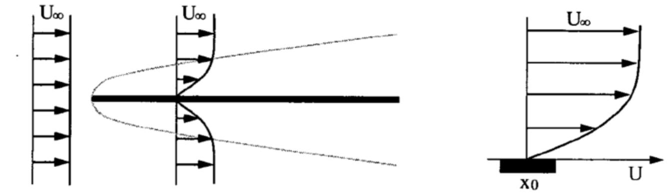
Most flows are unstable...

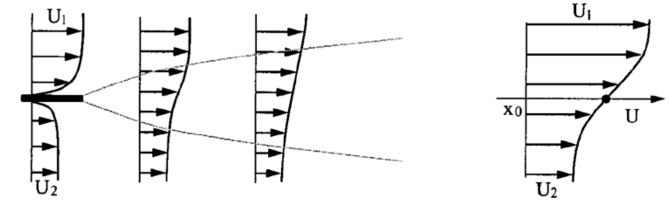
1. Intro-definitions
2. Rayleigh-Taylor
3. Rayleigh Plateau (destabilization through surface tension)
4. Rayleigh-Benard (convection)
5. Taylor Couette-Centrifugal instability
6. Kelvin-Helmholtz
7. Inflection point theorem Rayleigh ! Orr sommerfeld
8. transient growth
9. Spatial growth

SPATIALLY DEVELOPING SHEAR FLOWS

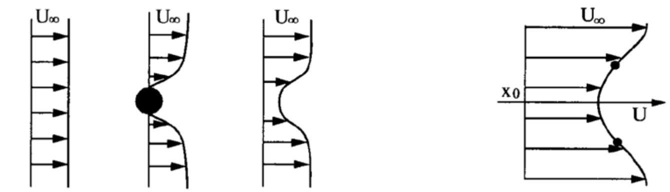
Flat plate boundary layer



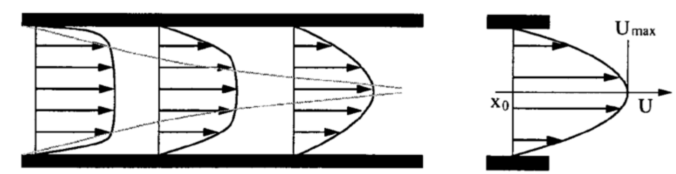
Mixing layer



Cylinder wake



Plane channel flow



2D jet

2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi=\frac{1}{Re}\nabla^4\Psi$$

Basic flow + perturbation

$$\Psi(x,t)=\int U(y)dy+\psi(x,y,t)$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial \psi}{\partial x}=\frac{1}{Re}\nabla^4\psi$$

Dispersion relation

$$D(k, \omega) = 0$$

Temporal approach:
k is real; ω is complex
Perturbation grow and
decay in time!

Spatial approach:
ω is real; k is complex
Perturbations grow and
decay in space!

Shear layer is inviscidly unstable!

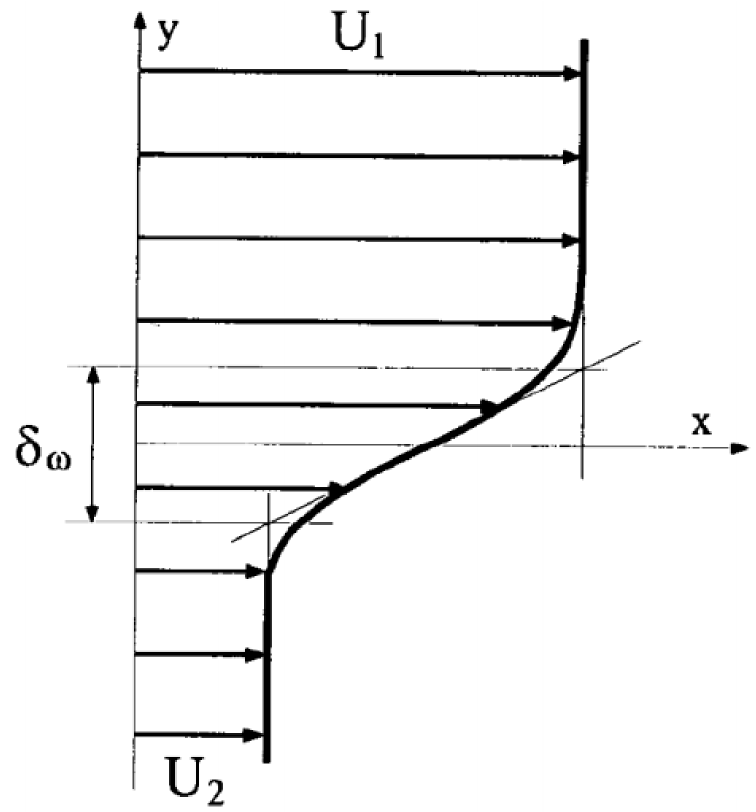
Hyperbolic tangent mixing layer

$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

Velocity ratio

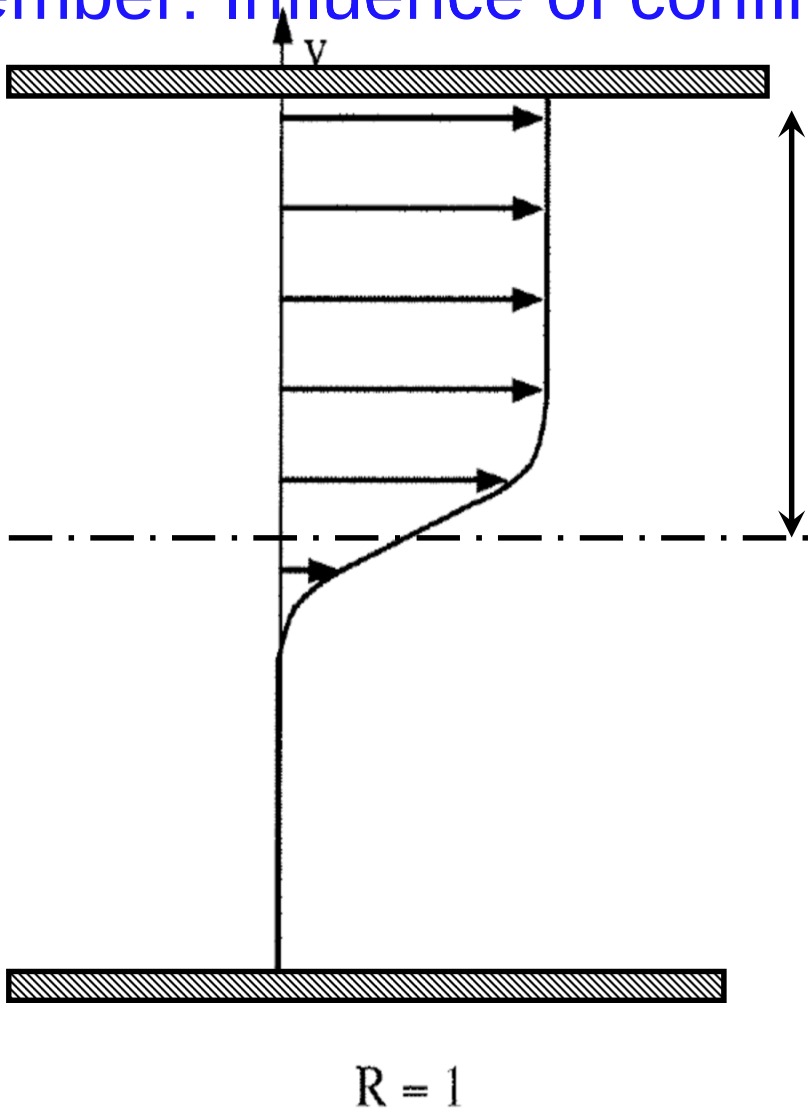
$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$



Why?

Only a necessary condition for instability!

Remember: Influence of confinement



2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

$$U(y;R) = 1 + R\tanh y$$

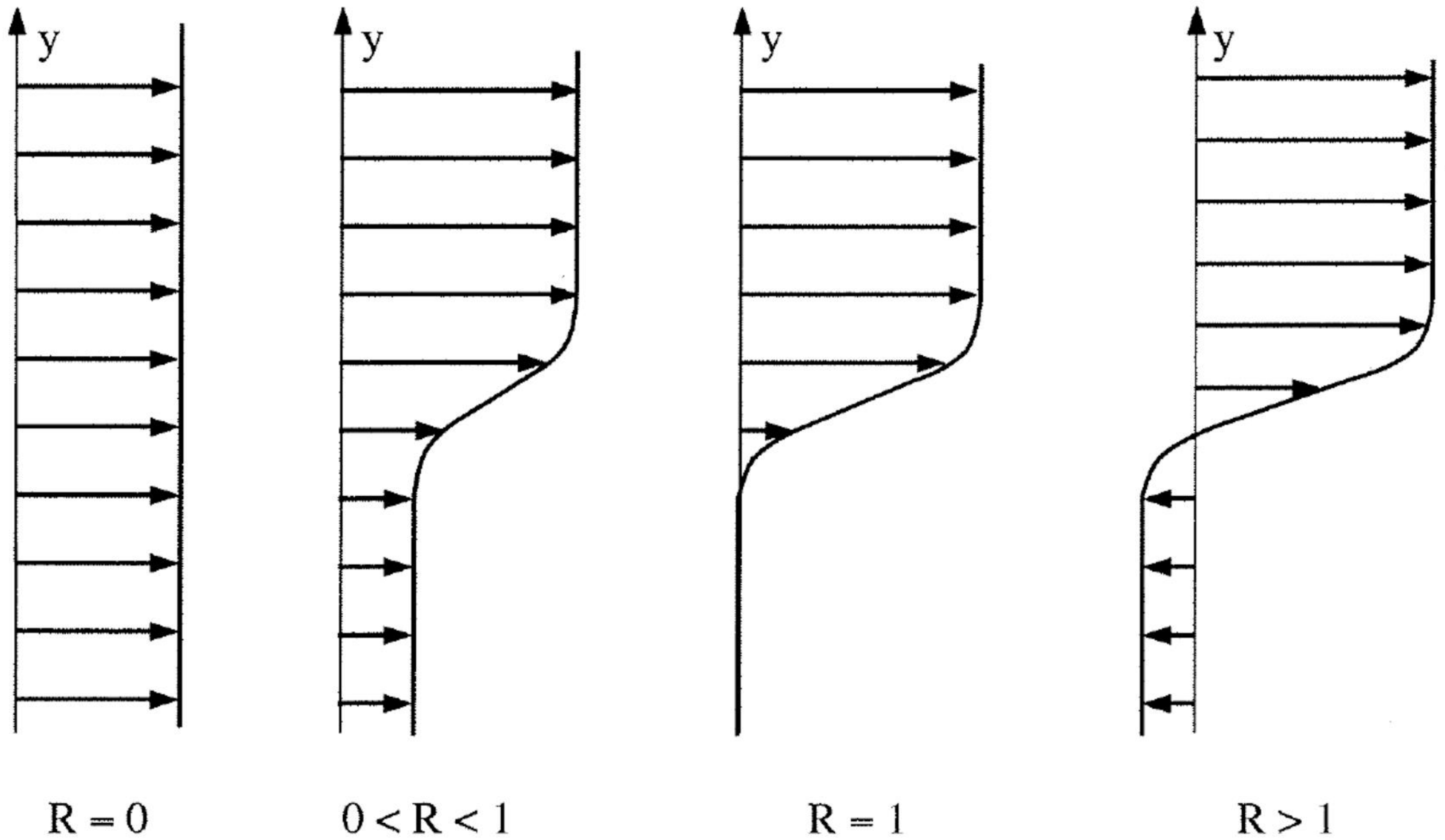
$$U_1(y) = \tanh\, y$$

Dispersion relation

$$\omega(k;R) = k + R\,\omega_1(k)$$

PARALLEL FLOW CONCEPTS

Effect of velocity ratio



2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

Temporal approach

$$\omega_1(k) = i\,\omega_{1,i}(k)$$

$$\omega_i(k;R) = R\,\omega_{1,i}(k)$$

$$c_r = \omega_r/k = 1$$

Temporal approach: k is real; ω is complex

2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

Spatial approach

$$k + R\,\omega_1(k) = \omega$$

$$R \ll 1$$

$$\boxed{-k_i(\omega, R) \sim R\,\omega_{1,i}(\omega)}$$

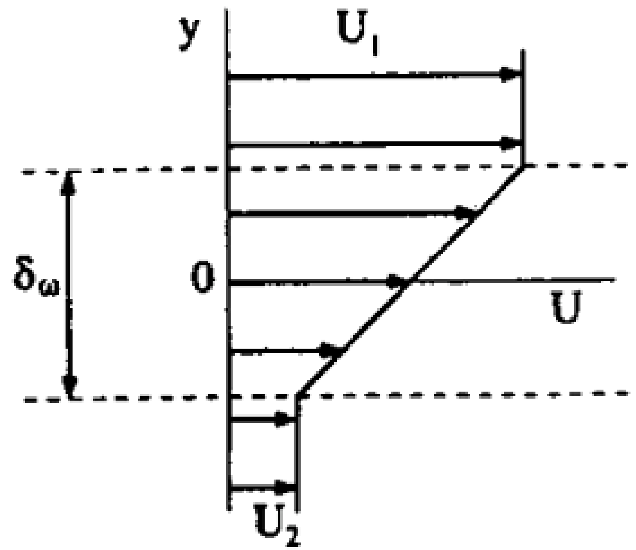
Gaster transformation

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$U(y) = \begin{cases} U_1\,, & y > \delta_\omega/2 \\ (U_1+U_2)/2 + (U_1-U_2)y/\delta_\omega\,, & |y| < \delta_\omega/2 \\ U_2\,, & y < -\delta_\omega/2 \end{cases}$$

$$\phi'' - k^2\phi = 0$$



$$\phi_1(y) = A_1\,\mathrm{e}^{-ky}, \quad y > \delta_\omega/2\,,$$
$$\phi_2(y) = B_2\,\mathrm{e}^{ky}, \quad y < -\delta_\omega/2\,,$$
$$\phi_0(y) = A_0\,\mathrm{e}^{-ky} + B_0\,\mathrm{e}^{ky}, \quad |y| < \delta_\omega/2$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$
\begin{aligned}
A_1\, e^{-k\delta_\omega/2} &= A_0\, e^{-k\delta_\omega/2} + B_0\, e^{k\delta_\omega/2}\,, \\
B_2\, e^{-k\delta_\omega/2} &= A_0\, e^{k\delta_\omega/2} + B_0\, e^{-k\delta_\omega/2}\,, \\
-k(U_1 - c)A_1\, e^{-k\delta_\omega/2} &= k(U_1 - c)(-A_0\, e^{-k\delta_\omega/2} + B_0\, e^{k\delta_\omega/2}) \\
&\quad - \frac{\Delta U}{\delta_\omega}(A_0\, e^{-k\delta_\omega/2} + B_0\, e^{k\delta_\omega/2})\,, \\
k(U_2 - c)B_2\, e^{-k\delta_\omega/2} &= k(U_2 - c)(-A_0\, e^{k\delta_\omega/2} + B_0\, e^{-k\delta_\omega/2}) \\
&\quad - \frac{\Delta U}{\delta_\omega}(A_0\, e^{k\delta_\omega/2} + B_0\, e^{-k\delta_\omega/2})\,.
\end{aligned}
$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$-\frac{\Delta U}{\delta_\omega} A_0 \, e^{-k\delta_\omega/2} + \left[2k(U_1 - c) - \frac{\Delta U}{\delta_\omega}\right] B_0 \, e^{k\delta_\omega/2} = 0$$

$$\left[2k(U_2 - c) + \frac{\Delta U}{\delta_\omega}\right] A_0 \, e^{k\delta_\omega/2} + \frac{\Delta U}{\delta_\omega} B_0 \, e^{-k\delta_\omega/2} = 0$$

2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$4(k\delta_\omega)^2(c-\bar{U})^2 - \left[(k\delta_\omega - 1)^2 - e^{-2k\delta_\omega}\right]\Delta U^2 = 0$$

$$k\delta_\omega \mapsto 2k,\; c/\bar{U} \mapsto c$$

$$4k^2(c-1)^2 - R^2\left[(2k-1)^2 - e^{-4k}\right] = 0$$

$$\boxed{c \equiv \frac{\omega}{k} = 1 \pm \frac{R}{2k}\left[(2k-1)^2 - e^{-4k}\right]^{1/2}}$$

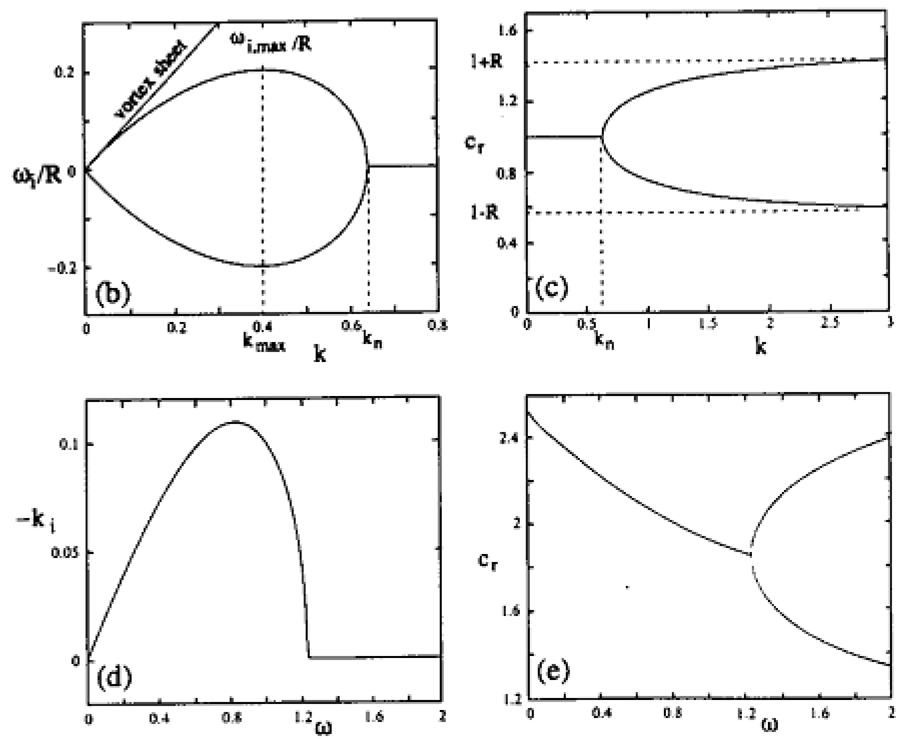
2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

Temporal approach

$$2k_n - 1 = e^{-2k_n}$$

Spatial approach
R = 0.5

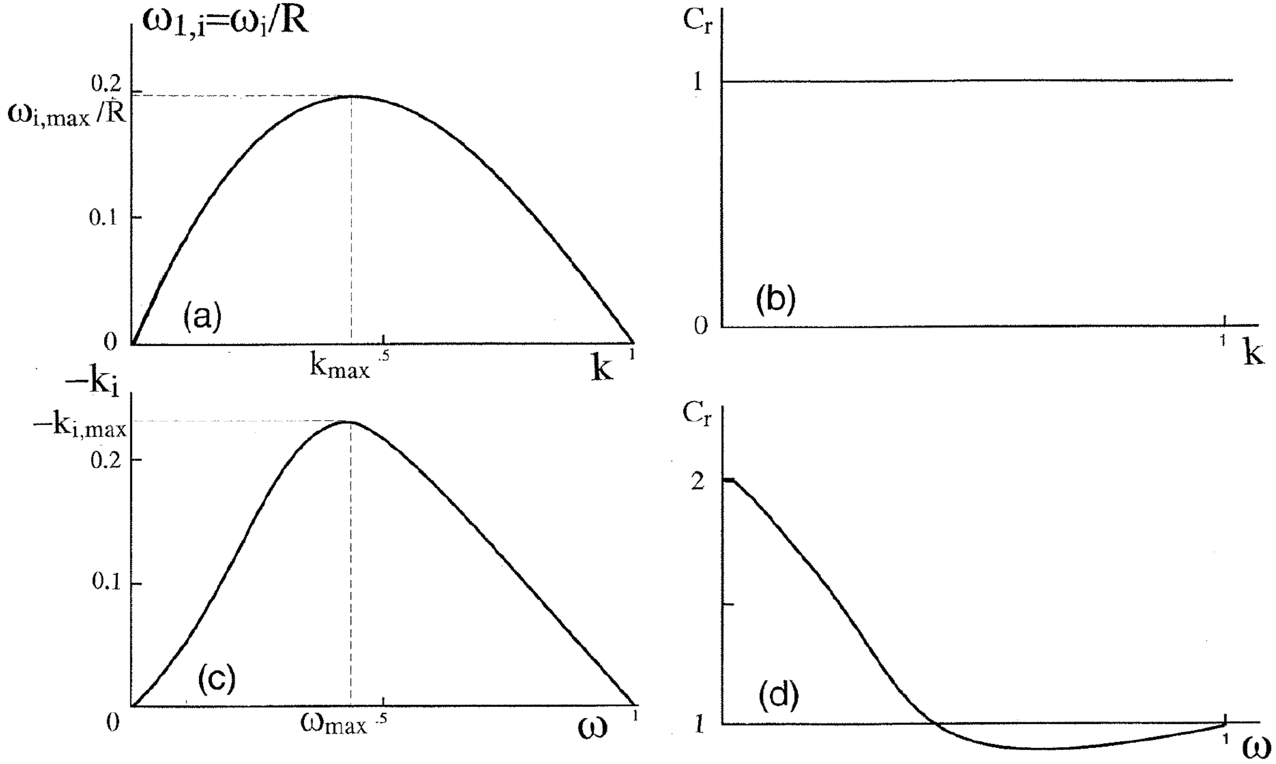


2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

Temporal

Spatial



Michalke (1964, 65)

Solving a spatial instability problem
ex: Rayleigh equation

Back to temporal stability analysis!
How to solve Rayleigh equation for real k and complex ω?

We fix k, we need to find all ω and ψ such that

$$\boxed{\begin{gathered} k\left(U\left(\frac{d^2}{dy^2}-k^2\right)-U''(y)\right)\psi = \omega\left(\frac{d^2}{dy^2}-k^2\right)\psi \\ \psi(-L)=\psi(L)=0 \end{gathered}}$$

Formally,

$$\boxed{\mathcal{A}\psi = c\,\mathcal{E}\psi}$$

c=ω/k

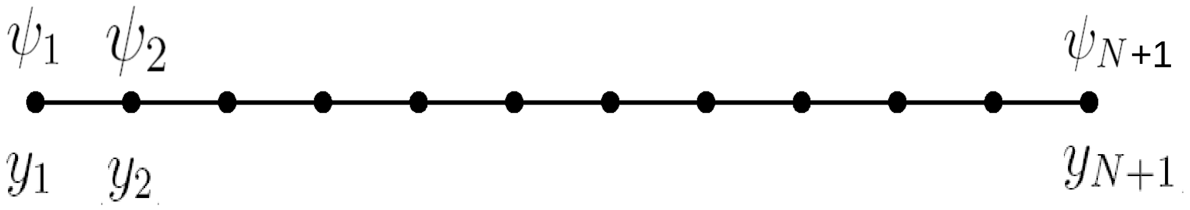
Discretize

$$\boxed{\mathbf{A}\Psi = c\mathbf{E}\Psi}$$

Generalized eigenvalue problem

How to solve Rayleigh equation for real k and complex ω?

Finite differences of order 1



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \\ \psi(y_{N+1}) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \psi_{N+1} \end{pmatrix} \qquad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \\ \psi''(y_{N+1}) \end{pmatrix}$$

How to solve Rayleigh equation for real k and complex ω?

Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

How to solve Rayleigh equation for complex k and real ω?

We fix ω, we need to find all k and ψ such that

$$k\left(U\left(\frac{d^2}{dy^2}-k^2\right)-U''(y)\right)\psi = \omega\left(\frac{d^2}{dy^2}-k^2\right)\psi$$

$$\psi(-L)=\psi(L)=0$$

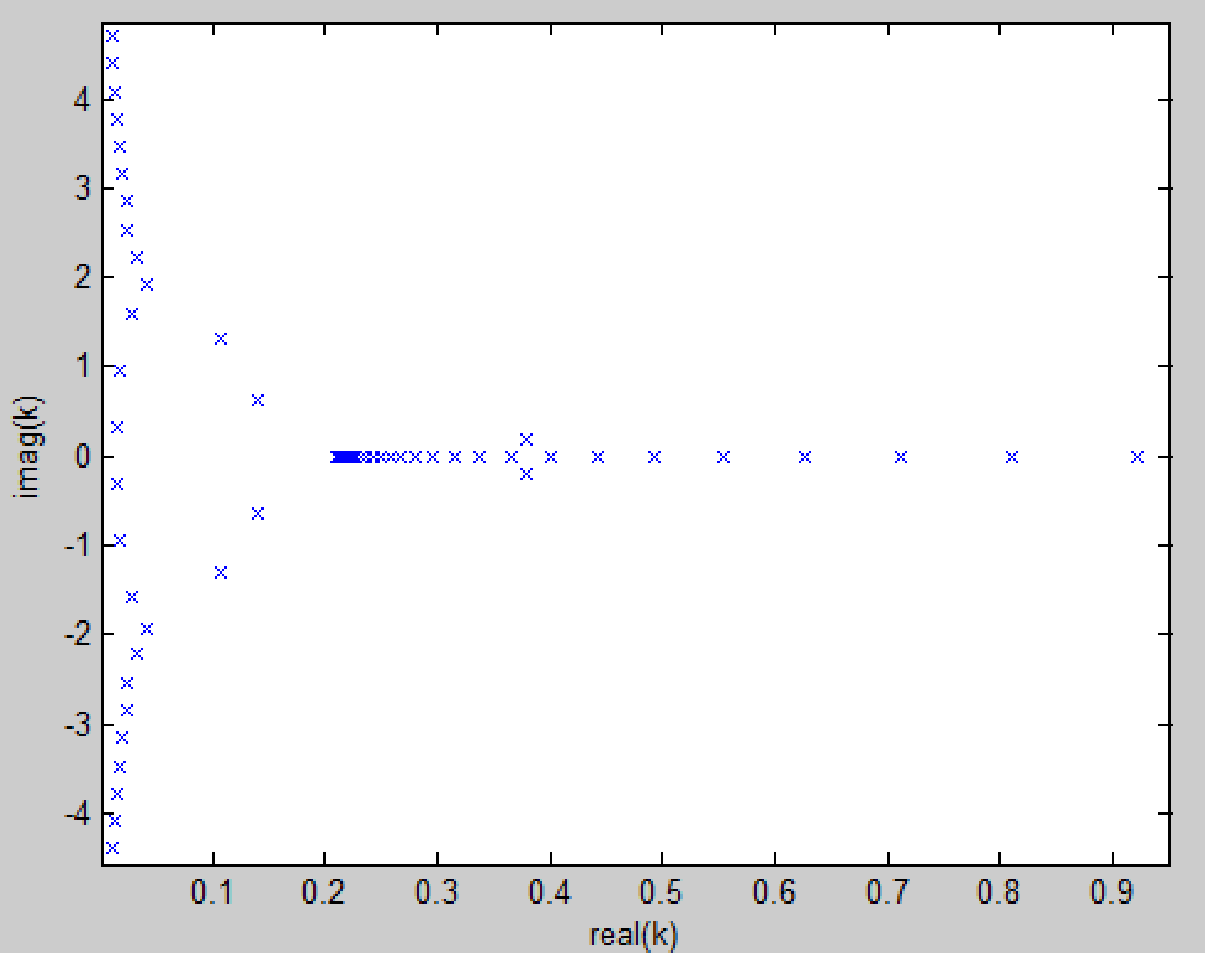
Formally,

$$\boxed{(A_0(\omega,y)+kA_1(\omega,y)+k^2A_2(\omega,y)+k^3A_3(\omega,y))\quad \psi = 0}$$

Polynomial eigenvalue problem

Many more eigenvalues
(for Rayleigh equation: 3 x more!)

U=1+0.9*tanh(y); ω=0.4; L=5



Which of these waves are unstable?

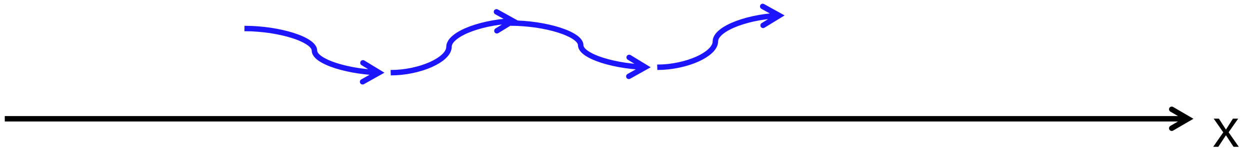
$\mathrm{Im}(k)<0$?

$\mathrm{Im}(k)>0$?

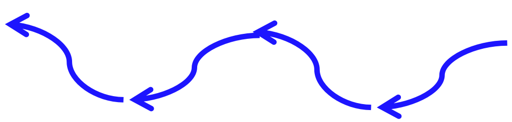
Recall : $\exp(i(kx-\omega t))$

The stability of a spatial wave can be only determined if one knows in which direction it propagates!

k^+ waves propagate towards positive x



k^- waves propagate towards negative x

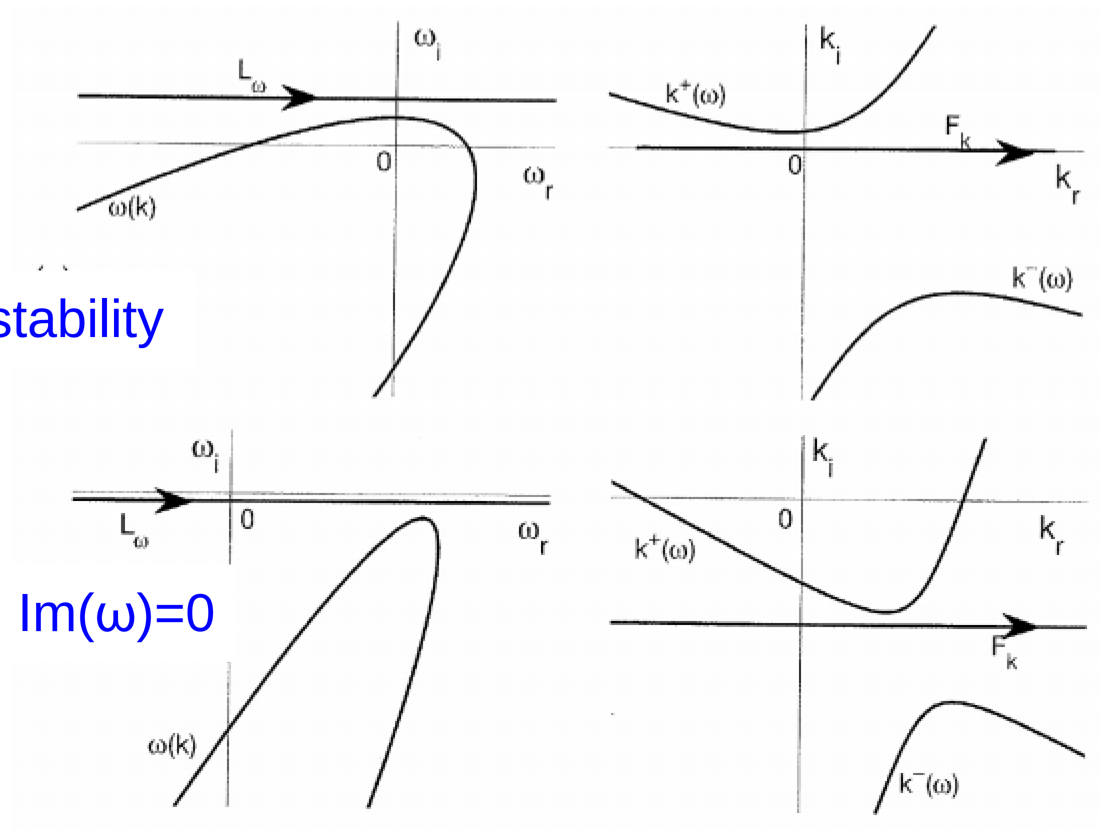


However, determining this direction of propagation is particularly difficult, except in the case of a temporally stable flow.

The addition of a positive imaginary offset to the frequency makes the temporal problem stable!
This separates the spatial waves into k+ and k- waves.

offset spatial stability

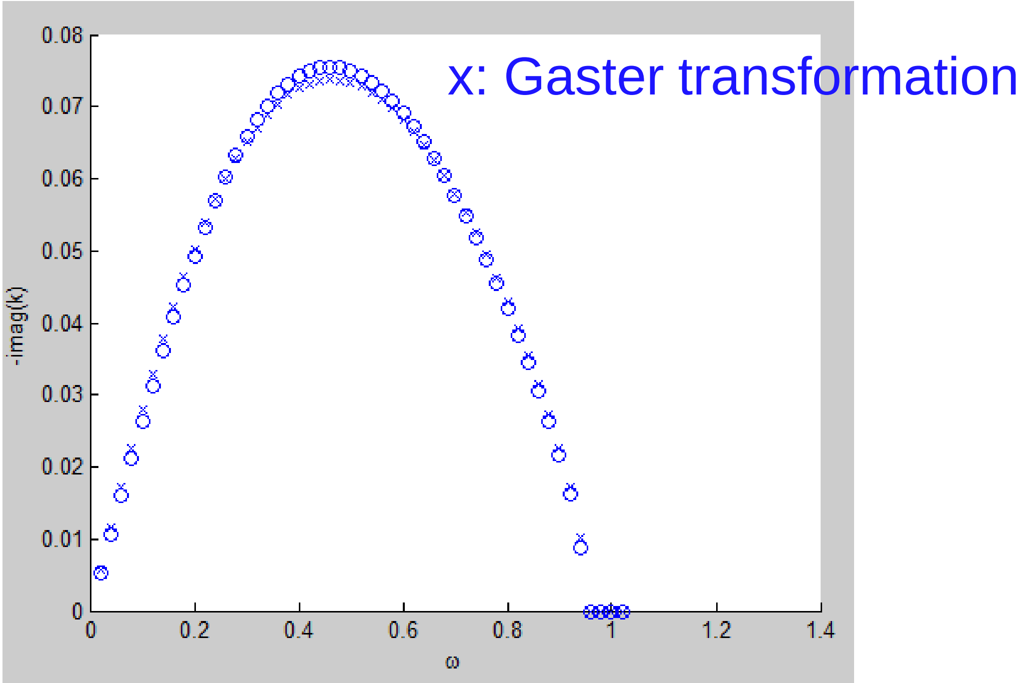
spatial stability: Im(ω)=0



The branches are then followed by continuity

Validity of Gaster transformation?

R=0.4



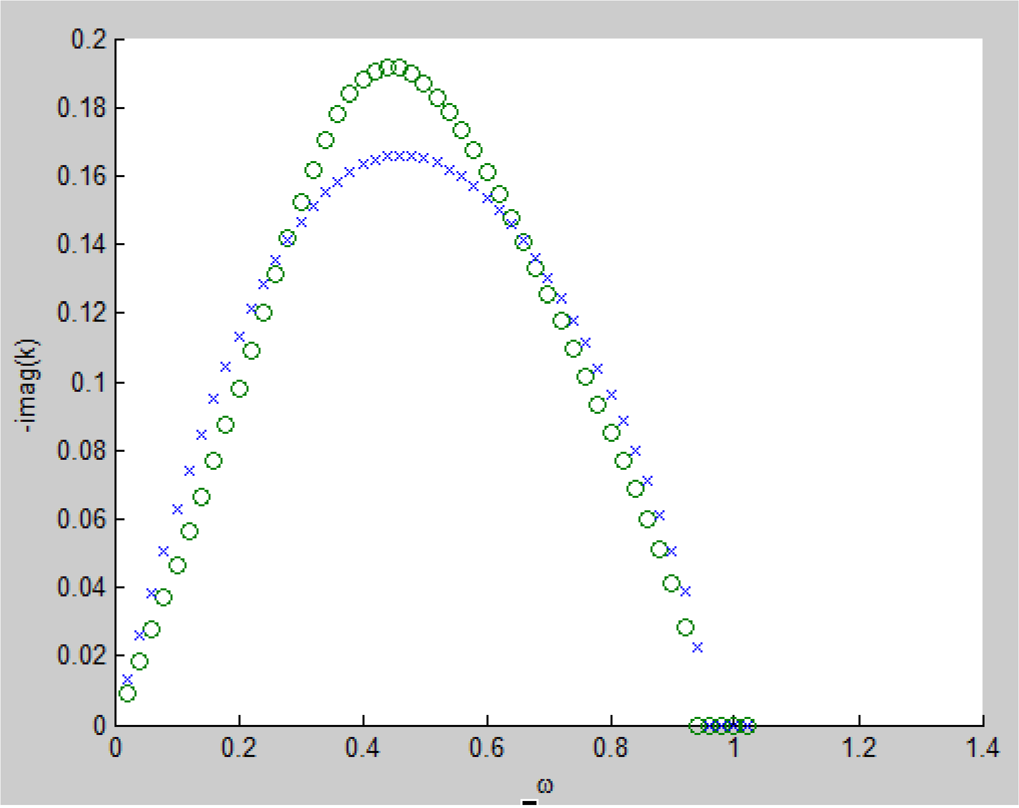
$$R \ll 1$$

$$\boxed{-k_i(\omega, R) \sim R\omega_{1,i}(\omega)}$$

Validity of Gaster transformation?

R=0.9

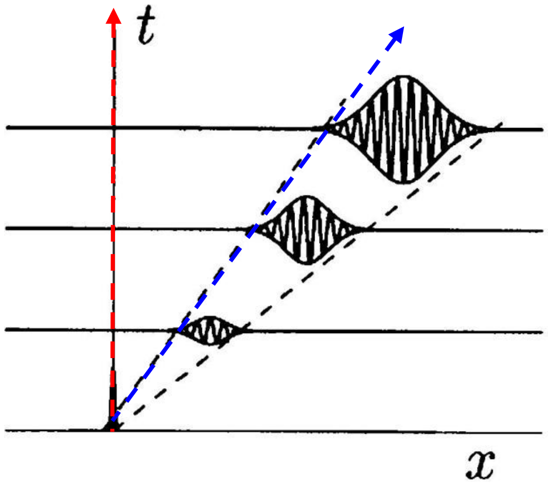
x: Gaster transformation



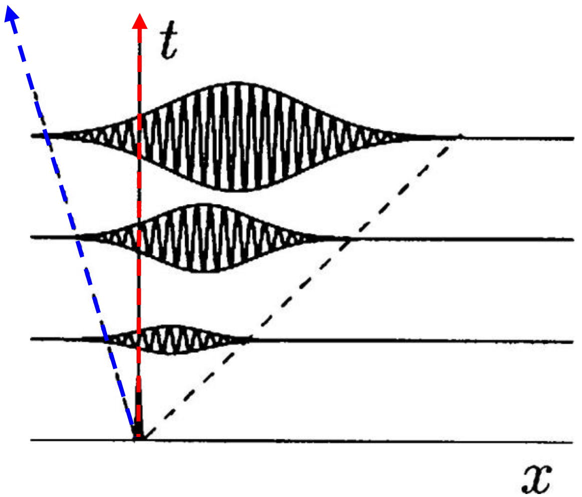
$$-k_i(\omega, R) \sim R\omega_{1,i}(\omega)$$

Spatio-temporal instability theory

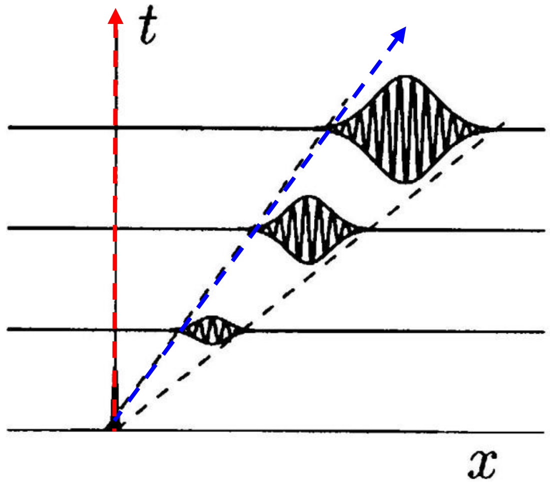
☞ Convective instability



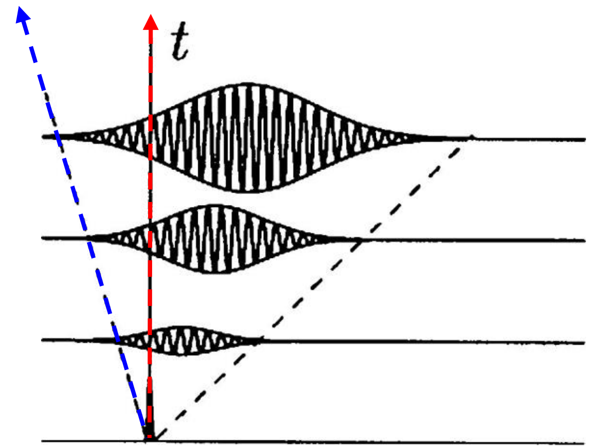
F Absolute instability



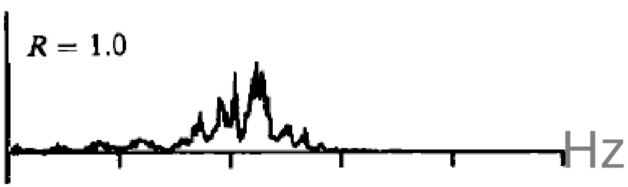
☞Convective instability

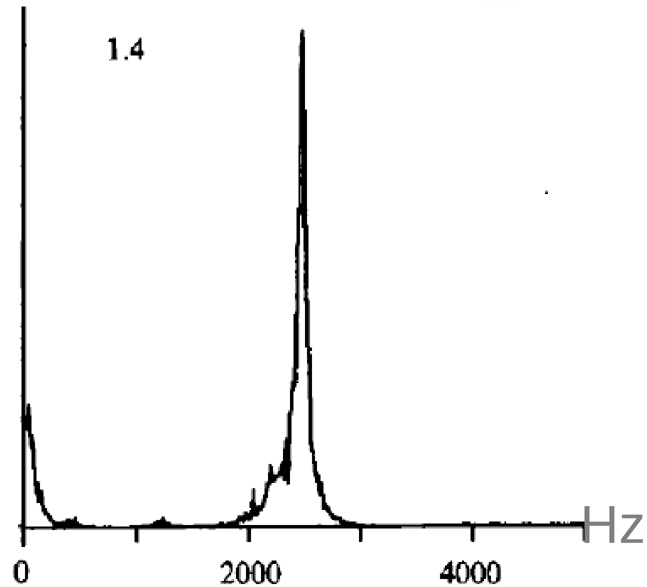


F Absolute instability



Spectrum





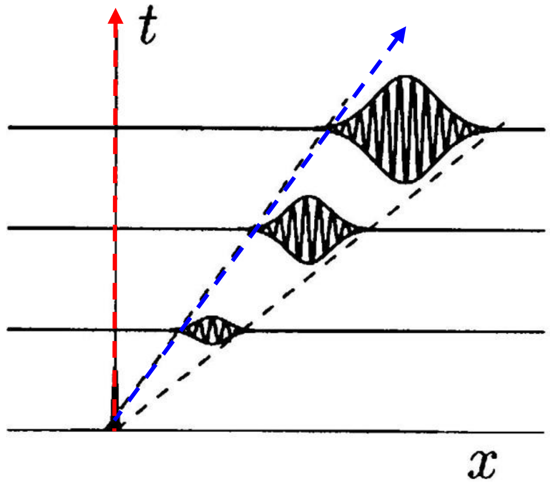
☞amplifier

☞oscillator

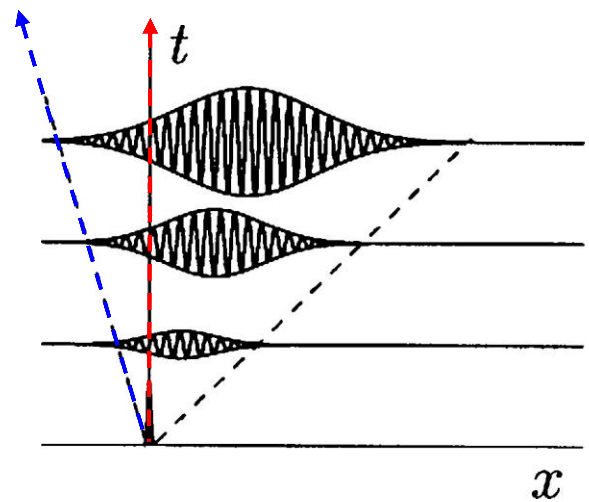
Mixing layer experiments by Strikowsky and Niccum (1991)

Spatio-temporal instability theory

☞Convective instability



F Absolute instability



We need to generalize the concept of group velocity since ω (and why not k) is complex

For neutral waves, the group velocity is dω/dk
Here this quantity is the derivative of a complex function with respect to a complex variable. Cauchy-Rieman conditions apply.

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}\,(k,\omega)\, e^{i(kx-\omega t)}\, dk\, d\omega$$

$$\hat{u}(k,\omega) = (2\pi)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u\,(x,t)\, e^{-i(kx-\omega t)}\, dx\, dt$$

Direct Fourier Transform

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k,\omega)\, e^{i(kx-\omega t)}\, dk\, d\omega$$

Use dispersion relation ω(k)!

Spectral analysis

Fourier transform: $$\boxed{u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + c.c.}$$

Carrier/enveloppe

Carrier/enveloppe : $$\boxed{u(x,t) = \frac{1}{2}A(x,t)\,\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.}$$

Spectral analysis

Fourier transform:

$$\boxed{u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)} \mathrm{d}k + c.c.}$$

Carrier/enveloppe :

$$\boxed{u(x,t) = \frac{1}{2} A(x,t)\, \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + c.c.}$$

Spectral analysis

Fourier transform:

$$\boxed{u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + c.c.}$$

Carrier/enveloppe :

$$\boxed{u(x,t) = \frac{1}{2}A(x,t)\,\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.}$$

Enveloppe :

$$\boxed{A(x,t) = \int_0^{\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.}$$

Spectral analysis at time=0

Fourier transform:

$$\boxed{u(x,0) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(kx)}\,\mathrm{d}k + \text{c.c.}}$$

$\hat{u}(k)$ is given by Fourier transform at time t=0

Enveloppe :

$$\boxed{A(x,0) = \int_0^{\infty} \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(k-k_0)x}\,\mathrm{d}k + \text{c.c.}}$$

Spectral analysis

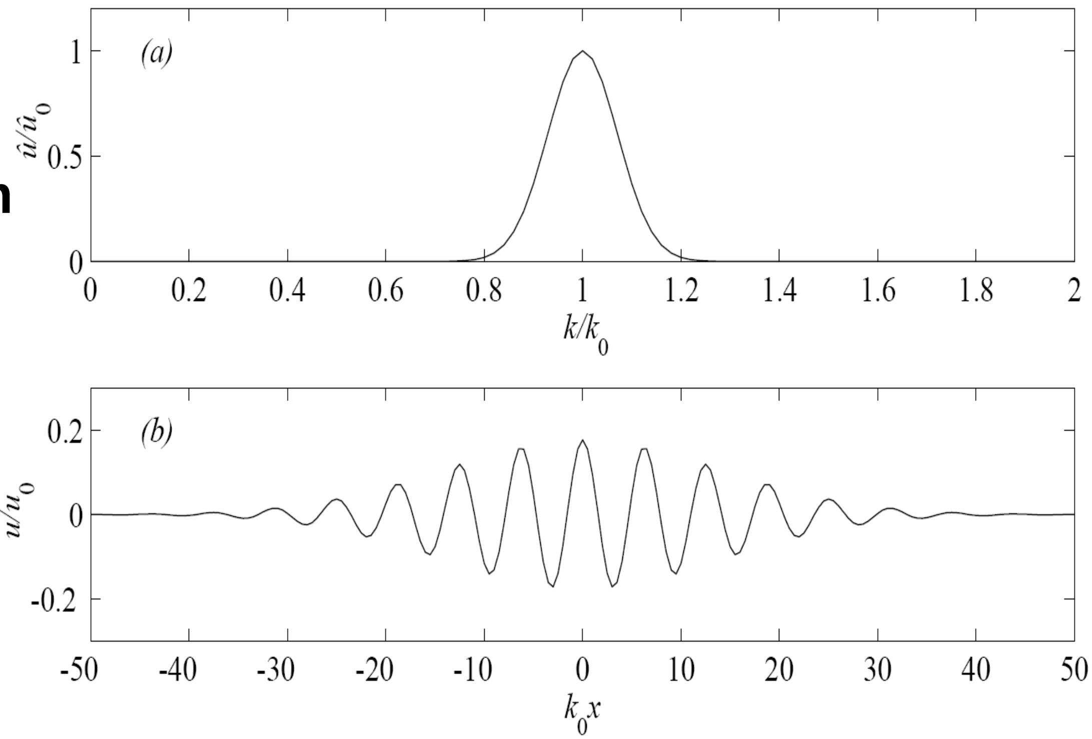
Gaussian spectrum: $$\hat{u}(k) = u_0\,\mathrm{e}^{-\sigma^2 (k-k_0)^2}$$

Initial enveloppe : $$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

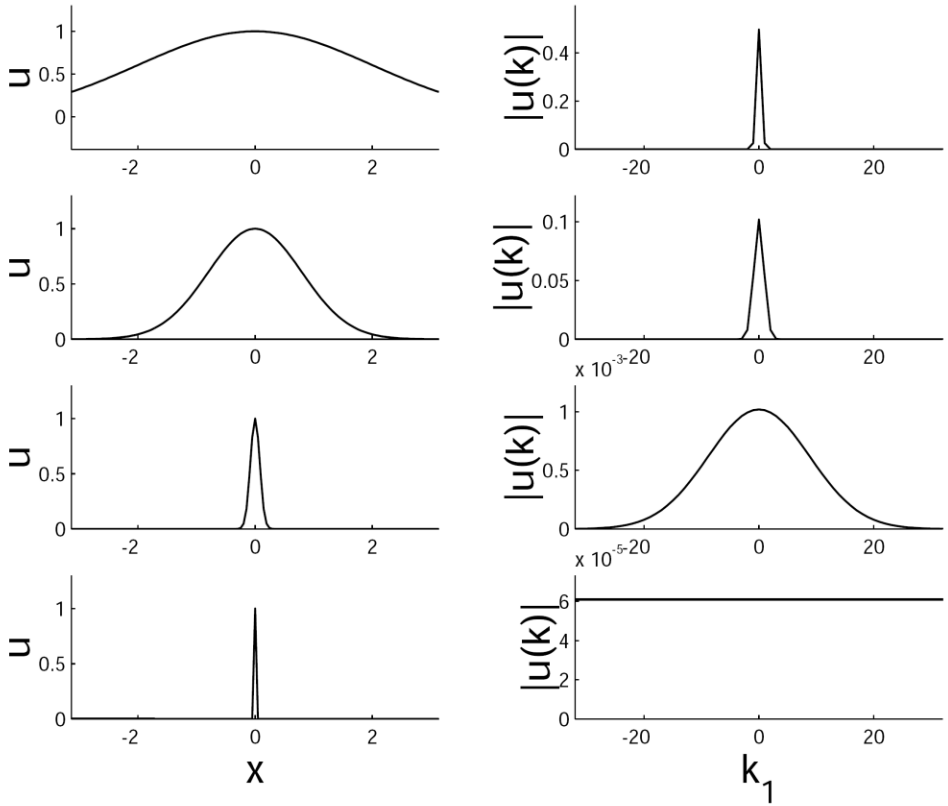
Gaussian spectrum

spectrum

wave



Gaussian wavepackets



$u_0(x)$	$\widehat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\,\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}}\ \exp\left(-\frac{\sigma^2\,k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Spectral analysis

Initial enveloppe :
$$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:
$$\hat{u}(k) = u_0\,\mathrm{e}^{-\sigma^2(k-k_0)^2}$$

Spectral analysis

Initial enveloppe :
$$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:
$$\hat{u}(k) = u_0\,\mathrm{e}^{-\sigma^2(k-k_0)^2}$$

Evolution of enveloppe :
$$\boxed{A(x,t) = \int_0^\infty \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.}$$

Spectral analysis

Initial enveloppe :

$$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:

$$\hat{u}(k) = u_0\,\mathrm{e}^{-\sigma^2(k-k_0)^2}$$

Evolution of enveloppe :

$$\boxed{A(x,t) = \int_0^\infty \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.}$$

Definition group velocity

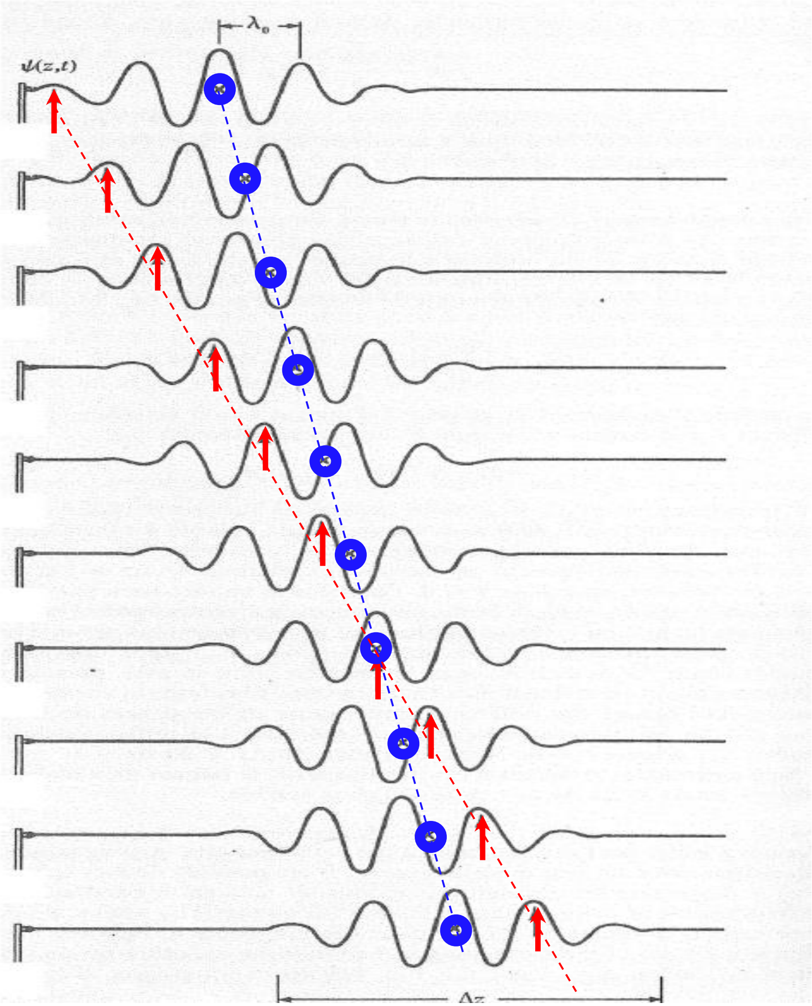
$$\boxed{\omega-\omega_0 = c_g(k-k_0), \qquad c_g = \frac{\partial\omega}{\partial k}(k_0)}$$

Spectral analysis

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \qquad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$\boxed{A(x,t) = \frac{u_0\sqrt{\pi}}{2\sigma}\,\mathrm{e}^{-\frac{(x-c_g t)^2}{4\sigma^2}}}$$

Group velocity



Wavepacket

Spectral analysis

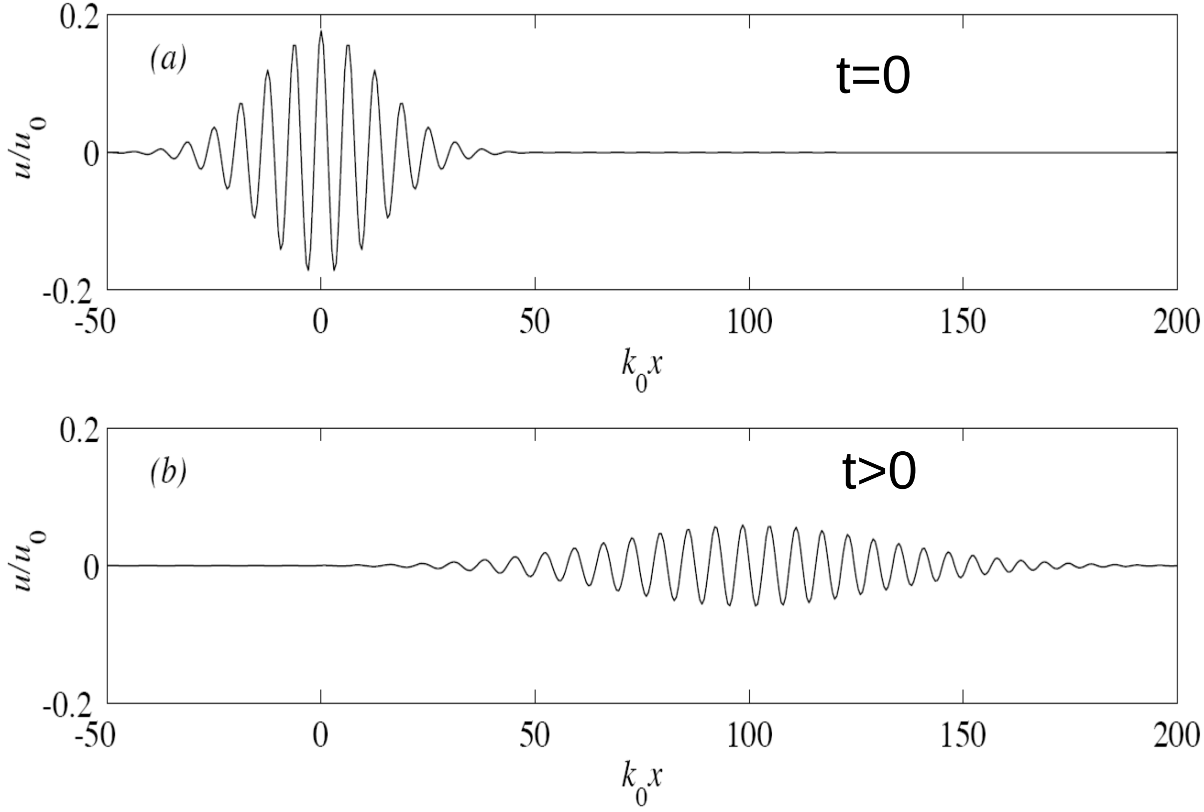
Higher order development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$

$$c_g = \frac{\partial \omega}{\partial k}(k_0), \qquad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

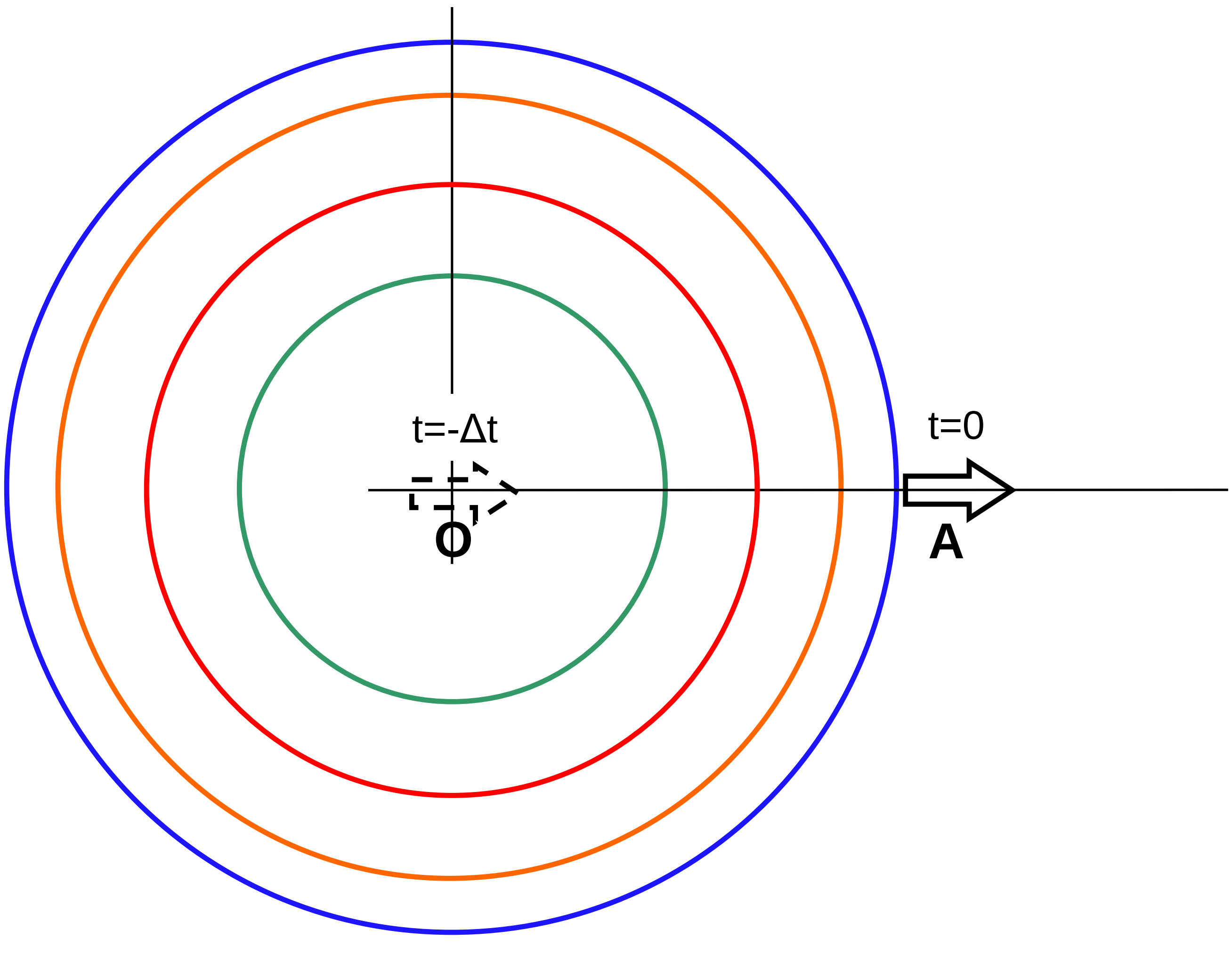
$$A(x,t) = \frac{u_0}{2}\sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}\mathrm{i}\omega_0'' t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}\mathrm{i}\omega_0'' t)}\right)$$

Wave packet dispersion

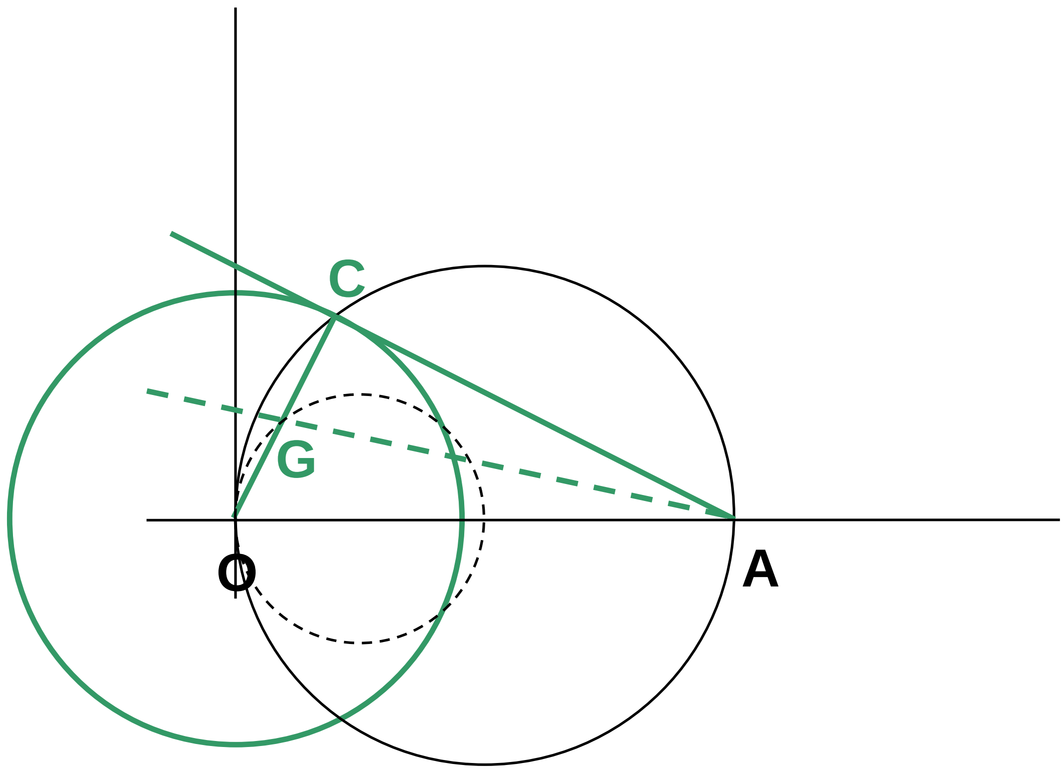


Onde correspondant à l'enveloppe *pour* $\sigma^{-1}k_0 = 0{,}1$ *et* $\omega_0'' = 4c_g/k_0$: *(a), instant initial* $t = 0$; *(b),* $c_g t = 100/k_0$.

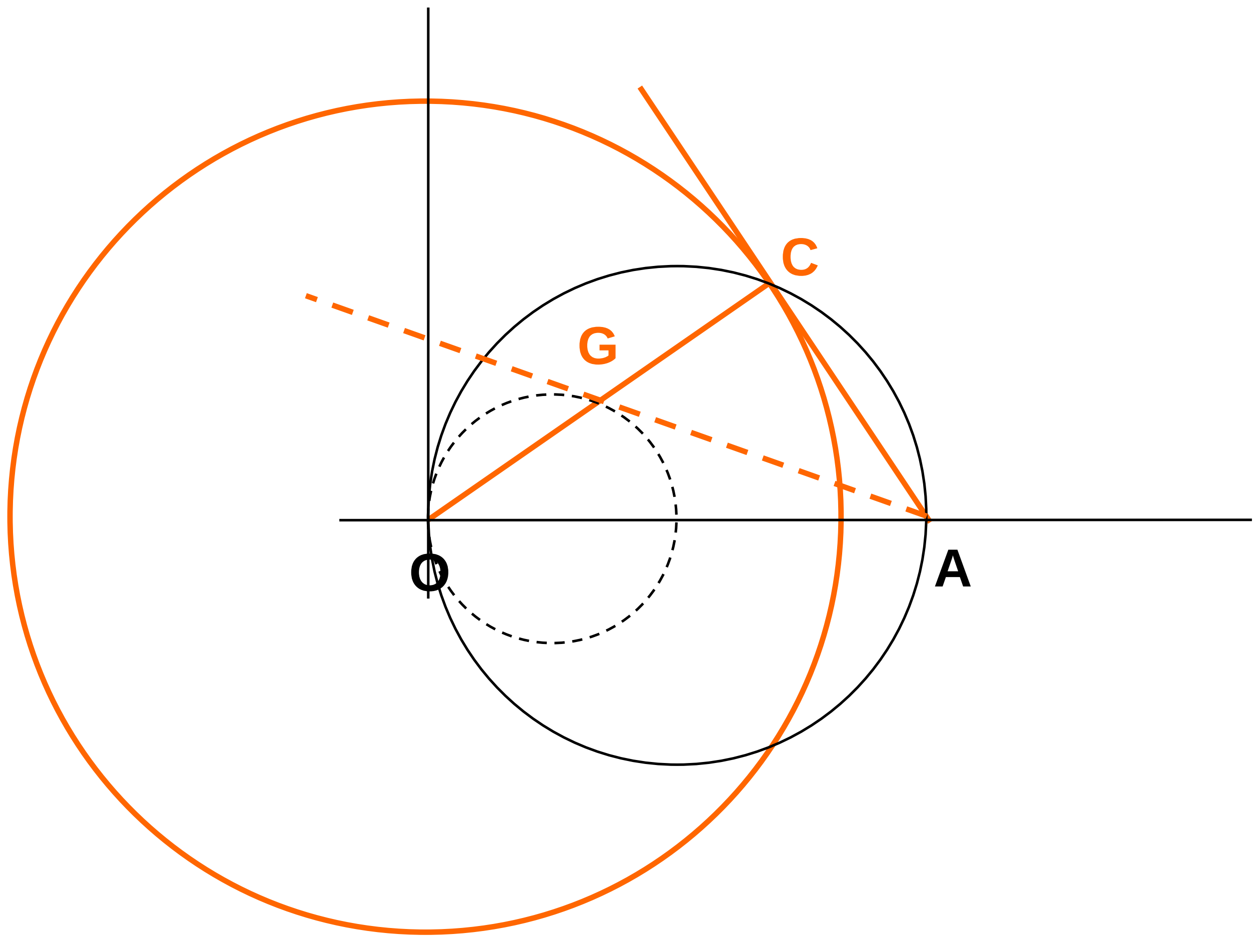
Kelvin's wake



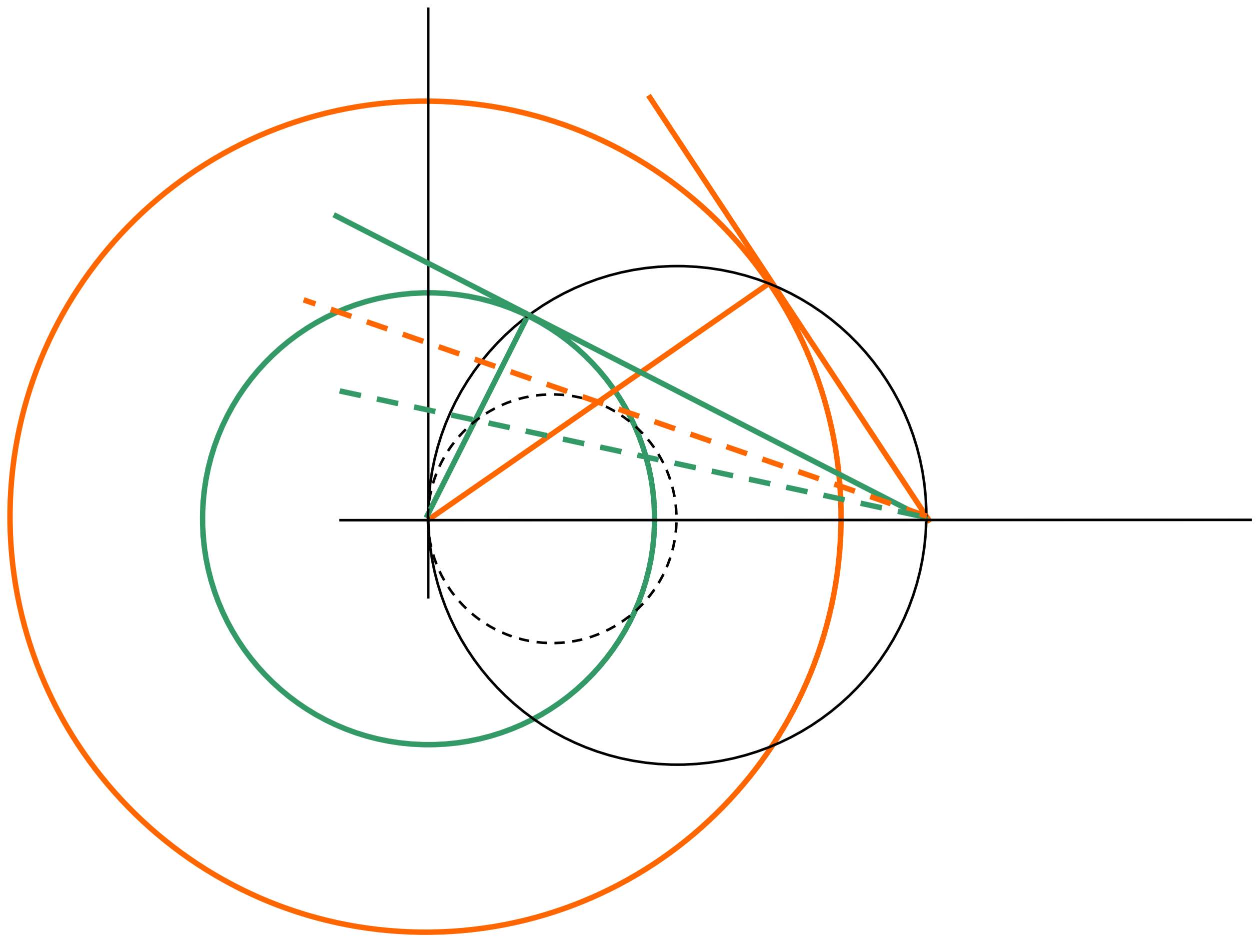
Gravity waves created by a ship



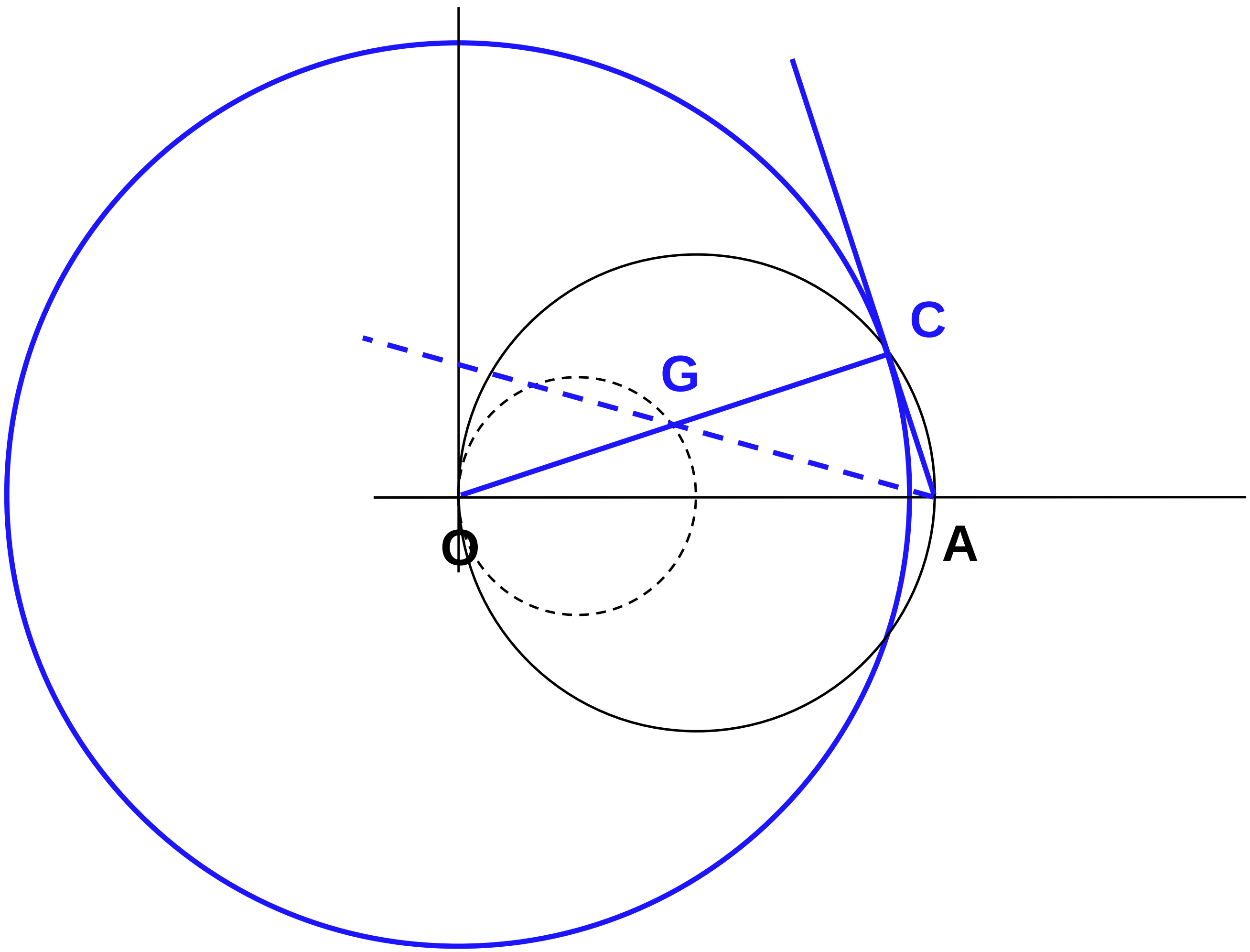
Gravity waves created by a ship



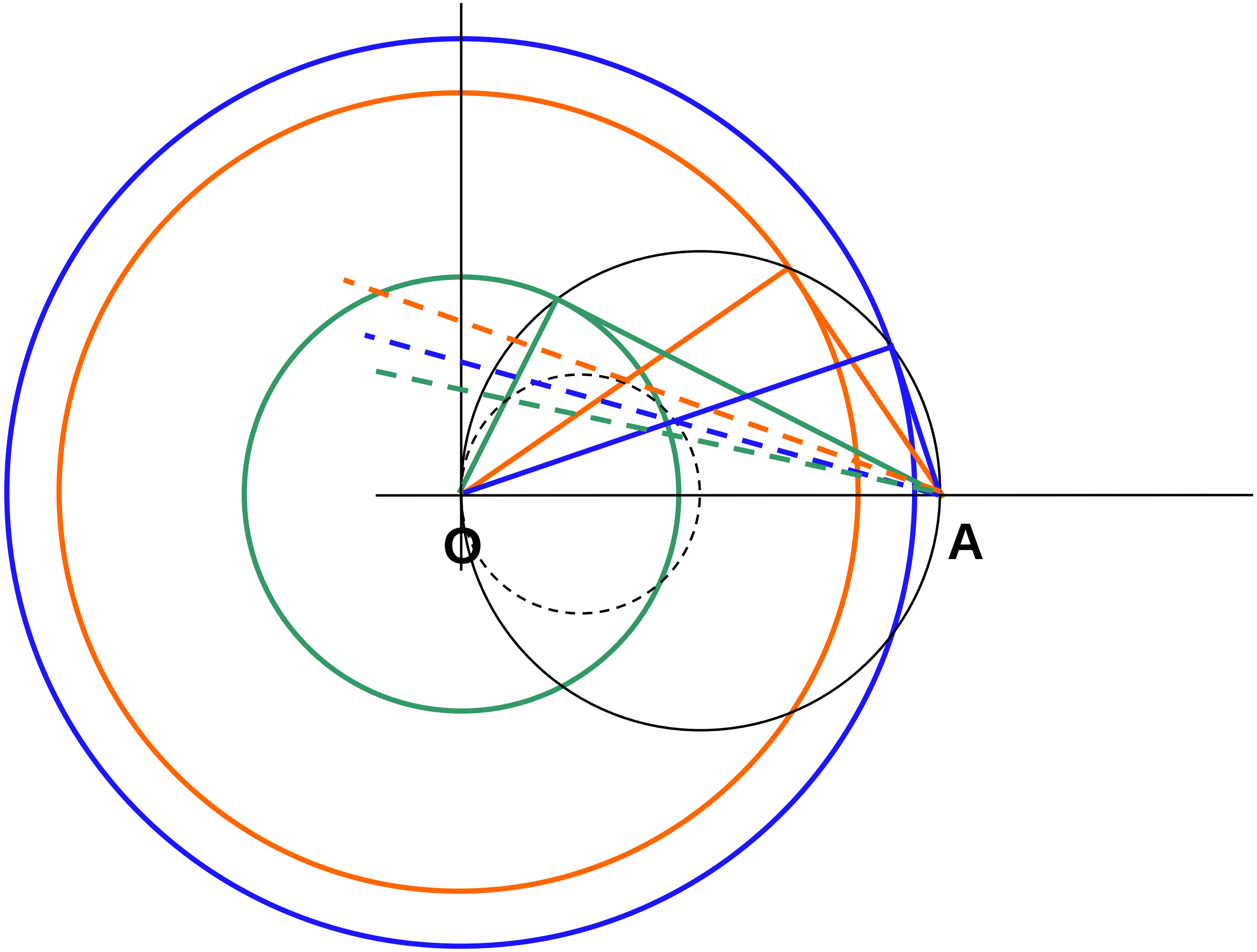
Gravity waves created by a ship



Gravity waves created by a ship



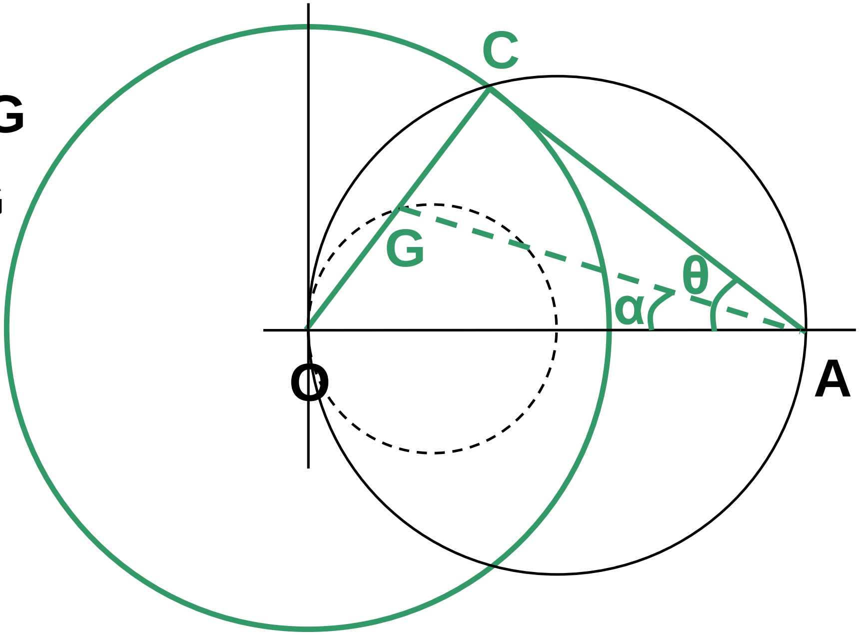
Gravity waves created by a ship



Gravity waves created by a ship

$\sin(\alpha)/OG=\cos(\theta)/AG$

$\sin(\theta-\alpha)AG=GC=OG$

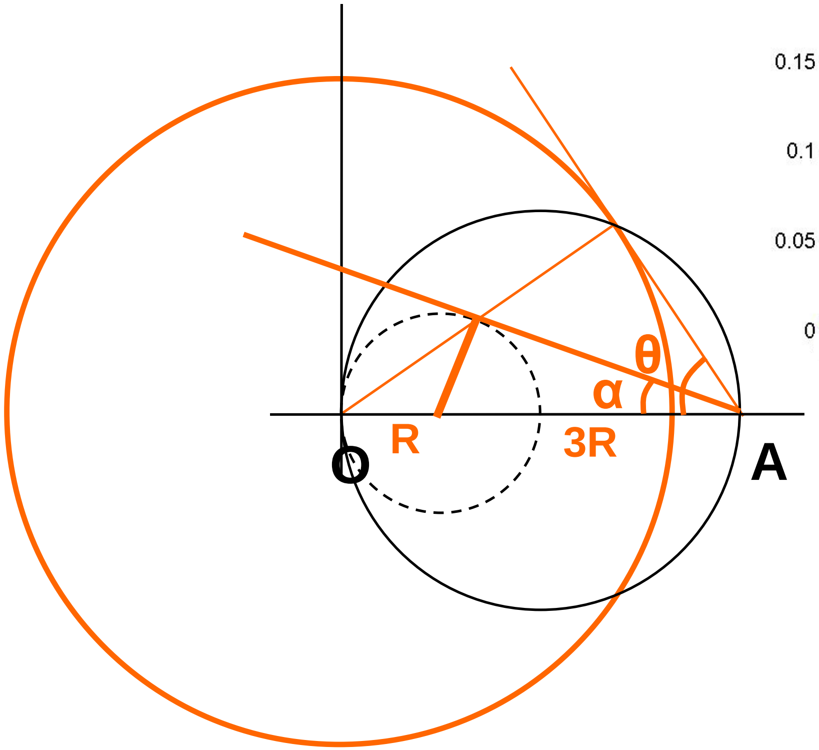


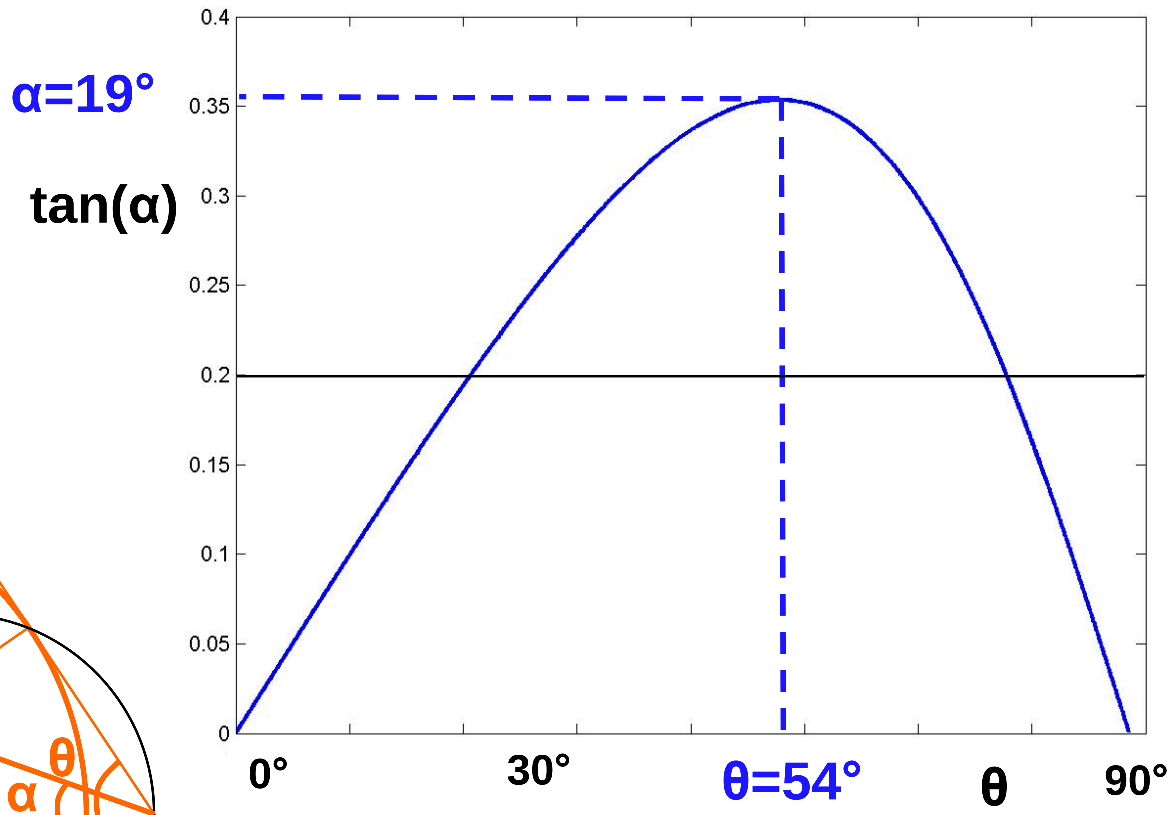
$\Rightarrow \sin(\alpha)=\cos(\theta)\sin(\theta-\alpha)$

$\Rightarrow \sin(\alpha)=\cos(\theta)(\sin(\theta)\cos(\alpha)+\cos(\theta)\sin(\alpha))$

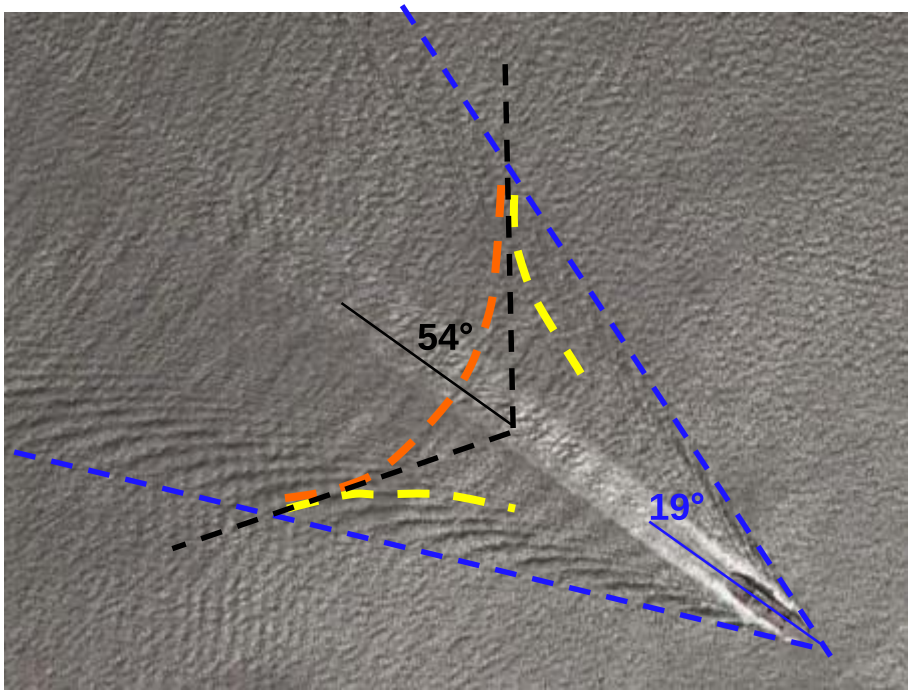
$\Rightarrow \tan(\alpha)=\cos(\theta)\sin(\theta)/(1+\cos^2(\theta))$

Gravity waves created by a ship

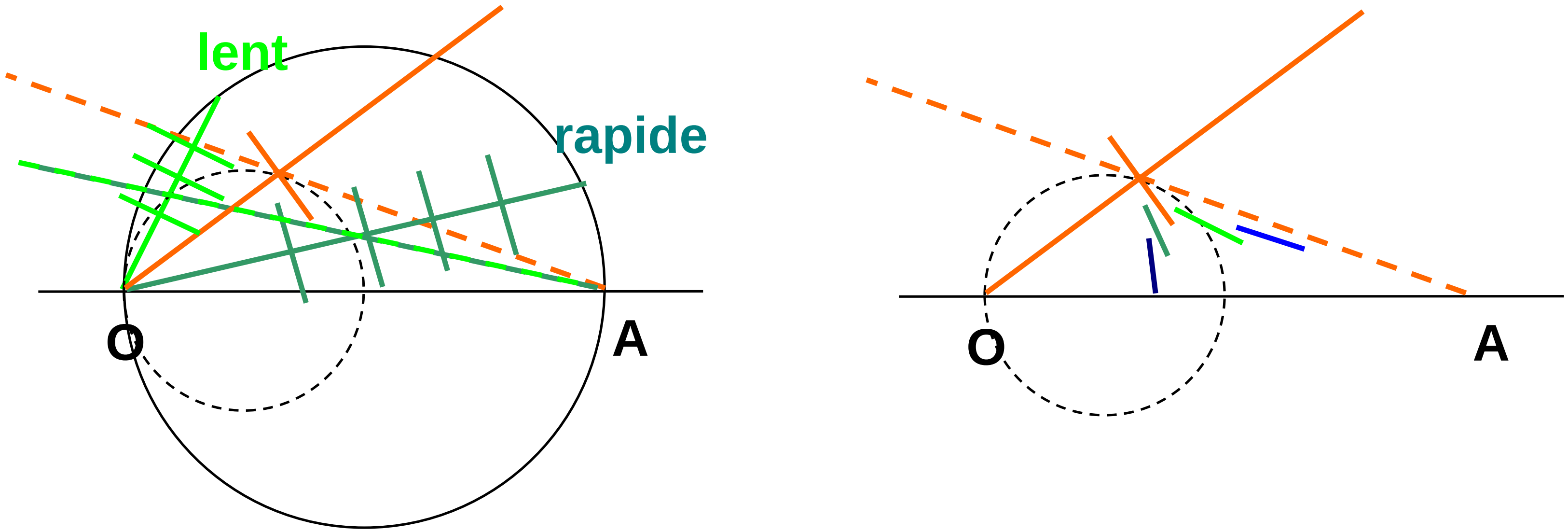




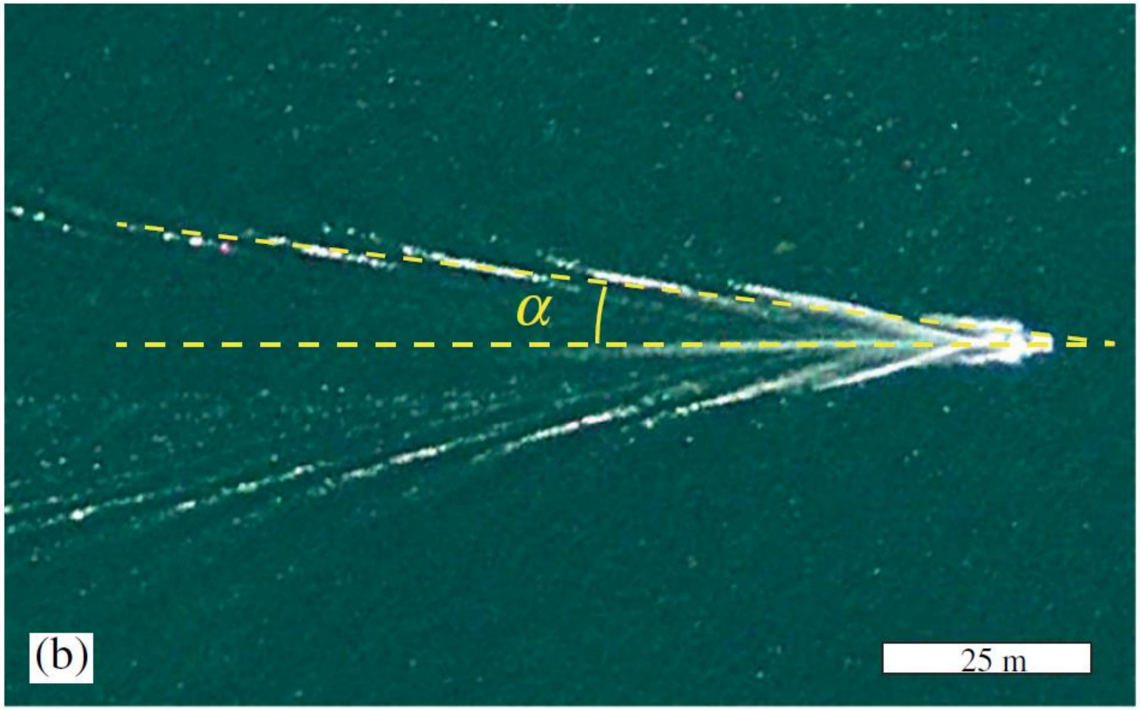
Gravity waves created by a ship



Gravity waves created by a ship

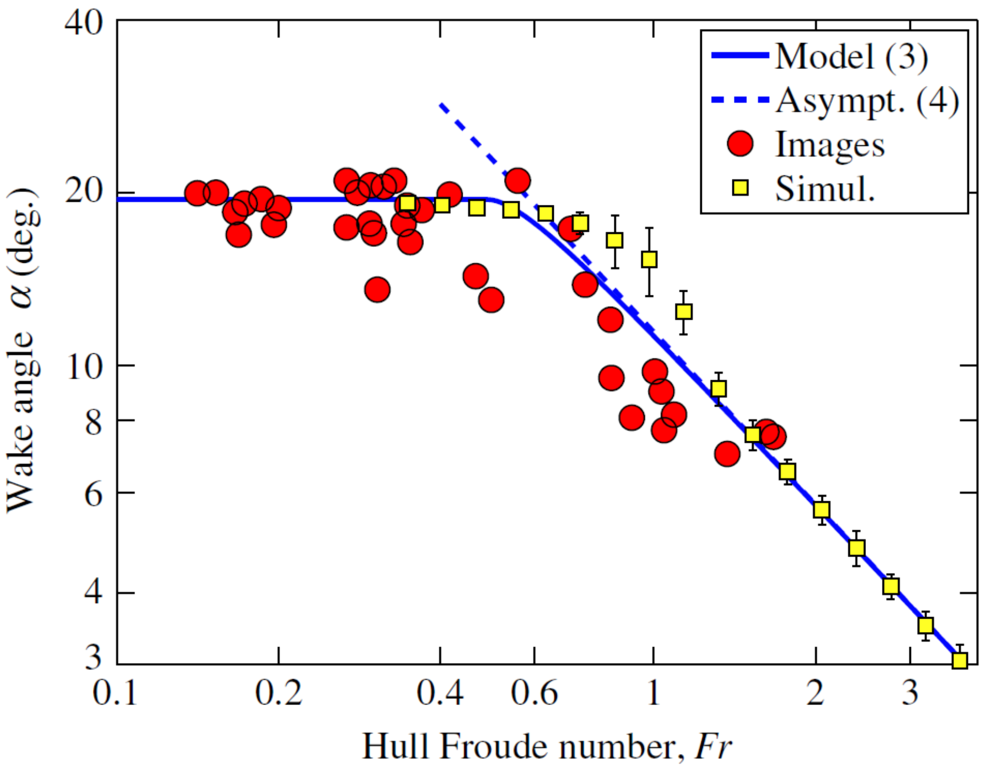


But observations show



Moisy and Rabaud 2013

But observations show

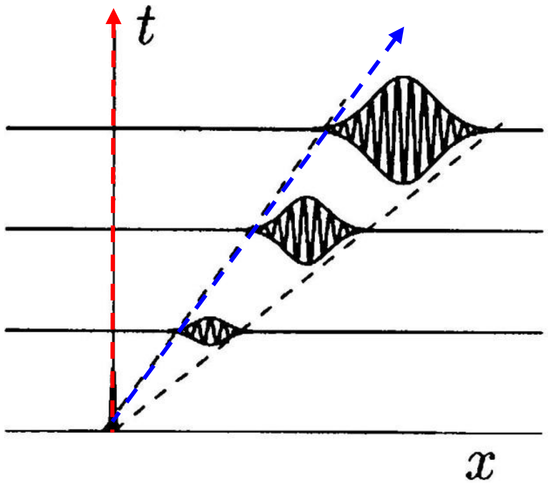


Moisy and Rabaud 2013

Generalization: Spatio-temporal instability theory

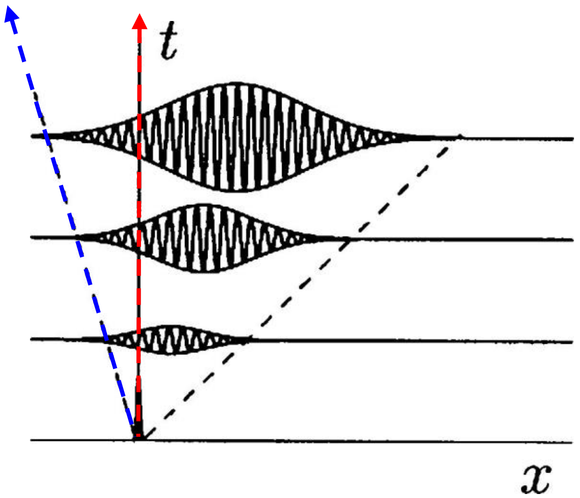
First find the zero group velocity wave: $d\omega/dk=0 \Rightarrow (k_0,\omega_0)$
and consider the sign of $\mathrm{Im}(\omega_0)$

☞ Convective instability



$\mathrm{Im}(\omega_0)<0$

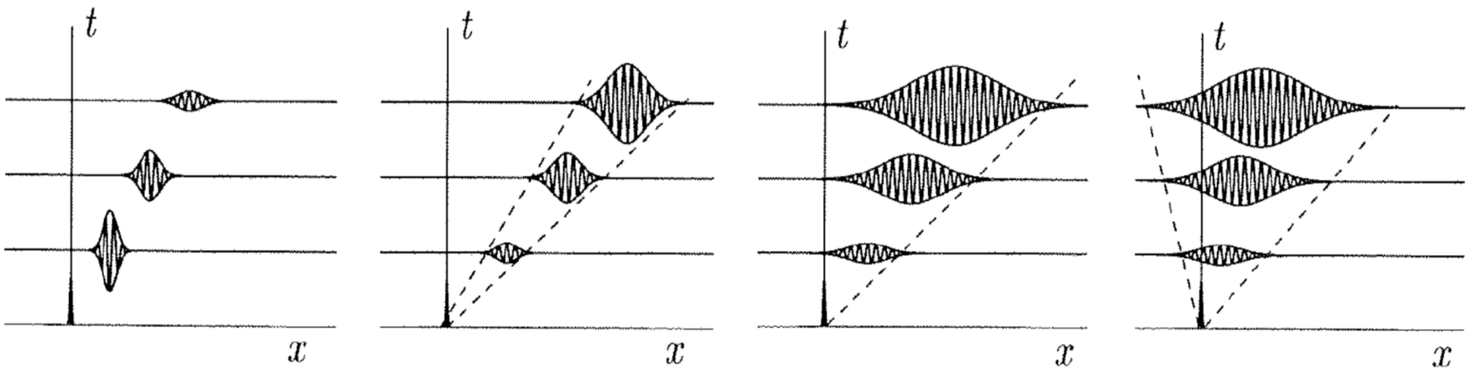
F Absolute instability



$\mathrm{Im}(\omega_0)>0$

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Green's function or impulse response



Stable Unstable Unstable Unstable

Briggs (1964) Bers (1983)

Huerre and Monkewitz (1985)

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Linearly stable flow

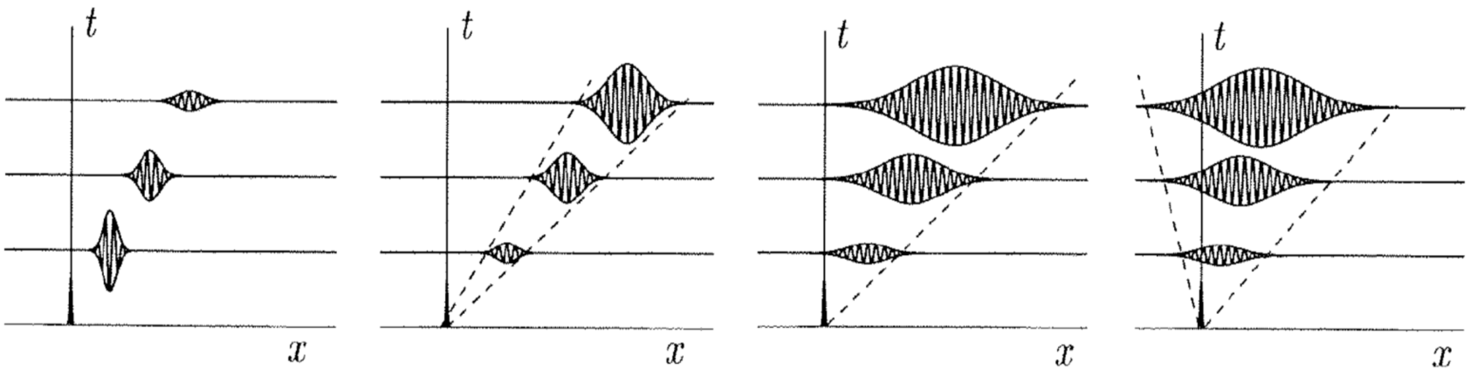
$$\lim_{t \to \infty} G(x,t) = 0$$ along all rays x/t = const.

Linearly unstable flow

$$\lim_{t \to \infty} G(x,t) = \infty$$ along at least one ray x/t = const.

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Green's function or impulse response



Stable Unstable Unstable Unstable

Briggs (1964) Bers (1983)

Huerre and Monkewitz (1985)

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Convectively unstable flow

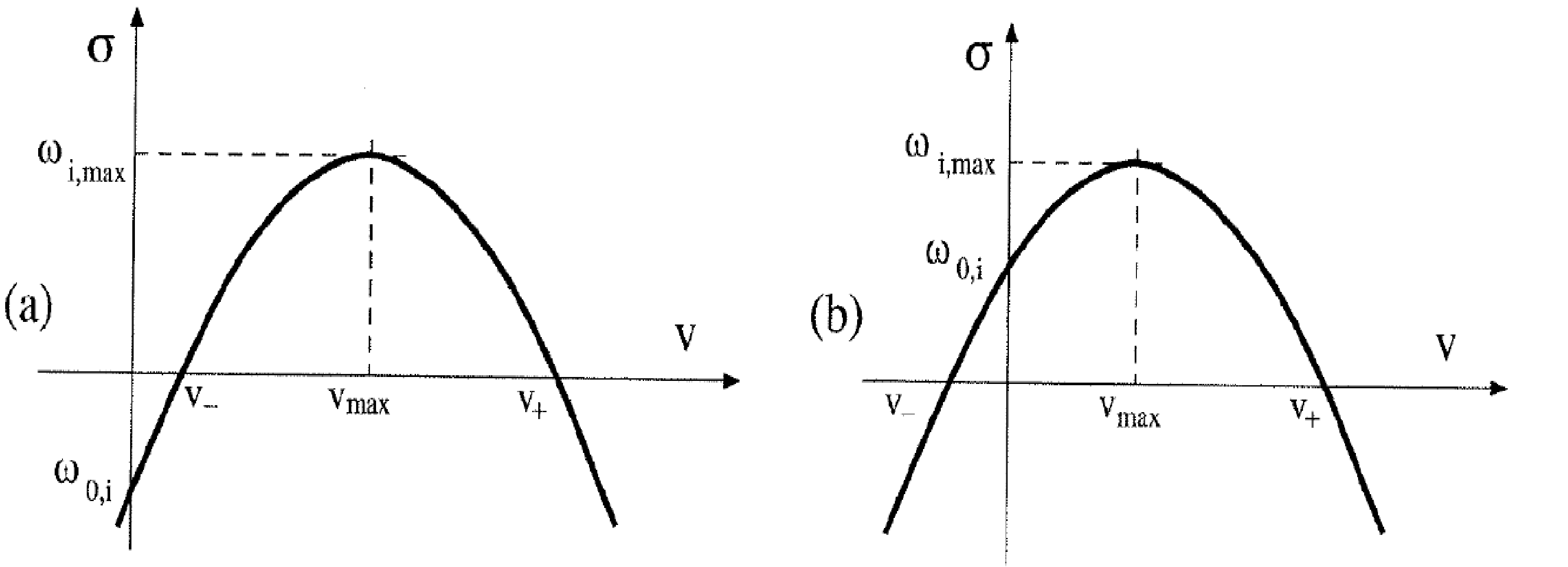
$$\lim_{t \to \infty} G(x,t) = 0$$ along the ray x/t = 0

Absolutely unstable flow

$$\lim_{t \to \infty} G(x,t) = \infty$$ along the ray x/t = 0

ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

Temporal growth rate « at velocity v »



Convective instability

Absolute instability

ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

Important notions

Absolute wavenumber k_0 and frequency $\omega_0 = \omega(k_0)$
observed along ray v = 0, i.e. for a stationary observer,
defined by

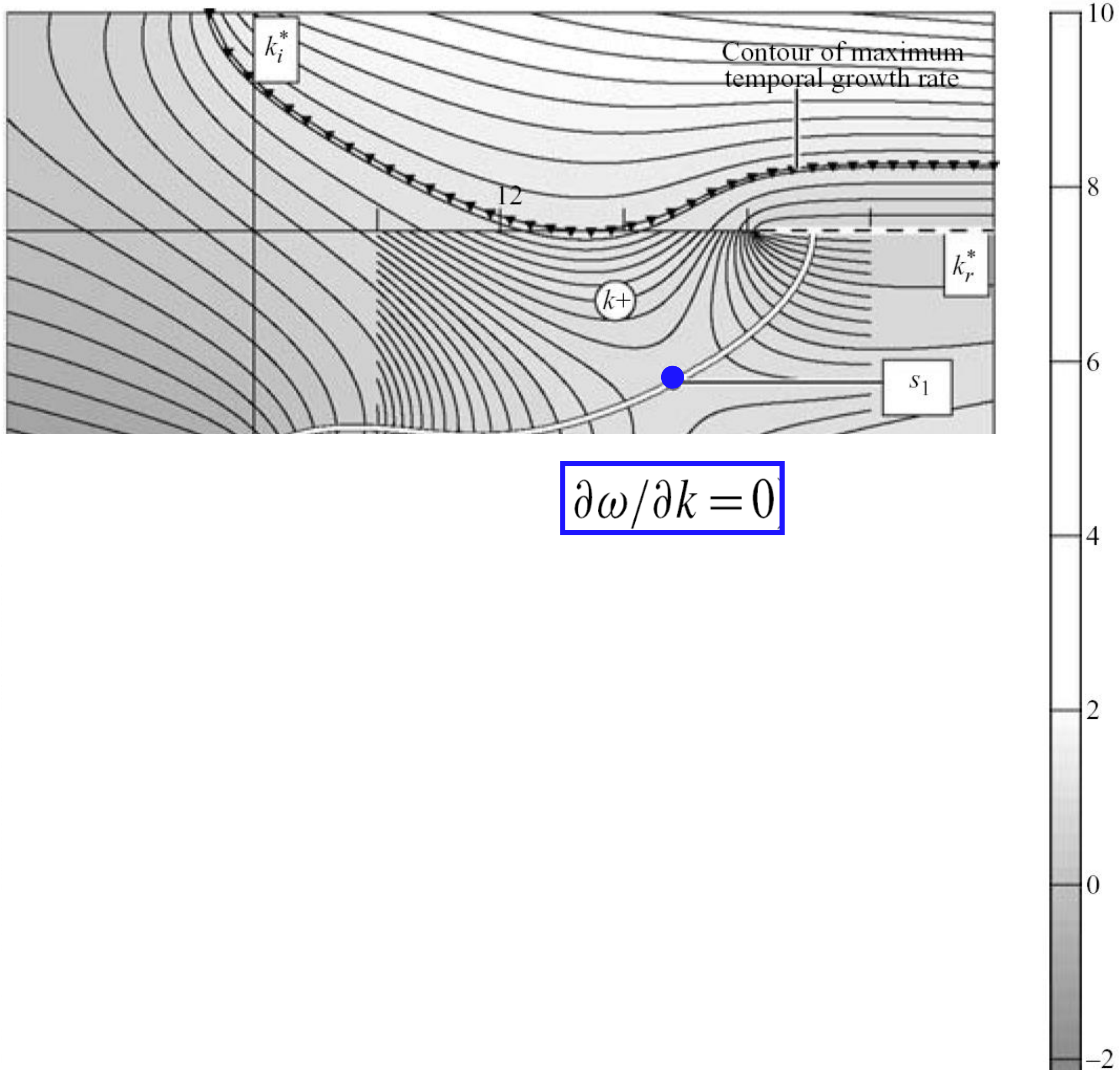
$$\frac{\partial \omega}{\partial k}(k_0) = 0$$

Absolute growth rate is

$$\sigma(0) = \omega_{0,i}$$

Isovaleurs de ω_i

Absolute frequency ω_{0}: Saddle point condition



$$\boxed{\partial\omega/\partial k = 0}$$

ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

Instability criteria

$\omega_{i,max} < 0$	linearly stable
$\omega_{i,max} > 0$	linearly unstable
$\omega_{0,i} < 0$	convectively unstable
$\omega_{0,i} > 0$	absolutely unstable

Hyperbolic tangent mixing layer

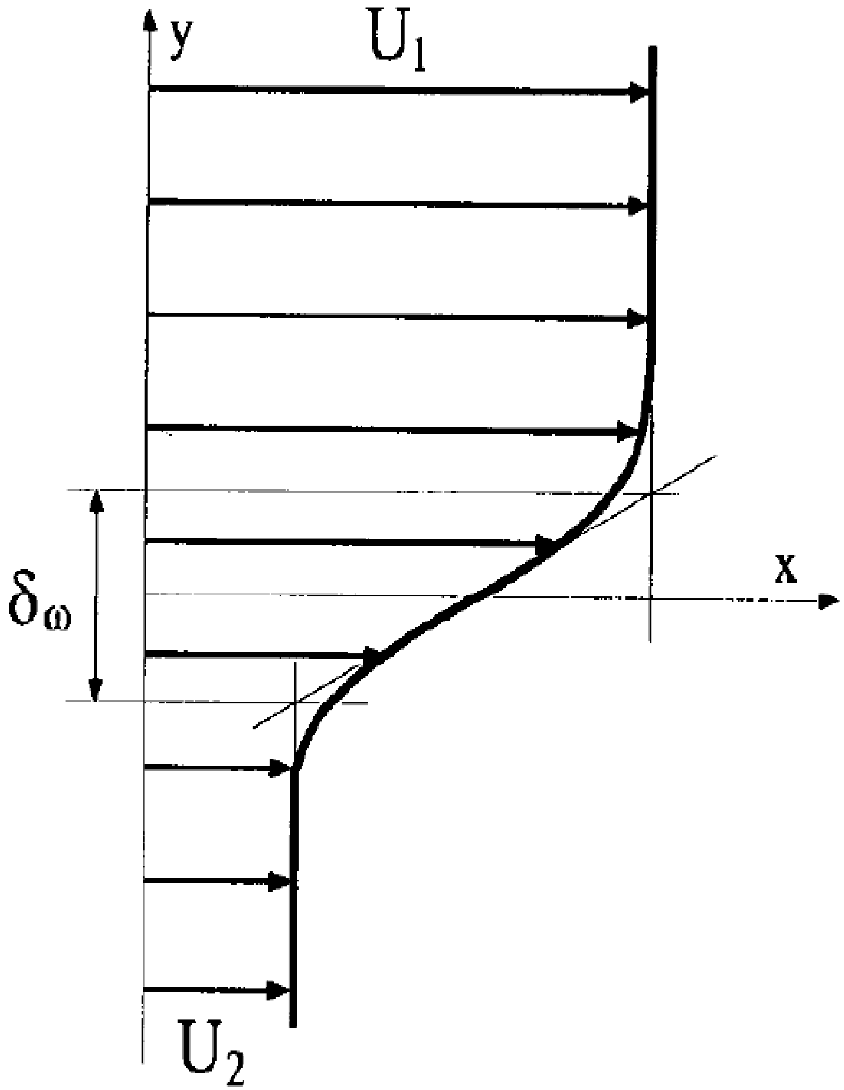
$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

Velocity ratio

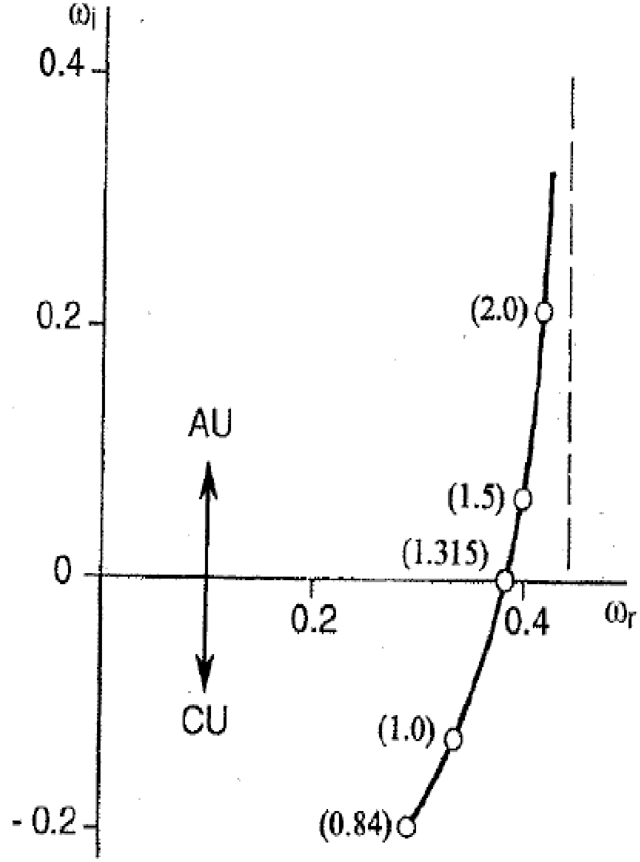
$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$

$$\boxed{U(y; R) = 1 + R \tanh y}$$

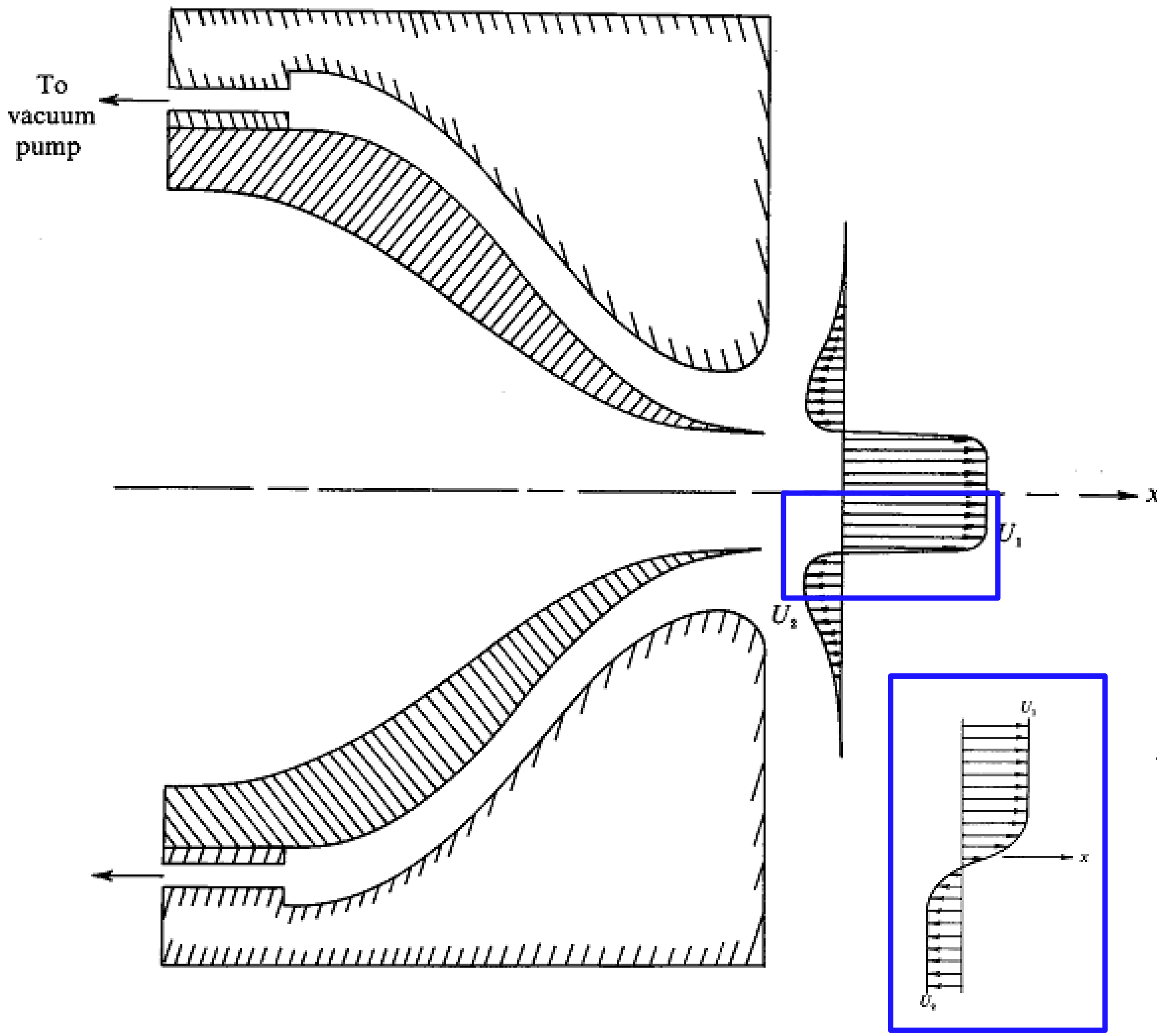


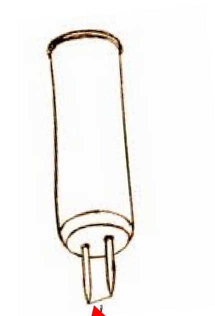
APPLICATION TO MIXING LAYERS

Locus of complex absolute frequency



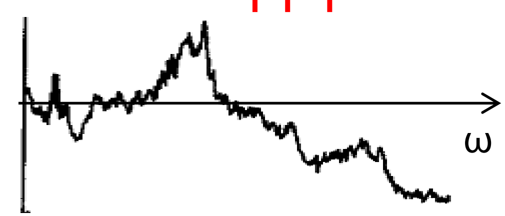
H.&Monkewitz (1985)



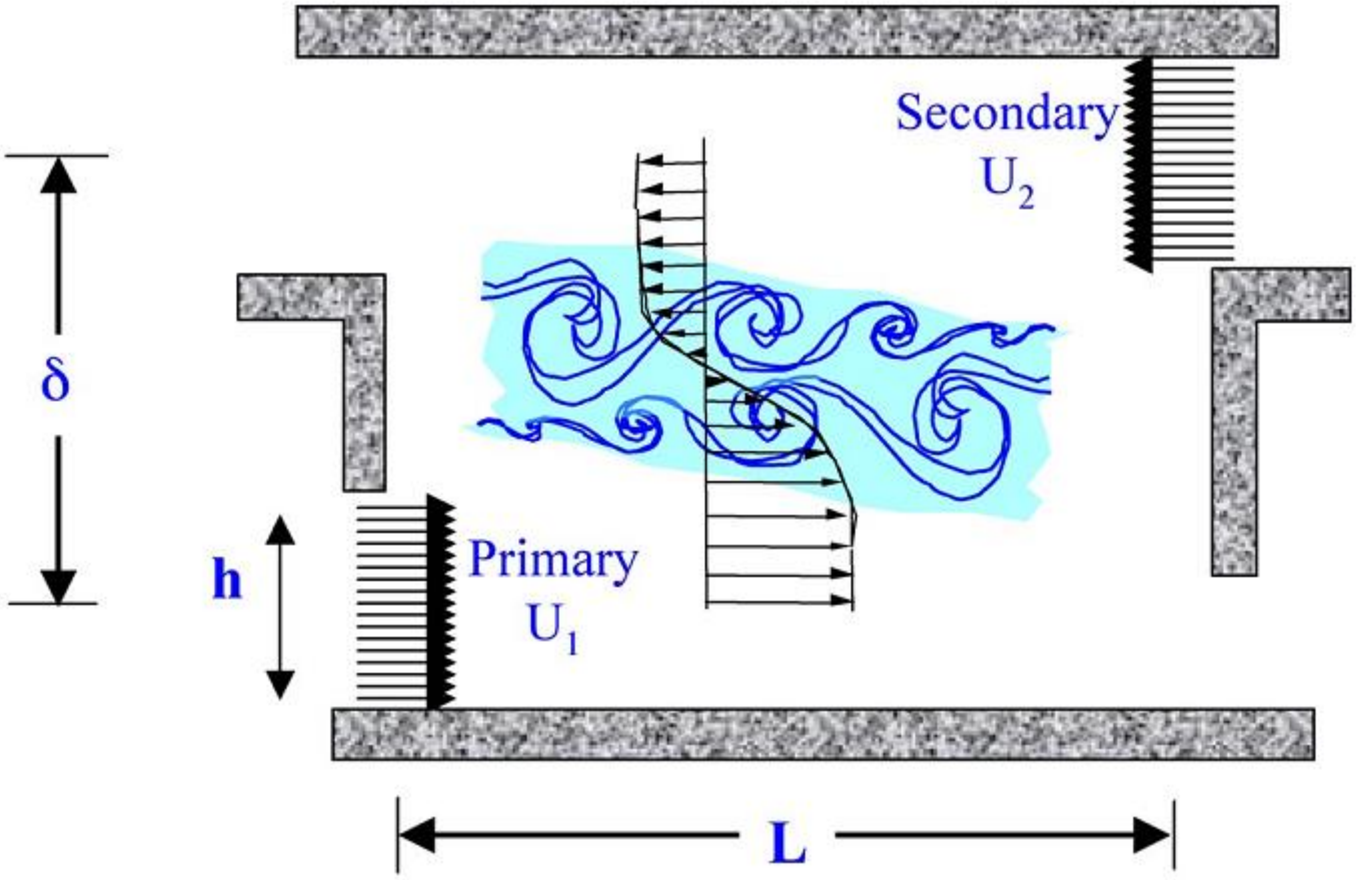


Hot wire measurement

FFT

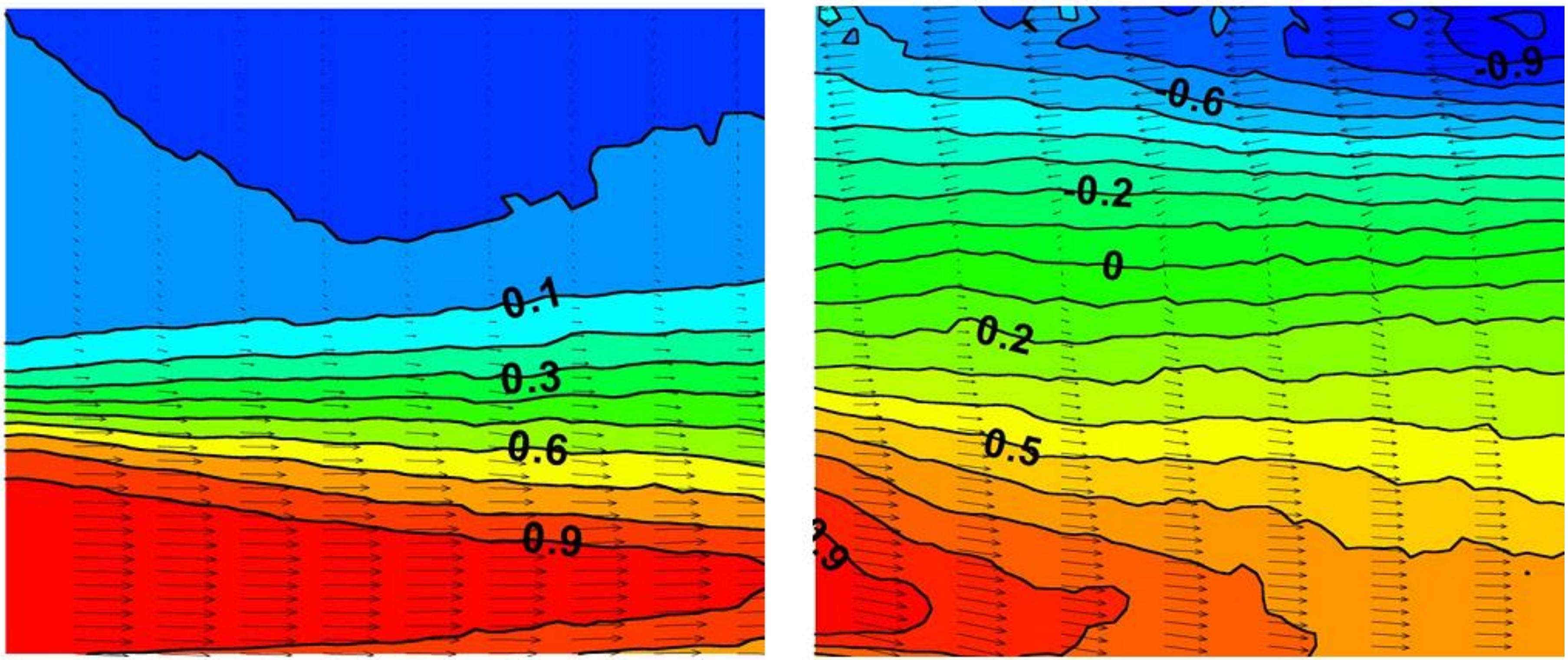


Influence of coutercurrent shear on turbulence level



Influence of coutercurrent shear on turbulence level

Base flow

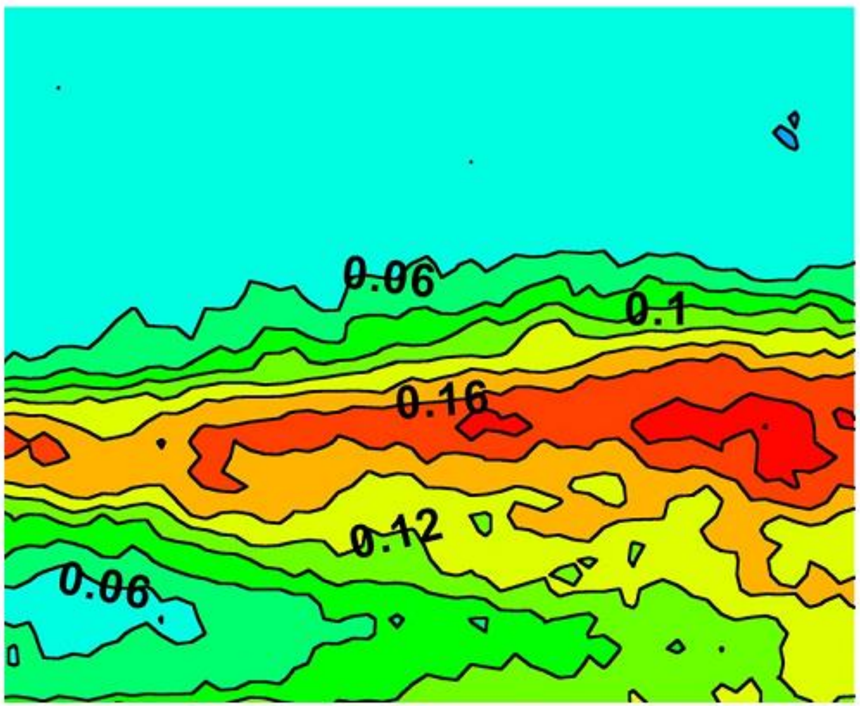


a) Single stream shear layer

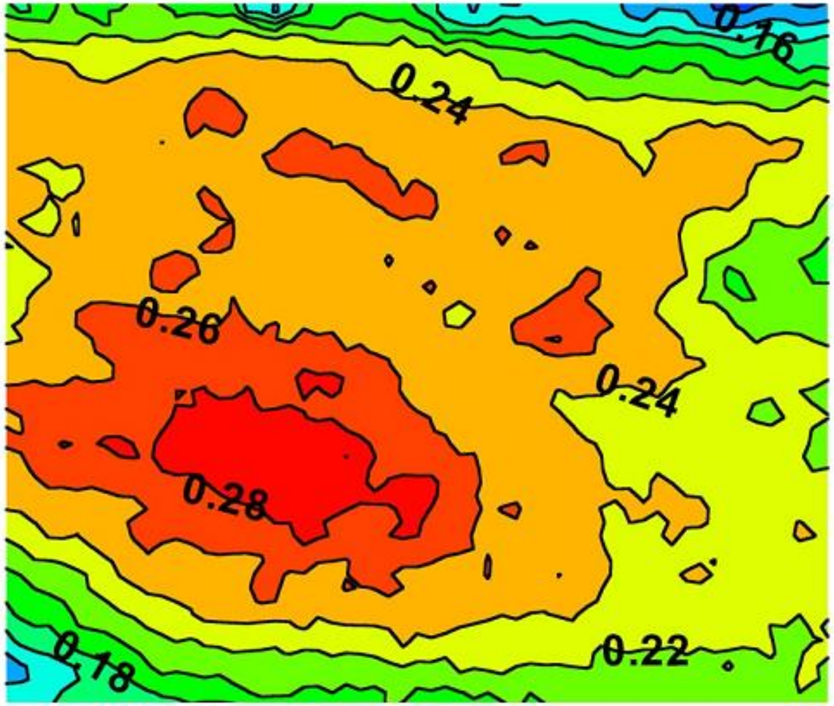
b) Countercurrent shear layer

Influence of coutercurrent shear on turbulence level

Turbulence intensity

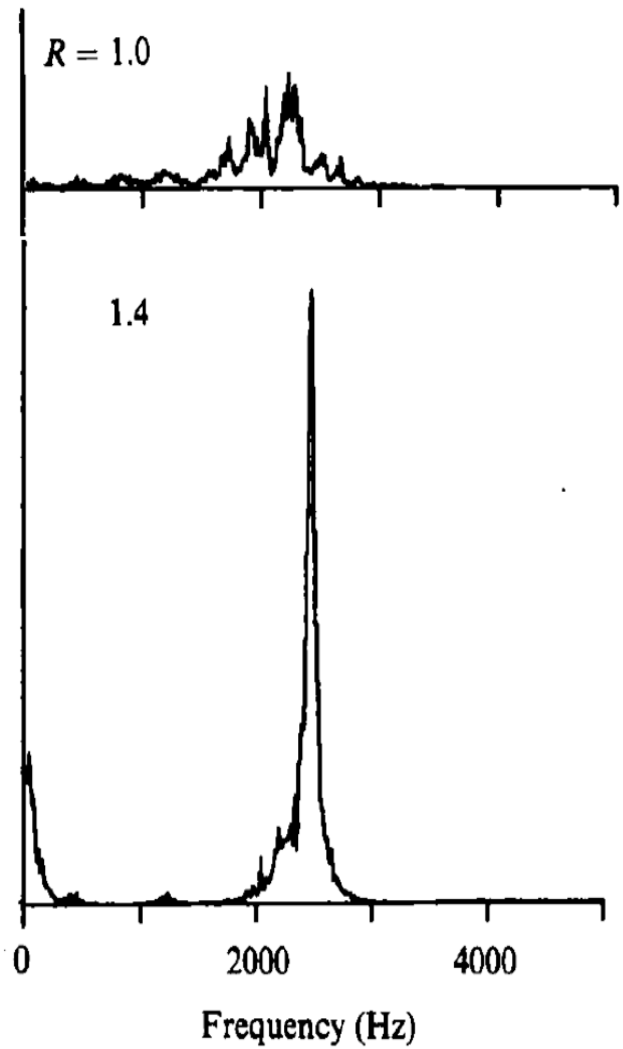


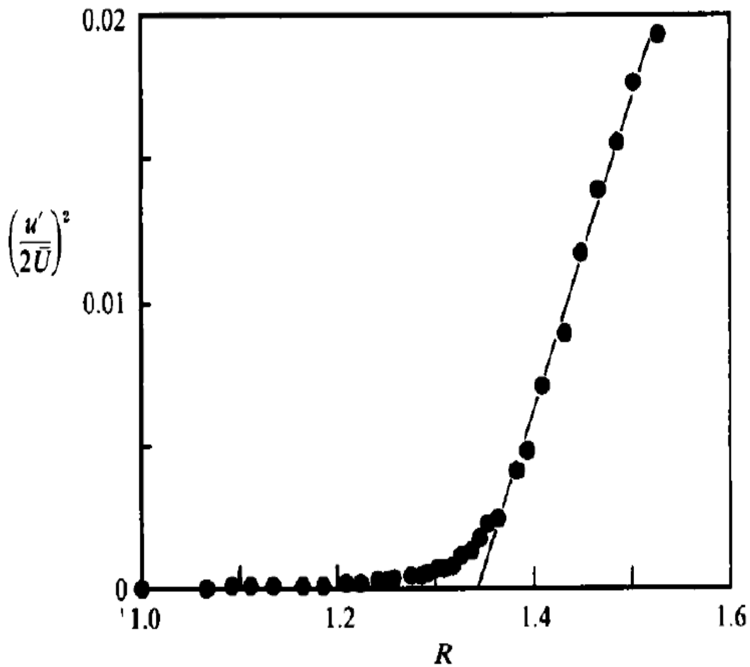
a) Single-stream shear layer



b) Countercurrent shear layer

THE MIXING LAYER: SHIFT TO OSCILLATOR !



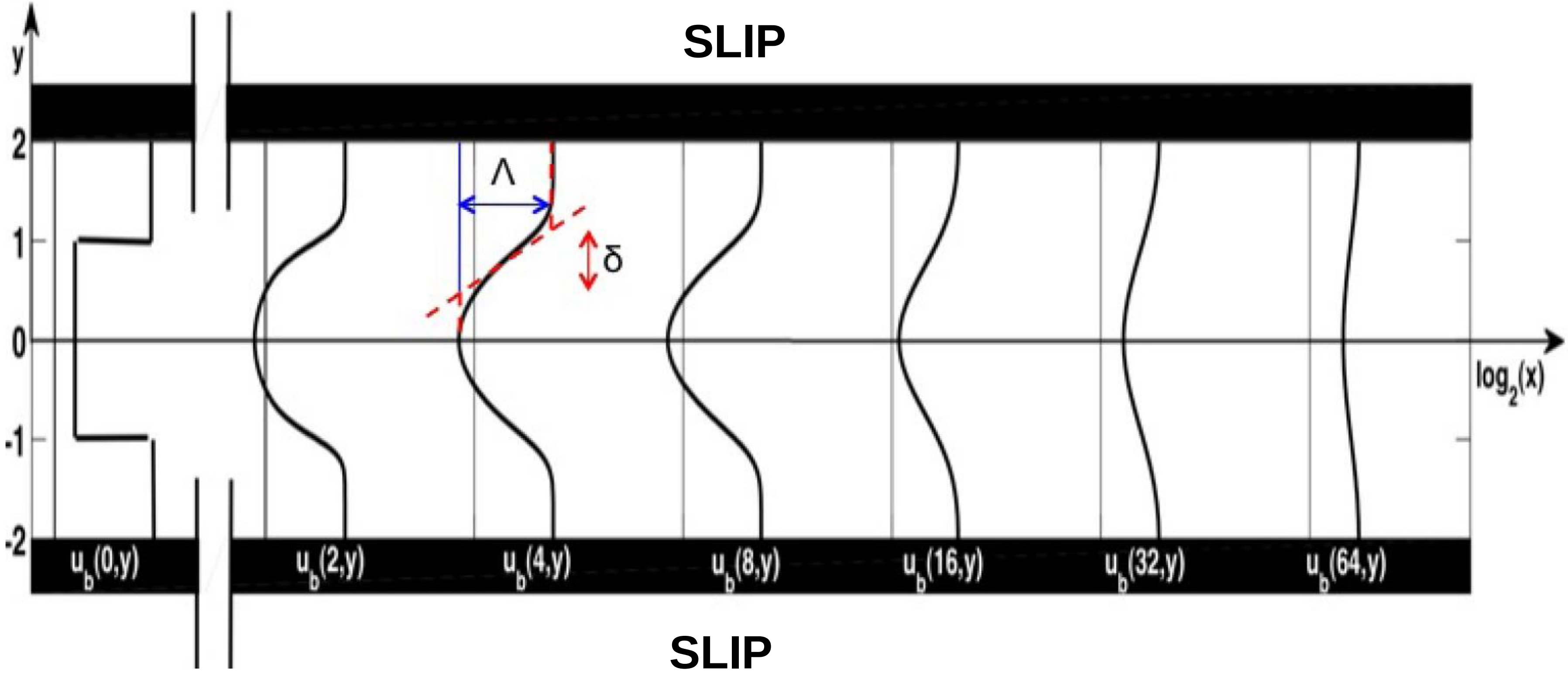


Strykowski & Niccum (1991)

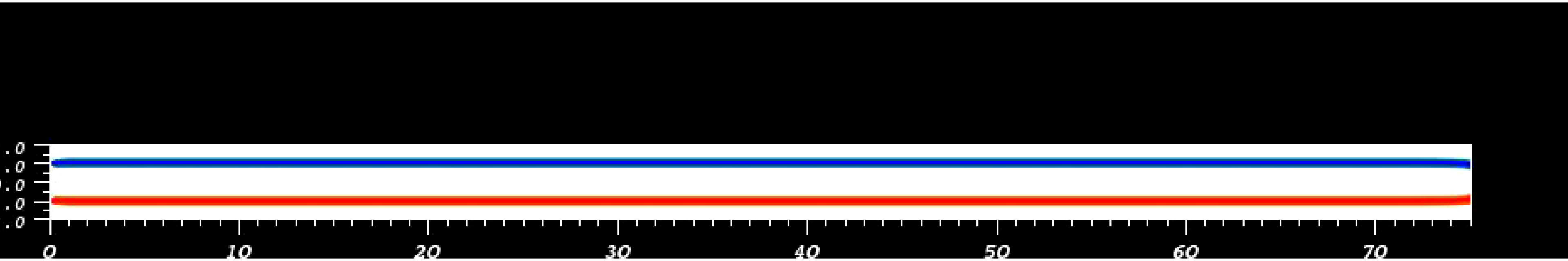
Direct Numerical Simulations with top-hat profile at inlet

Viscous diffusion $\longrightarrow$ Non-parallel flow

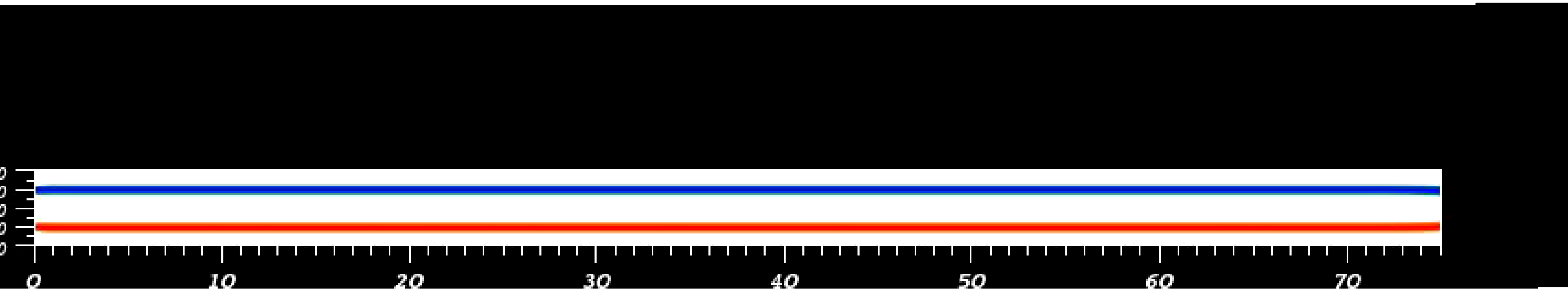
- $\Lambda_{loc} = (U_{max} - U_{min})/(U_{max} + U_{min})$
- $\delta = (U_{max} - U_{min})/(|dU/dy|_{max})$



Vorticity field: Re = 100, h = 1



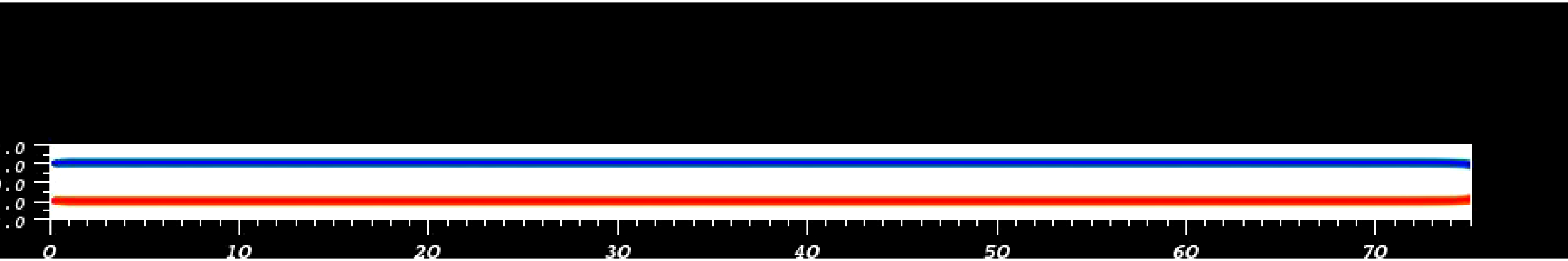
Λ = -0.739



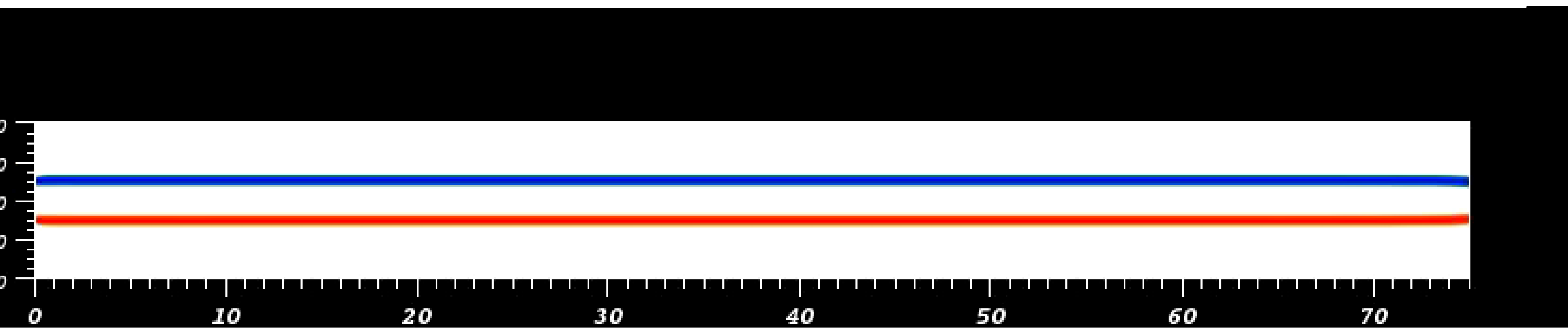
Λ = -0.667

An increase in Λ (more coflow) advects the perturbation

Vorticity field: Re = 100, Λ = -0.739



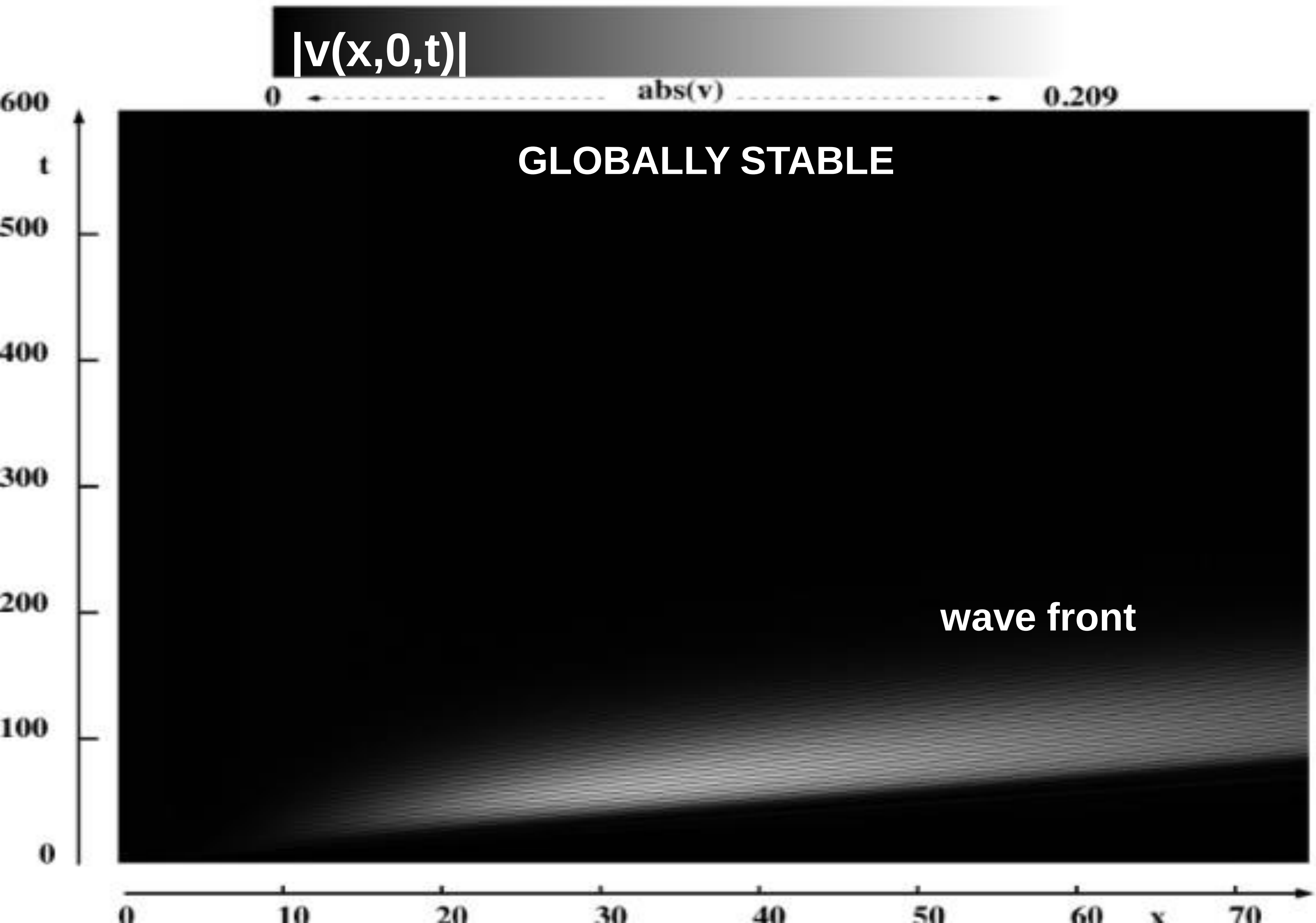
h=1



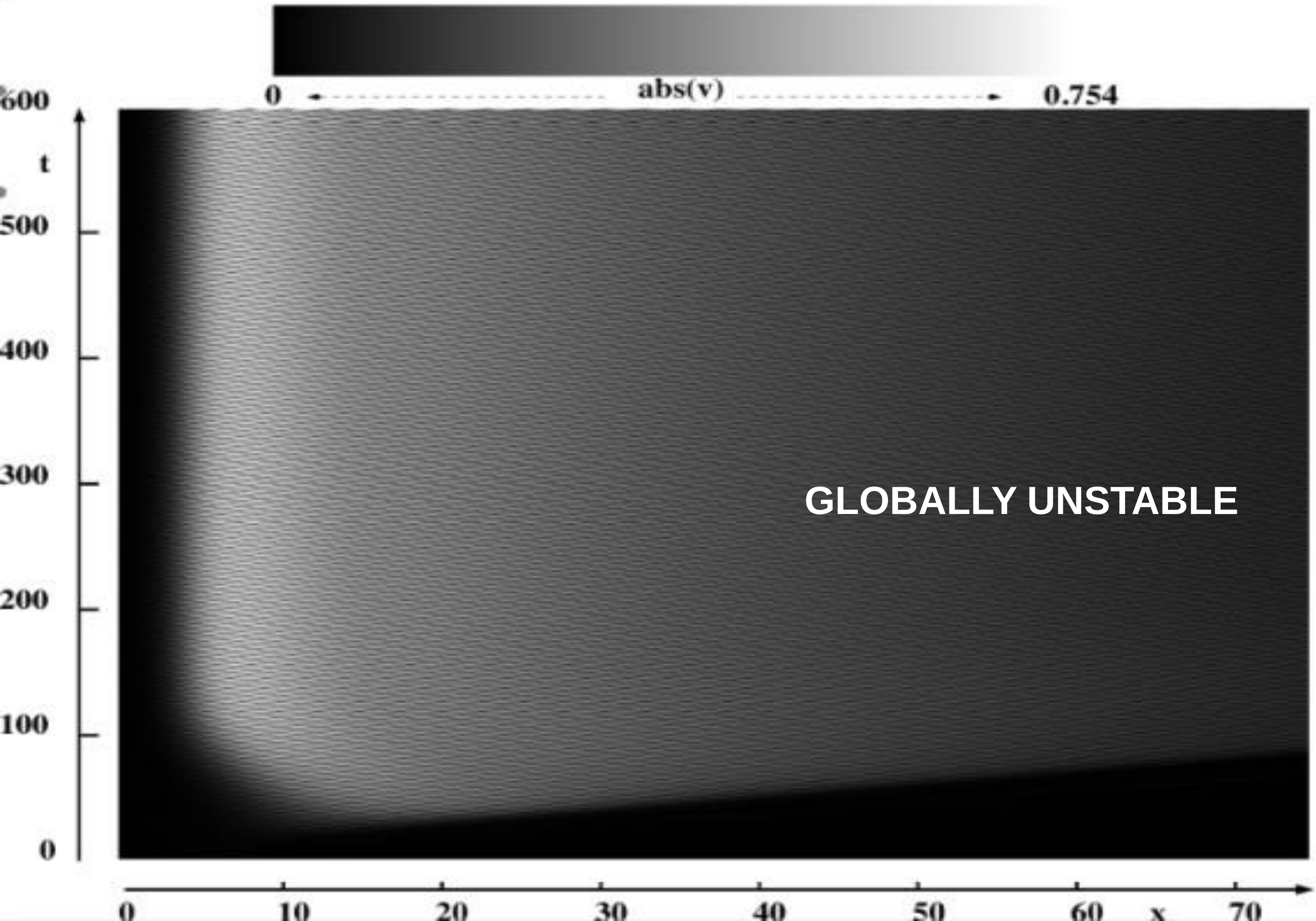
h=3

Destabilizing influence of confinement!

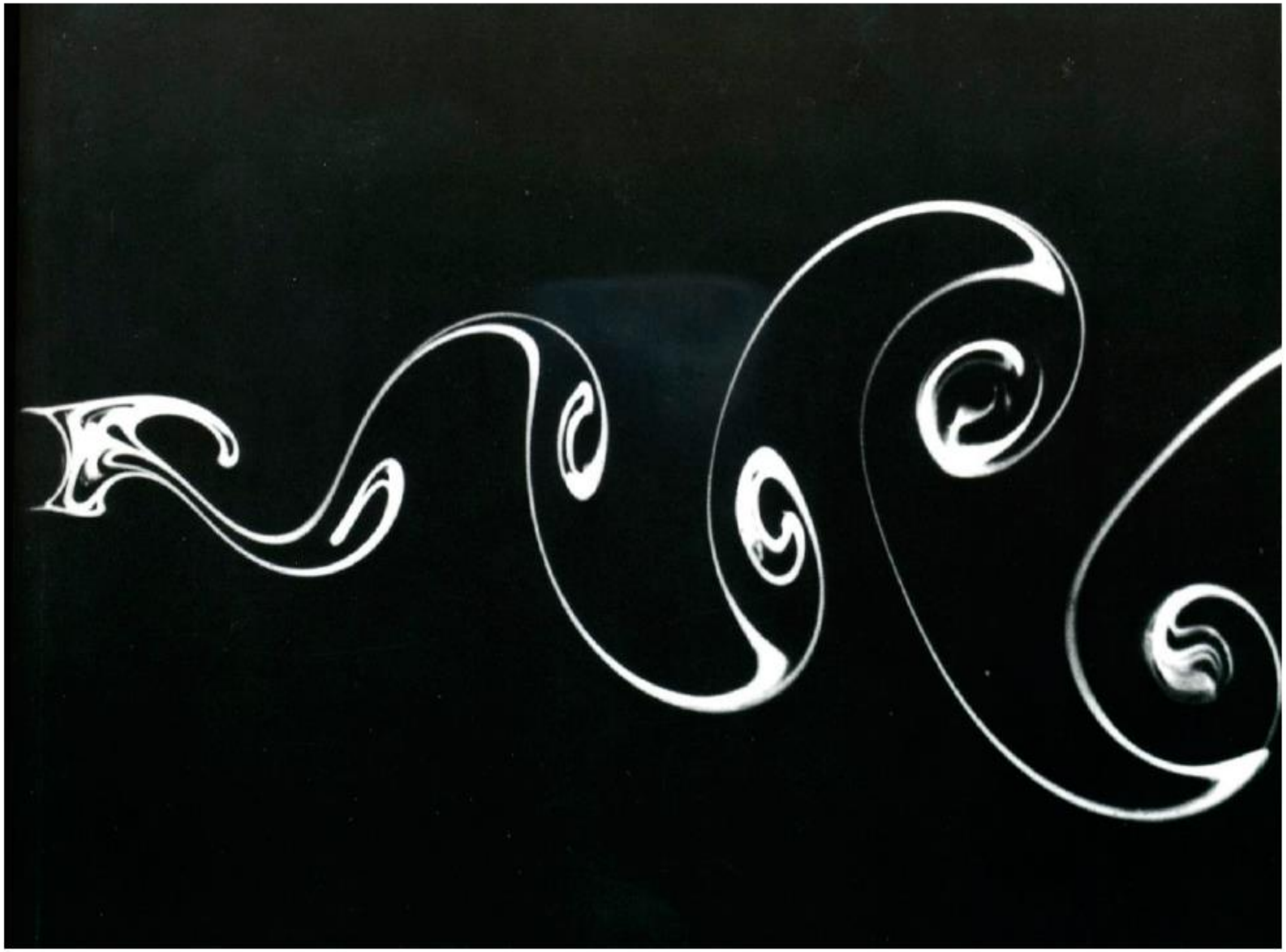
Spatio-temporal diagram, h=1 and Λ = -0.667



Spatio-temporal diagram, h=1 and Λ = -0.739



THE BLUFF BODY WAKE: A TYPICAL FLOW OSCILLATOR



Re = 140
Periodic
flow

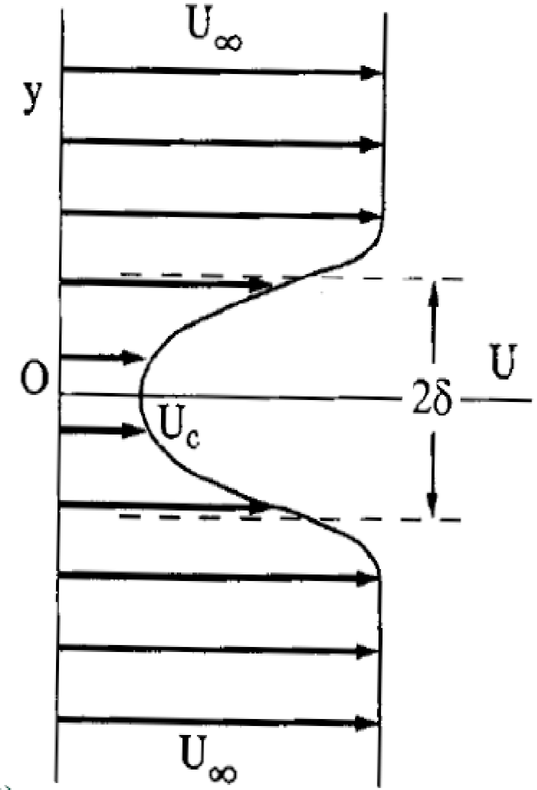
Taneda (1982)

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Family of wake profiles

$$U(y) = U_\infty + (U_c - U_\infty)\, U_1\left(\frac{y}{\delta}; N\right)$$

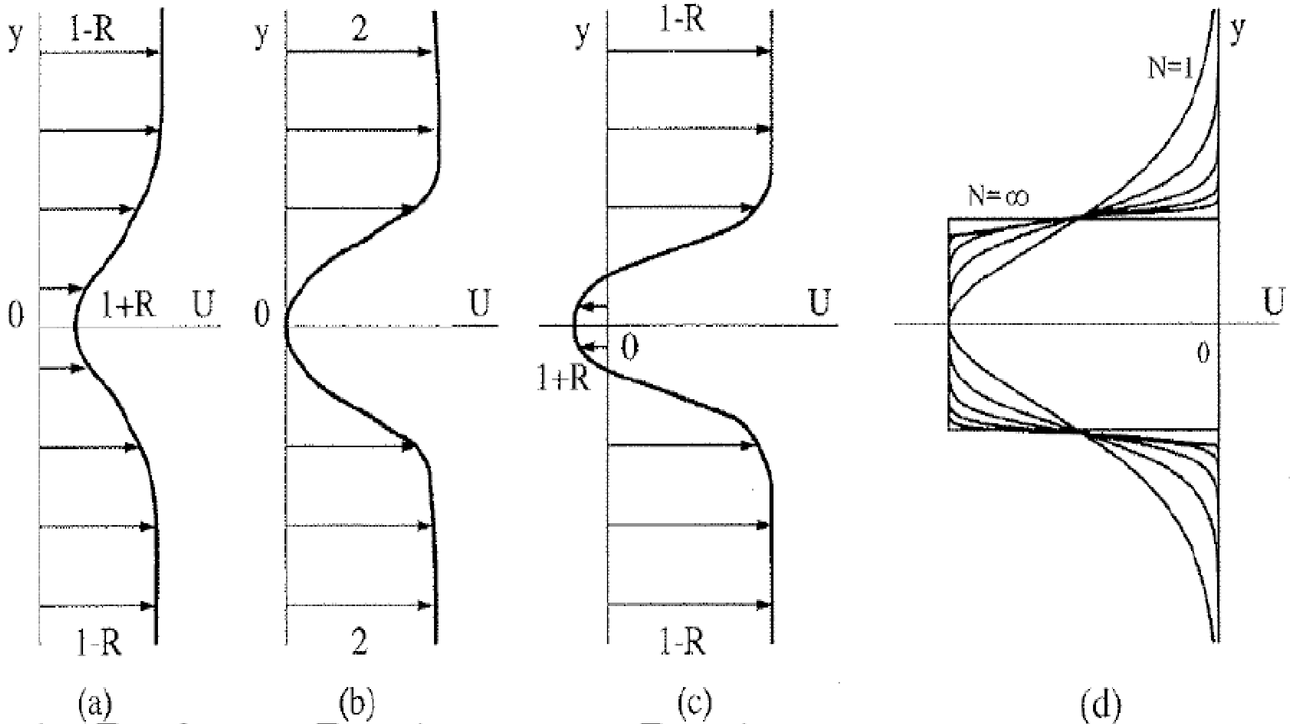
$$U_1(\xi; N) = [1 + \sinh^{2N}\{\xi \sinh^{-1}(1)\}]^{-1}$$



Monkewitz (1988)

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Family of wake profiles



$-1 < R < 0$ **$R = -1$** **$R < -1$**

Effect of velocity ratio R

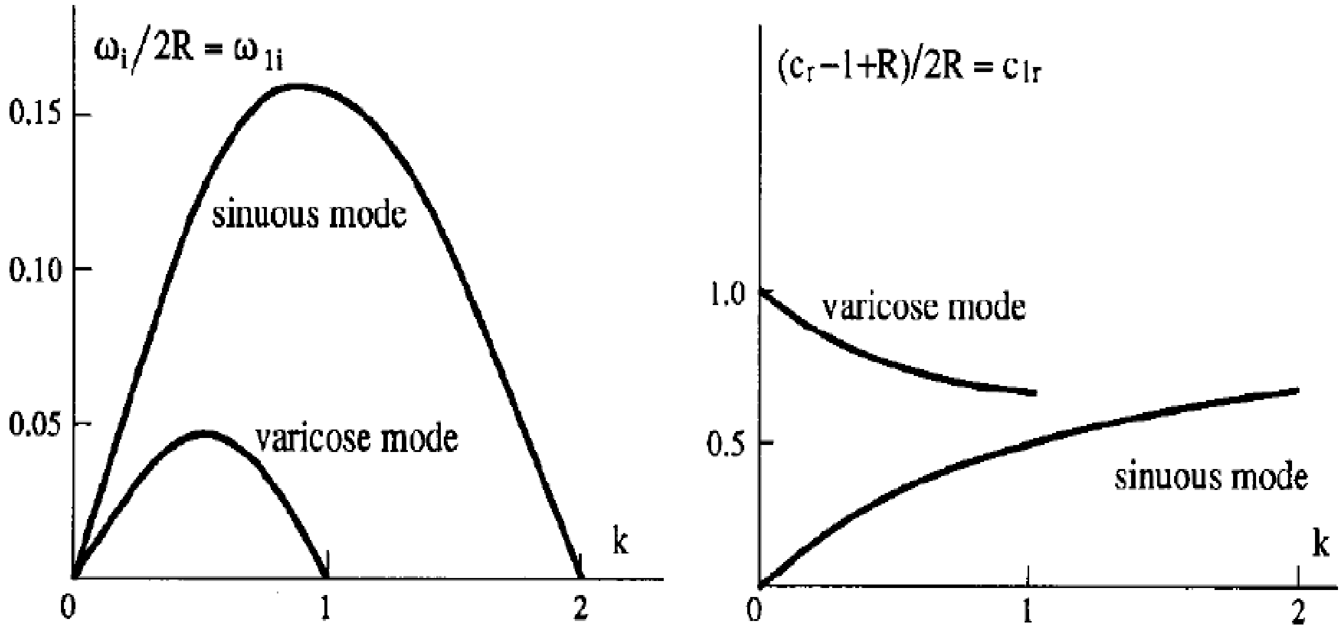
Effect of N

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL

2D PARALLEL FLOW CONCEPTS

$\mathrm{sech}^2\, y$ wake

Temporal approach

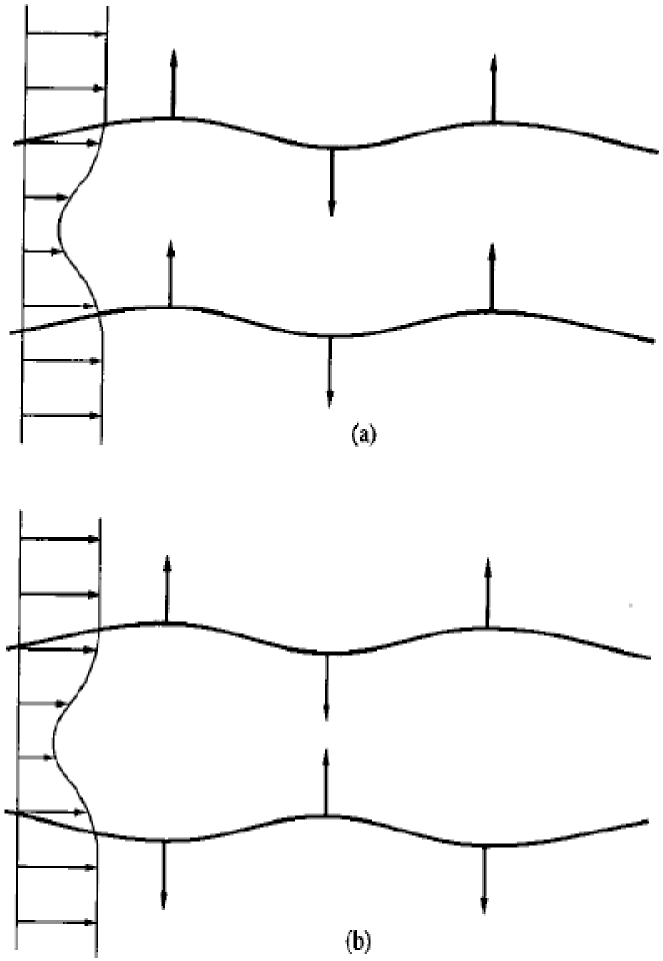


Betchov & Criminale (1966)

2D PARALLEL FLOW CONCEPTS

$\operatorname{sech}^2 y$ wake

Sinuous and varicose modes

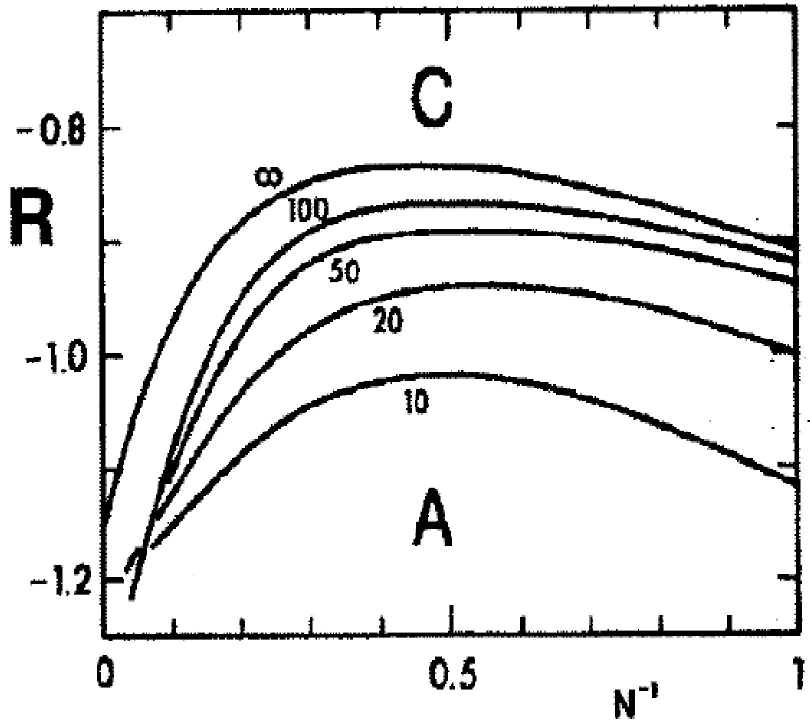


(a) sinuous

(b) varicose

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Effect of steepness, velocity ratio and Reynolds number



Monkewitz (1988)

LOCAL INSTABILITY BEHAVIOR OF CYLINDER WAKE

$5 < Re < 25$ **Convective instability**

$25 < Re < 48.5$ **Absolute instability**