Most flows are unstable...

1. Intro-definitions
2. Rayleigh-Taylor
3. Rayleigh Plateau (destabilization through surface tension)
4. Rayleigh-Benard (convection)
5. Benard-Marangoni
6. Taylor Couette-Centrifugal instability
7. Kelvin-Helmholtz
8. Inflection point theorem Rayleigh
9. Orr sommerfeld, transient growth
10. Spatial growth

SPATIALLY DEVELOPING SHEAR FLOWS

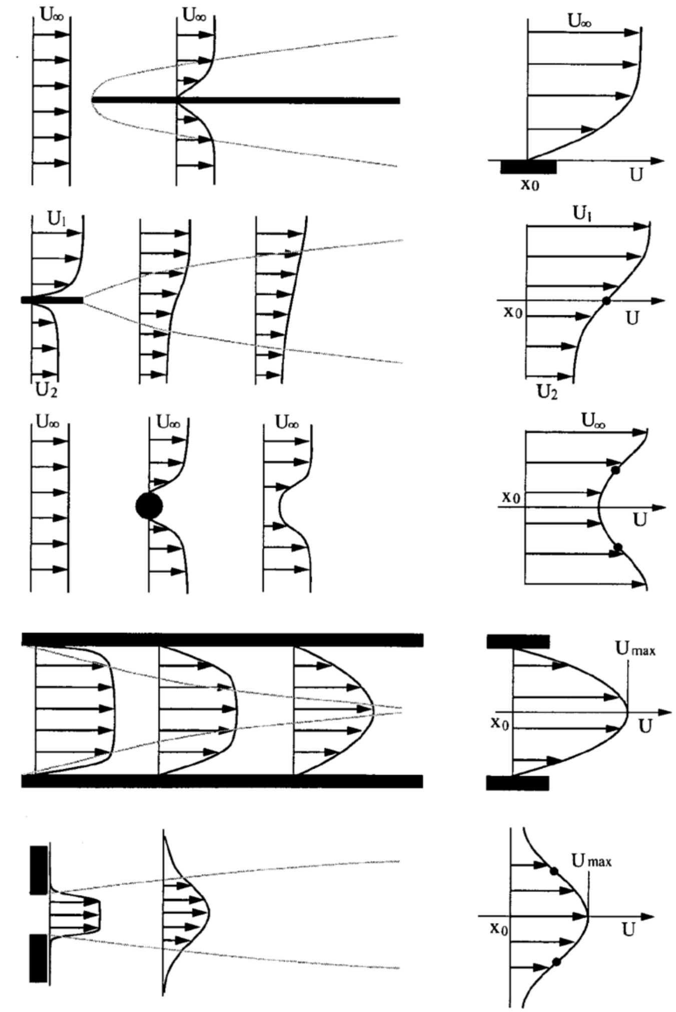
Flat plate boundary layer

Mixing layer

Cylinder wake

Plane channel flow

2D jet



2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi=\frac{1}{Re}\nabla^4\Psi$$

Basic flow + perturbation

$$\Psi(x,t)=\int U(y)dy+\psi(x,y,t)$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial \psi}{\partial x}=\frac{1}{Re}\nabla^4\psi$$

Dispersion relation

$$D(k, \omega) = 0$$

Temporal approach:
k is real; ω is complex
Perturbation grow and
decay in time!

Spatial approach:
ω is real; k is complex
Perturbations grow and
decay in space!

Shear layer is inviscidly unstable!

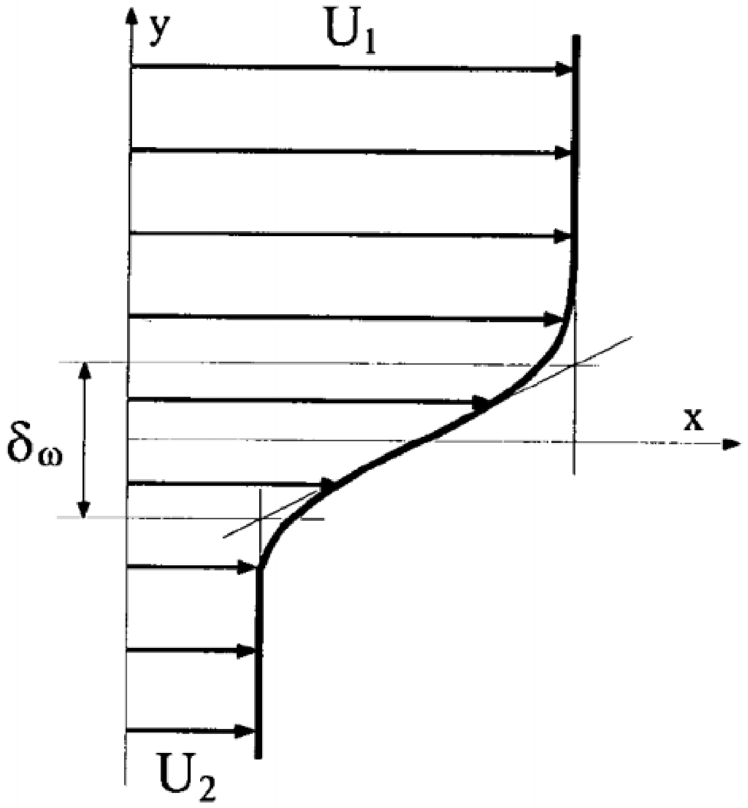
Hyperbolic tangent mixing layer

$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

Velocity ratio

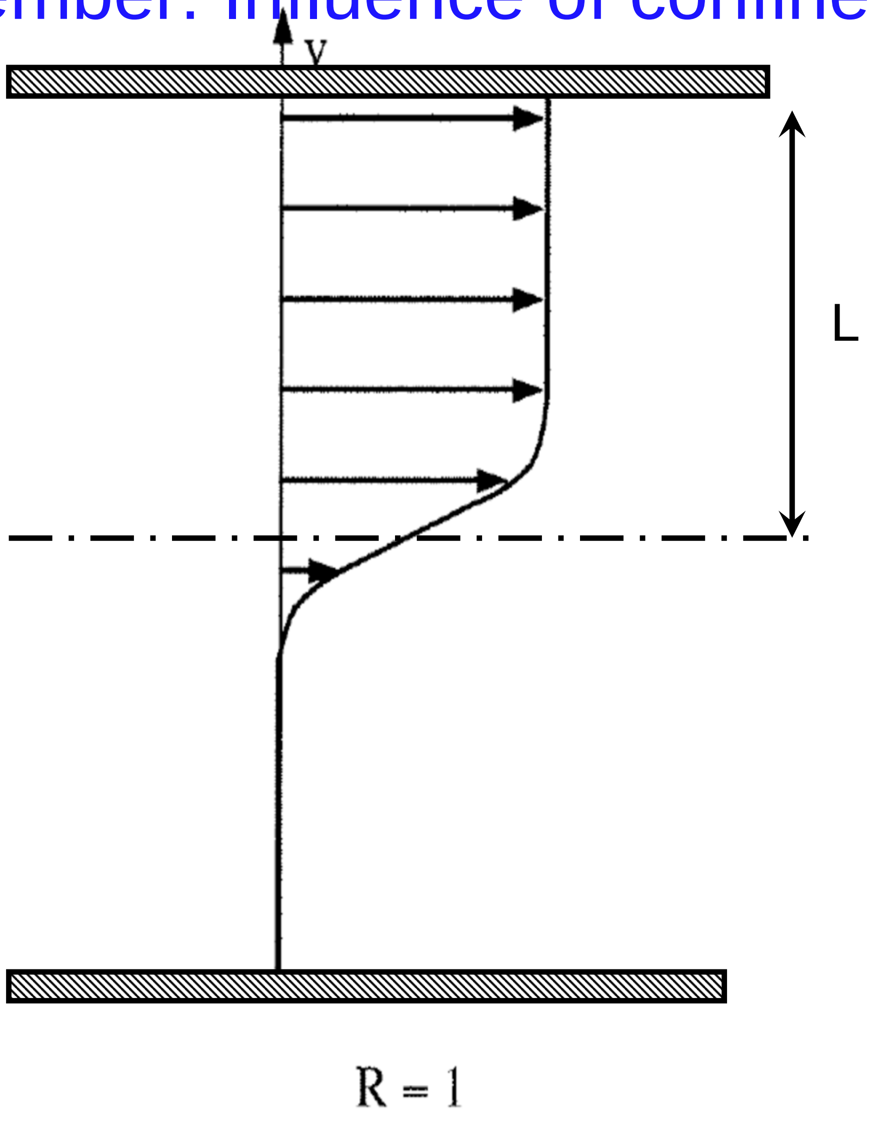
$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$



Why?

Only a necessary condition for instability!

Remember: Influence of confinement



2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

$$U(y;R) = 1 + R\tanh y$$

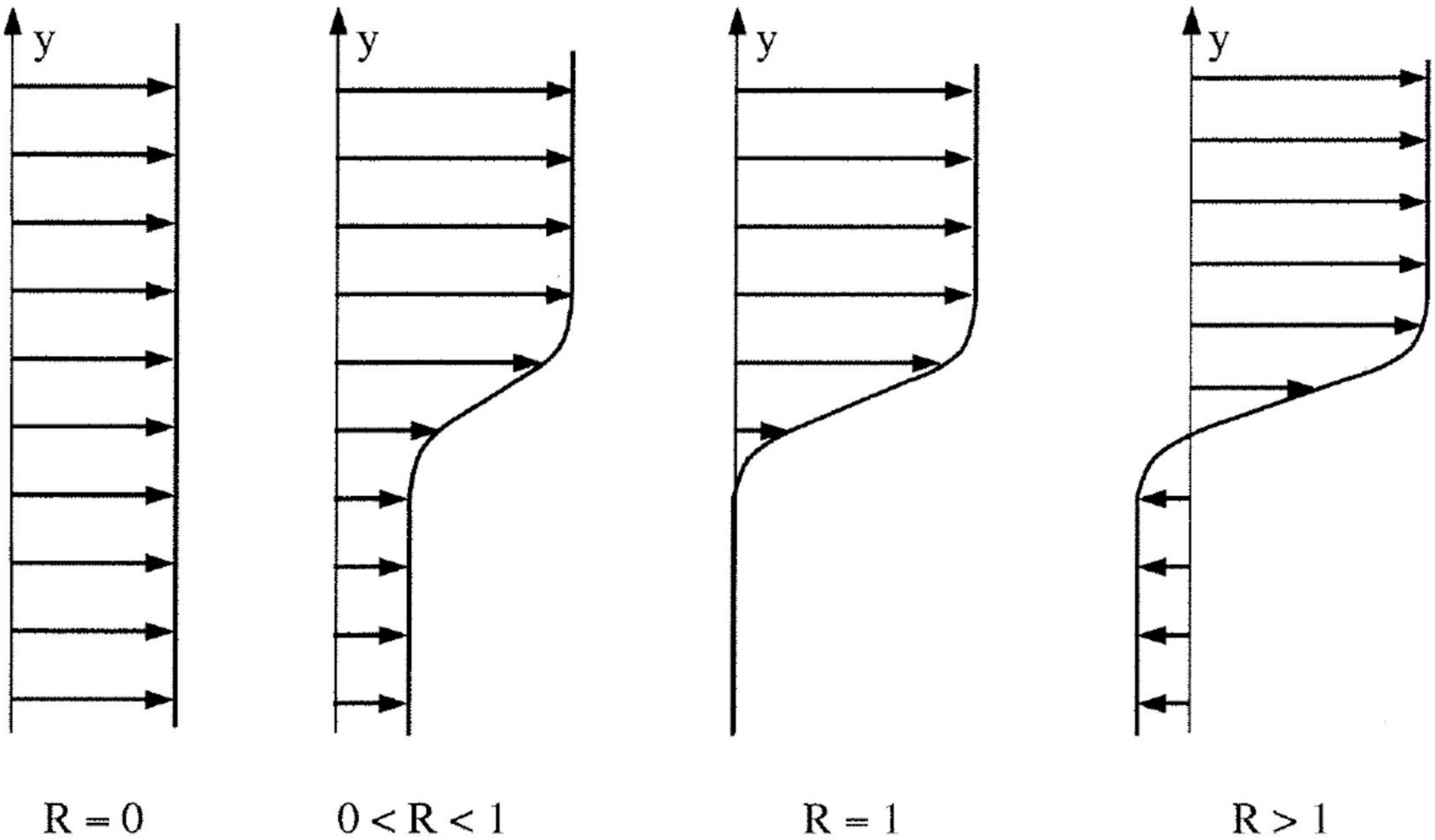
$$U_1(y) = \tanh y$$

Dispersion relation

$$\omega(k;R) = k + R\,\omega_1(k)$$

PARALLEL FLOW CONCEPTS

Effect of velocity ratio



2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

Temporal approach

$$\omega_1(k) = i\,\omega_{1,i}(k)$$

$$\omega_i(k;R) = R\,\omega_{1,i}(k)$$

$$c_r = \omega_r/k = 1$$

Temporal approach: k is real; ω is complex

2D PARALLEL FLOW CONCEPTS
Hyperbolic tangent mixing layer

Spatial approach

$$k + R\,\omega_1(k) = \omega$$

$$R \ll 1$$

$$\boxed{-k_i(\omega, R) \sim R\,\omega_{1,i}(\omega)}$$

Gaster transformation

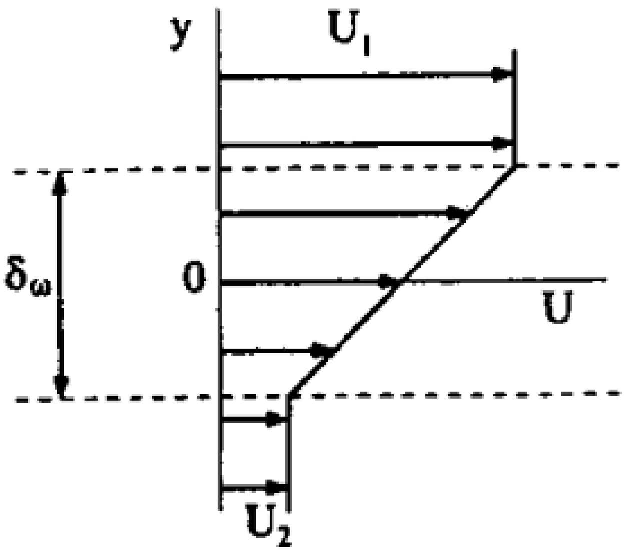
2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$U(y) = \begin{cases} U_1 \,, & y > \delta_\omega/2 \\ (U_1 + U_2)/2 + (U_1 - U_2)y/\delta_\omega \,, & |y| < \delta_\omega/2 \\ U_2 \,, & y < -\delta_\omega/2 \end{cases}$$

$$\phi'' - k^2\phi = 0$$

$$\phi_1(y) = A_1\,\mathrm{e}^{-ky}, \quad y > \delta_\omega/2\,,$$
$$\phi_2(y) = B_2\,\mathrm{e}^{ky}, \quad y < -\delta_\omega/2\,,$$
$$\phi_0(y) = A_0\,\mathrm{e}^{-ky} + B_0\,\mathrm{e}^{ky}, \quad |y| < \delta_\omega/2$$



2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$
\begin{aligned}
A_1\,\mathrm{e}^{-k\delta_\omega/2} &= A_0\,\mathrm{e}^{-k\delta_\omega/2} + B_0\,\mathrm{e}^{k\delta_\omega/2}\,,\\
B_2\,\mathrm{e}^{-k\delta_\omega/2} &= A_0\,\mathrm{e}^{k\delta_\omega/2} + B_0\,\mathrm{e}^{-k\delta_\omega/2}\,,\\
-k(U_1 - c)A_1\,\mathrm{e}^{-k\delta_\omega/2} &= k(U_1 - c)(-A_0\,\mathrm{e}^{-k\delta_\omega/2} + B_0\,\mathrm{e}^{k\delta_\omega/2})\\
&\quad - \frac{\Delta U}{\delta_\omega}(A_0\,\mathrm{e}^{-k\delta_\omega/2} + B_0\,\mathrm{e}^{k\delta_\omega/2})\,,\\
k(U_2 - c)B_2\,\mathrm{e}^{-k\delta_\omega/2} &= k(U_2 - c)(-A_0\,\mathrm{e}^{k\delta_\omega/2} + B_0\,\mathrm{e}^{-k\delta_\omega/2})\\
&\quad - \frac{\Delta U}{\delta_\omega}(A_0\,\mathrm{e}^{k\delta_\omega/2} + B_0\,\mathrm{e}^{-k\delta_\omega/2})\,.
\end{aligned}
$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$-\frac{\Delta U}{\delta_\omega} A_0\, e^{-k\delta_\omega/2} + \left[2k(U_1 - c) - \frac{\Delta U}{\delta_\omega}\right] B_0\, e^{k\delta_\omega/2} = 0$$

$$\left[2k(U_2 - c) + \frac{\Delta U}{\delta_\omega}\right] A_0\, e^{k\delta_\omega/2} + \frac{\Delta U}{\delta_\omega} B_0\, e^{-k\delta_\omega/2} = 0$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$4(k\delta_\omega)^2(c-\bar{U})^2 - \left[(k\delta_\omega - 1)^2 - e^{-2k\delta_\omega}\right]\Delta U^2 = 0$$

$$k\delta_\omega \mapsto 2k,\ c/\bar{U} \mapsto c$$

$$4k^2(c-1)^2 - R^2\left[(2k-1)^2 - e^{-4k}\right] = 0$$

$$\boxed{c \equiv \frac{\omega}{k} = 1 \pm \frac{R}{2k}\left[(2k-1)^2 - e^{-4k}\right]^{1/2}}$$

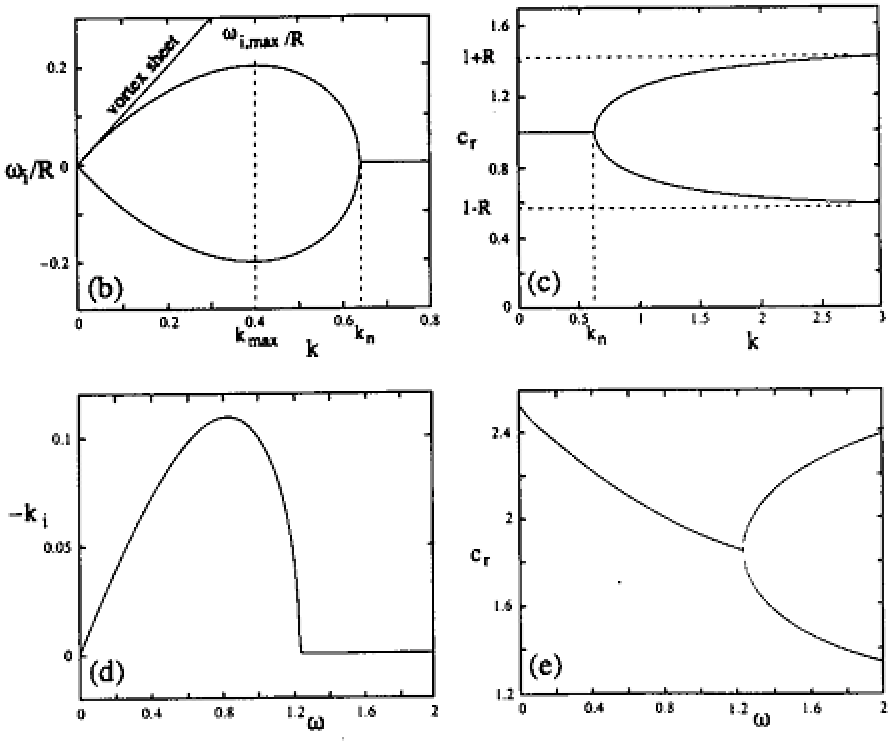
2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

Temporal approach

$$2k_n - 1 = e^{-2k_n}$$

Spatial approach
R = 0.5

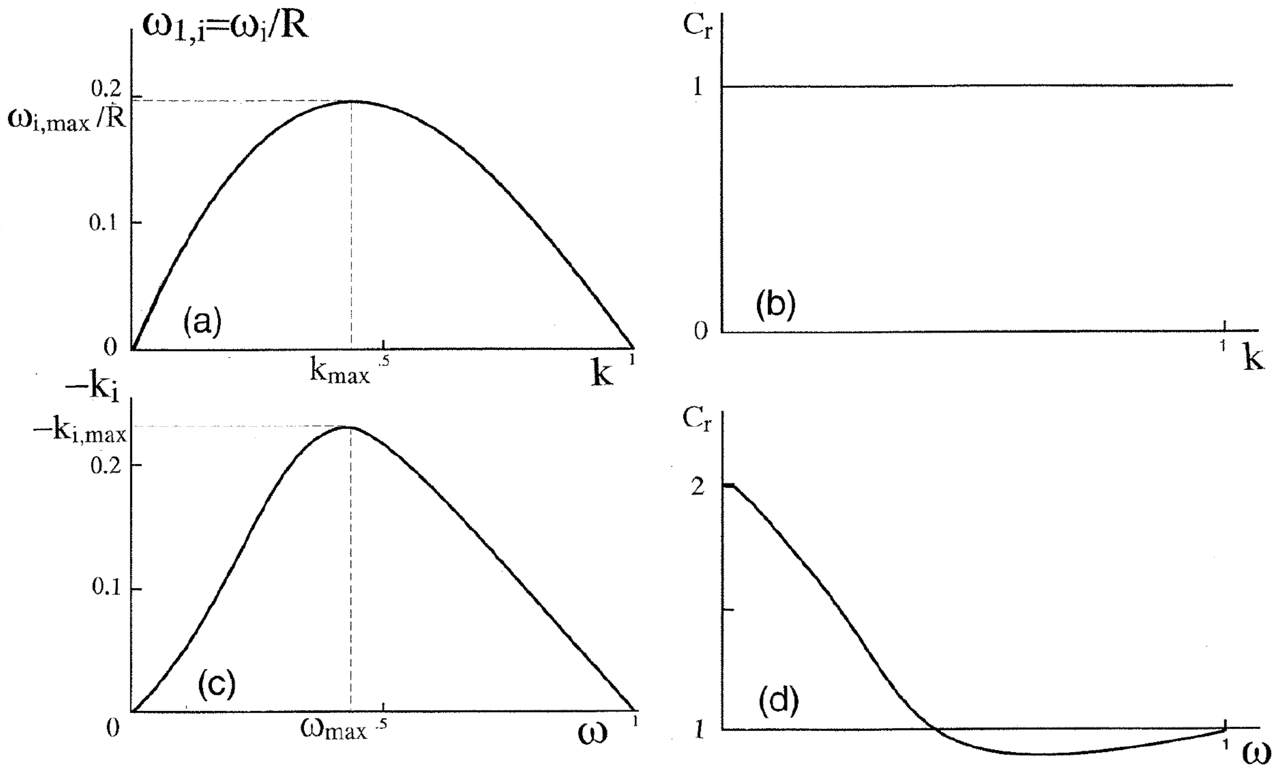


2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

Temporal

Spatial



Michalke (1964, 65)

Solving a spatial instability problem
ex: Rayleigh equation

Back to temporal stability analysis!
How to solve Rayleigh equation for real k and complex ω?

We fix k, we need to find all ω and ψ such that

$$k\left(U\left(\frac{d^2}{dy^2}-k^2\right)-U''(y)\right)\psi = \omega\left(\frac{d^2}{dy^2}-k^2\right)\psi$$

$$\psi(-L)=\psi(L)=0$$

Formally,

$$\boxed{\mathcal{A}\psi = c\,\mathcal{E}\psi}$$

$c=\omega/k$

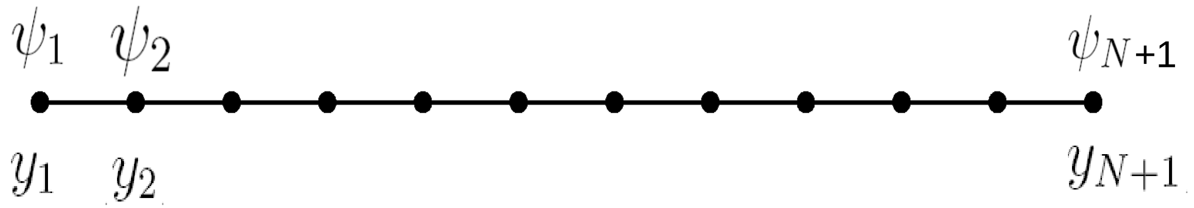
Discretize

$$\boxed{\mathbf{A}\Psi = c\mathbf{E}\Psi}$$

Generalized eigenvalue problem

How to solve Rayleigh equation for real k and complex ω?

Finite differences of order 1



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \\ \psi(y_{N+1}) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \psi_{N+1} \end{pmatrix} \qquad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \\ \psi''(y_{N+1}) \end{pmatrix}$$

How to solve Rayleigh equation for real k and complex ω?

Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

How to solve Rayleigh equation for complex k and real ω?

We fix ω, we need to find all k and ψ such that

$$k\left(U\left(\frac{d^2}{dy^2}-k^2\right)-U''(y)\right)\psi = \omega\left(\frac{d^2}{dy^2}-k^2\right)\psi$$

$$\psi(-L)=\psi(L)=0$$

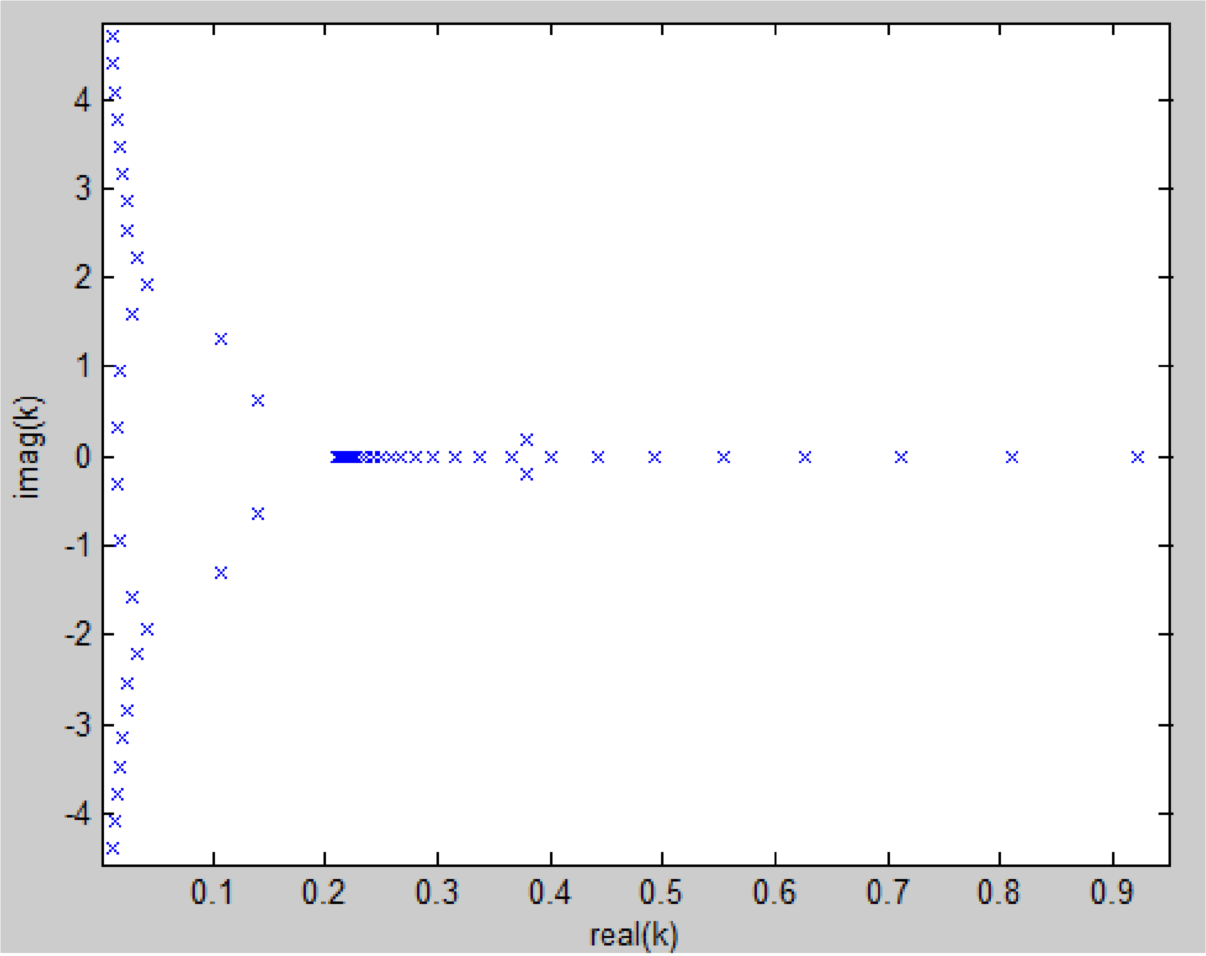
Formally,

$$(A_0(\omega,y)+kA_1(\omega,y)+k^2A_2(\omega,y)+k^3A_3(\omega,y))\quad\psi = 0$$

Polynomial eigenvalue problem

Many more eigenvalues
(for Rayleigh equation: 3 x more!)

U=1+0.9*tanh(y); ω=0.4; L=5



Which of these waves are unstable?

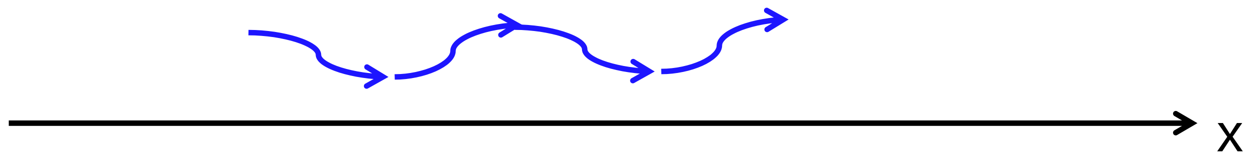
$\mathrm{Im}(k)<0$?

$\mathrm{Im}(k)>0$?

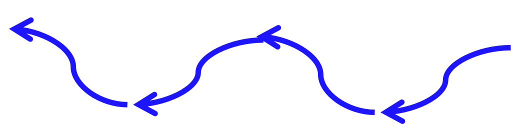
Recall : $\exp(i(kx-\omega t))$

The stability of a spatial wave can be only determined if one knows in which direction it propagates!

k^+ waves propagate towards positive x



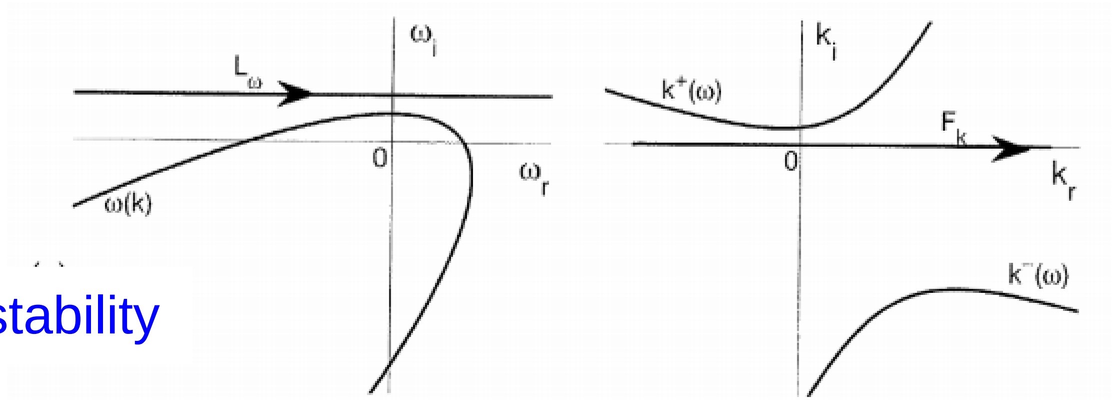
k^- waves propagate towards negative x



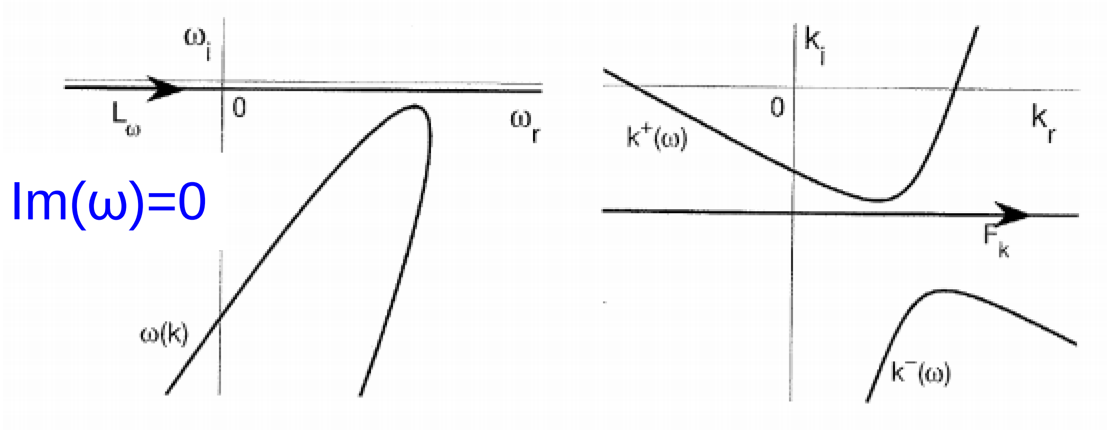
However, determining this direction of propagation is particularly difficult, except in the case of a temporally stable flow.

The addition of a positive imaginary offset to the frequency makes the temporal problem stable!
This separates the spatial waves into k+ and k- waves.

offset spatial stability



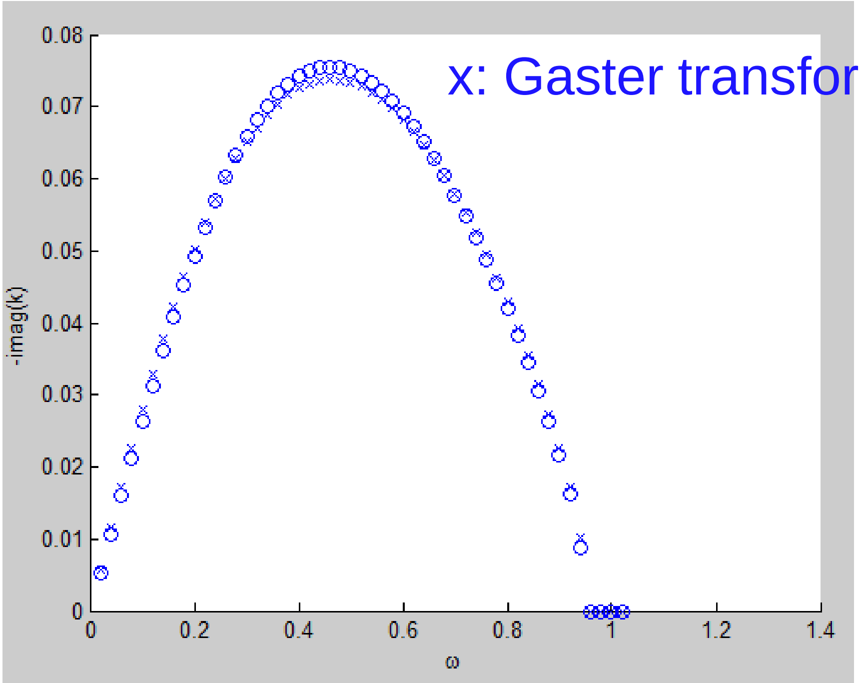
spatial stability: Im(ω)=0



The branches are then followed by continuity

Validity of Gaster transformation?

R=0.4

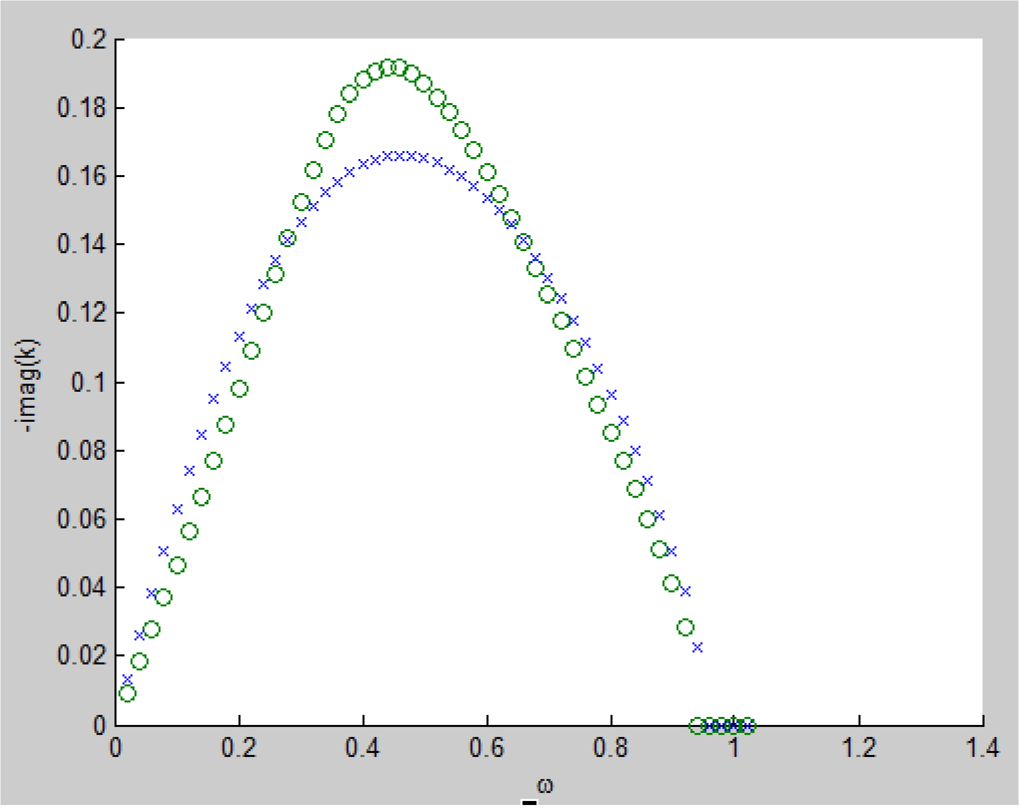


$$R \ll 1$$

$$\boxed{-k_i(\omega, R) \sim R\omega_{1,i}(\omega)}$$

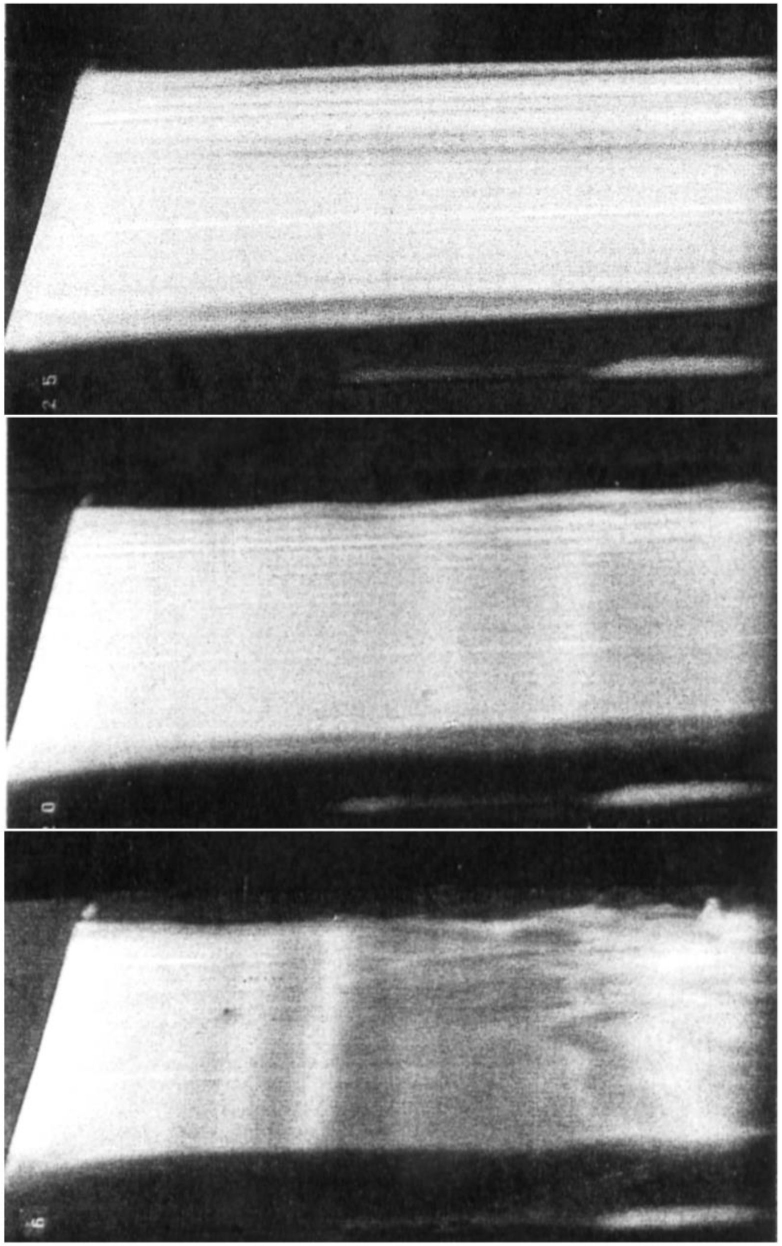
Validity of Gaster transformation?

R=0.9 x: Gaster transformation

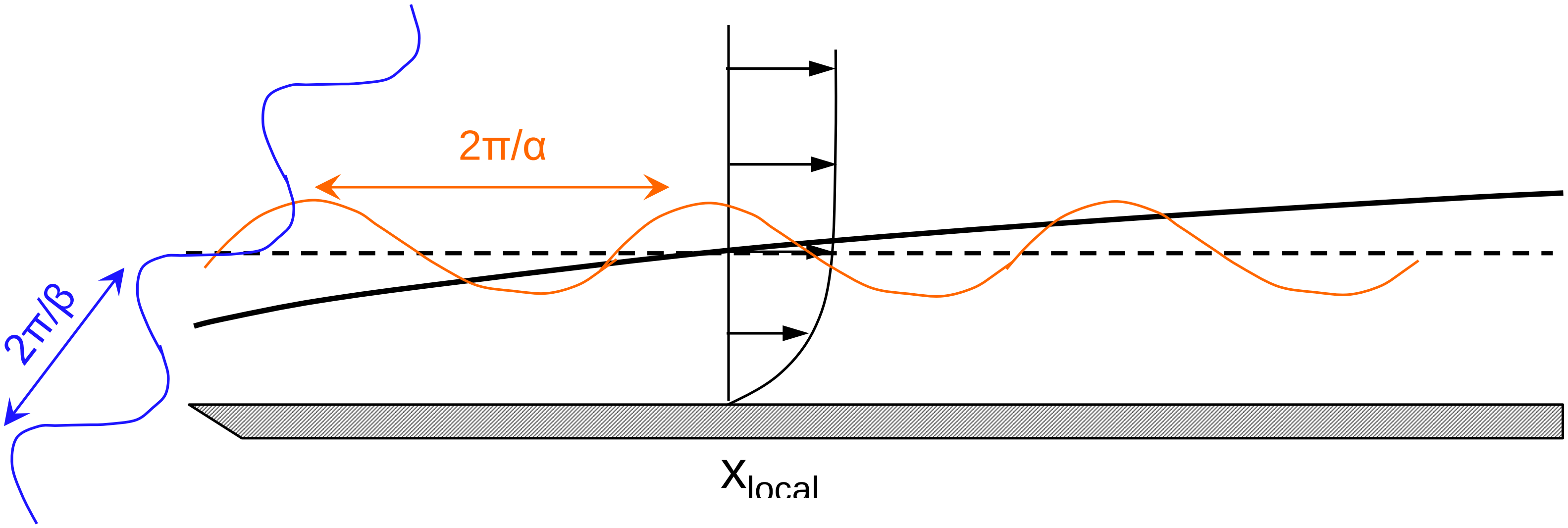


$$\boxed{-k_i(\omega, R) \sim R\omega_{1,i}(\omega)}$$

Boundary layer at increasing Reynolds number



Local parallel flow approximation



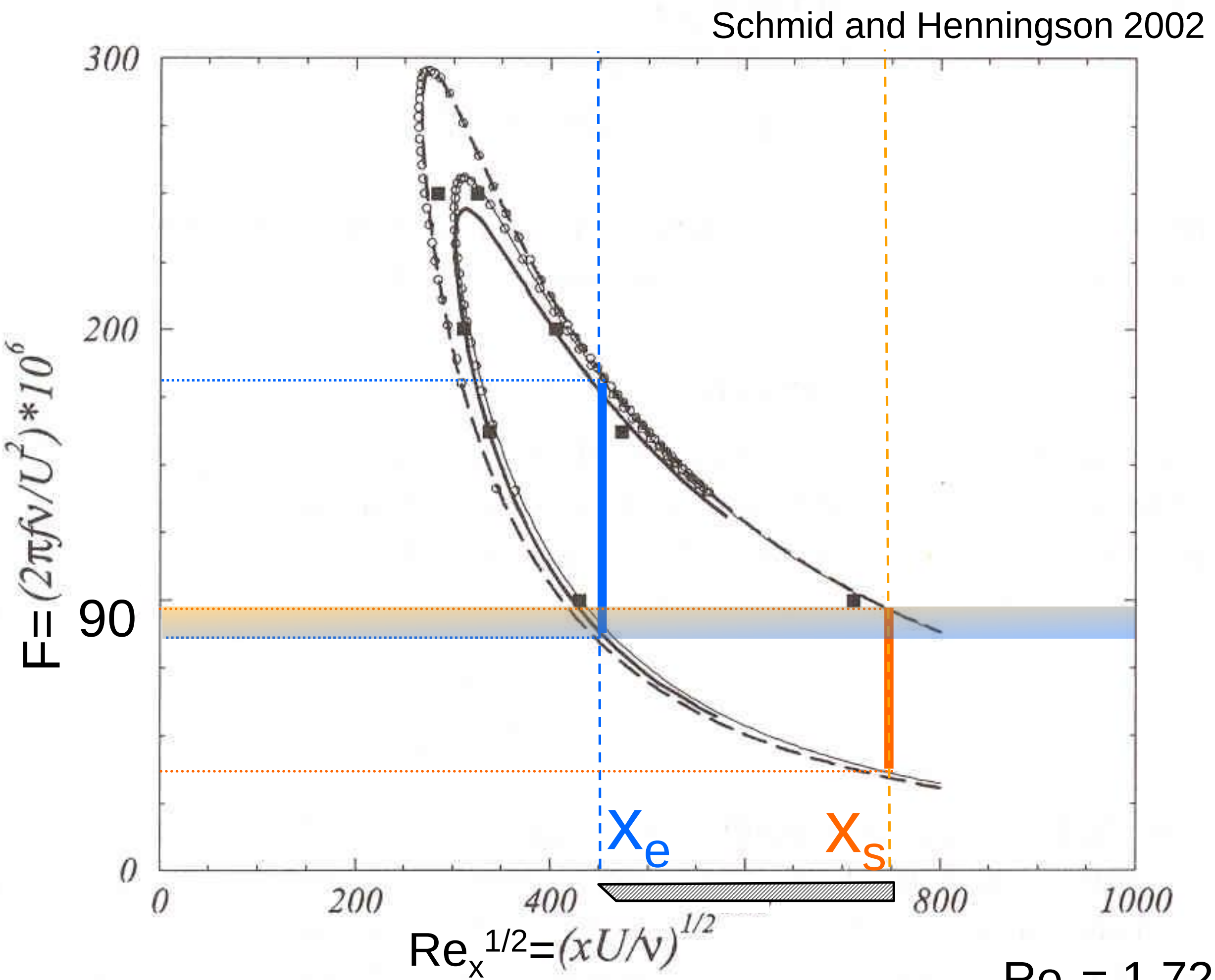
$$(\mathbf{u}, p) = (\mathbf{u}(y), p(y))\, \mathrm{e}^{\sigma t + i(\alpha x + \beta z)}$$

⇨Orr-Sommerfeld-Squire equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}(y)\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{U}(y) = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$

$$\nabla.\mathbf{u} = 0$$

Neutral curve



$Re_\delta = 1.72 Re_x^{1/2}$