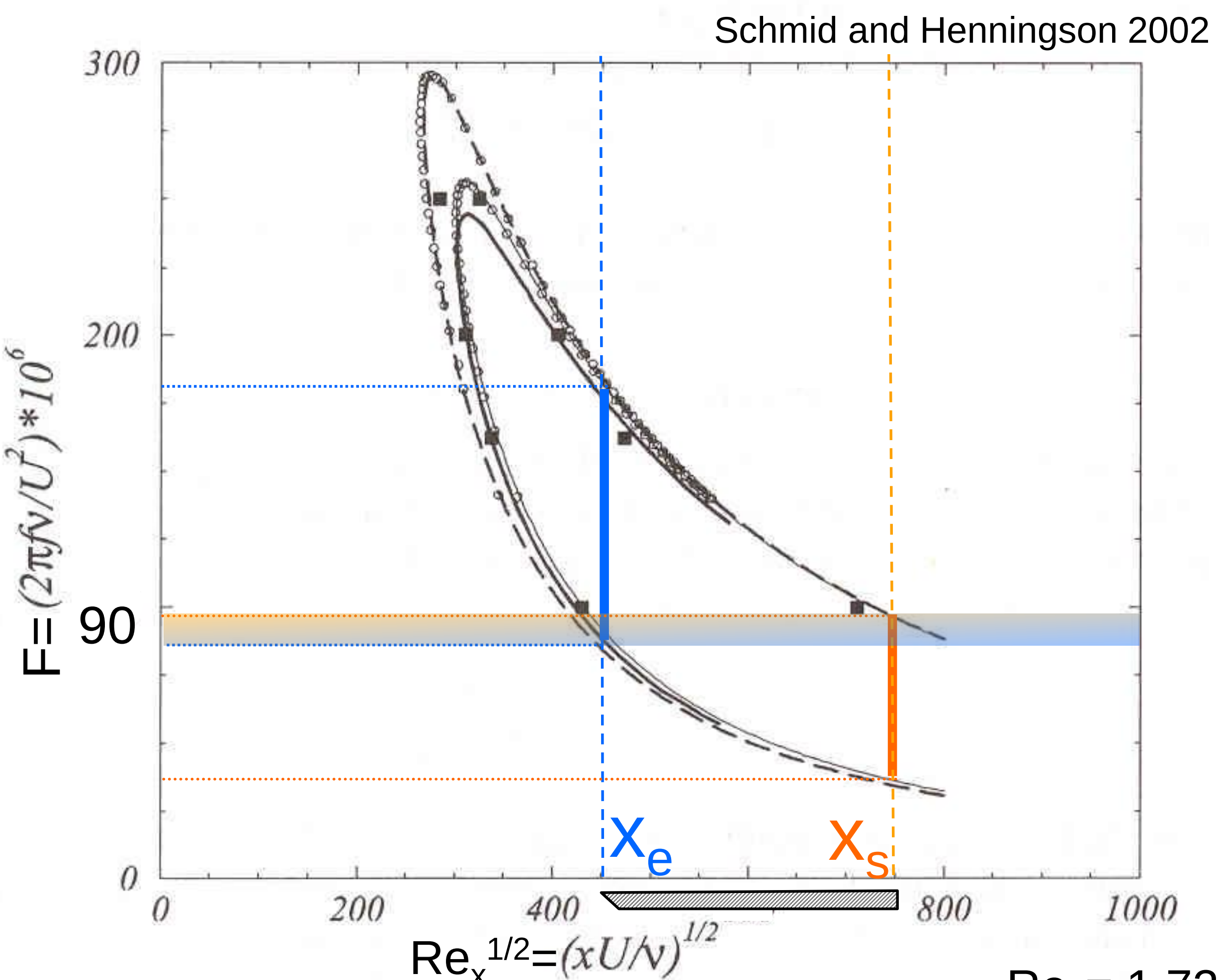
Validity of Gaster transformation?

R=0.9 x: Gaster transformation



$$-k_i(\omega, R) \sim R\omega_{1,i}(\omega)$$

Neutral curve



$Re_\delta = 1.72 Re_x^{1/2}$