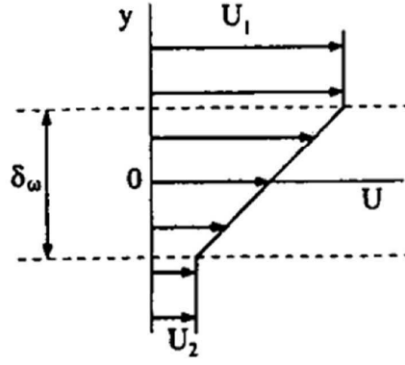


The 'broken line' profile.

We want to analyze the stability of a simplified model of a mixing layer. As illustrated in the figure 1, we consider a parallel flow $\mathbf{U} = U(y)\mathbf{e}_x$ consisting of three regions: two semi-infinite regions with constant velocity $U = U_1$ and $U = U_2$ connected by a region in which U varies linearly.



$$U(y) = \begin{cases} U_1, & y > \delta_\omega/2 \\ (U_1 + U_2)/2 + (U_1 - U_2)y/\delta_\omega, & |y| < \delta_\omega/2 \\ U_2, & y < -\delta_\omega/2 \end{cases}$$

Figure 1: Broken line profile.

For simplicity, we consider a single incompressible two-dimensional (2D) fluid of constant density ρ and zero viscosity. It can be easily verified that $\mathbf{U}$, together with an arbitrary constant pressure P_0 (or an hydrostatic pressure profile if one wants to account for gravity) is a stationary solution of the Euler equations. To analyze the stability of $\mathbf{U}$ we proceed, as usual, by considering the governing equations for the perturbed velocity $\mathbf{U} + \epsilon \mathbf{u}(x, y, t)$ and associated pressure $P_0 + \epsilon P(x, y, t)$. Dropping terms proportional to ϵ^2 , one obtains that the 2D Euler equations linearised around

the base flow $\mathbf{U} = U(y)\mathbf{e}_x$ are

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} - v \frac{dU}{dy}, \quad (1a)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}, \quad (1b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1c)$$

where u and v are respectively the velocities in the x and y directions, $p \equiv P/\rho$ is the pressure divided by the (constant) density and the explicit dependence on x, y and t is omitted. We look for solutions such that u, v go to zero as $|y|$ goes to infinity.

The three equations (1) governing u, v and p can be reduced to a single vorticity equation, which for the present base flow $U(y)$ (of figure 1) becomes

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \Delta \psi = 0, \quad (2)$$

where ψ is the stream function that satisfies

$$u = \frac{\partial \psi}{\partial y}, \quad (3a)$$

$$v = -\frac{\partial \psi}{\partial x}. \quad (3b)$$

Question 1. We look for plane waves solutions of the form

$$\psi(x, y, t) = \phi(y)e^{i(kx - \omega t)}, \quad (4)$$

where $k > 0$ and one should keep in mind that the solution will be given by the real part. Replace this expression in 2 to obtain an ordinary differential equation for ϕ . Express the boundary conditions for u and v when $y \rightarrow \pm\infty$ in terms of ϕ . We can assume in the following that $U \neq \omega/k$, so that the equation for ϕ becomes

$$\frac{d^2 \phi}{dy^2} - k^2 \phi = 0, \quad (5)$$

Question 2. Similar to the example shown in classes, equation 5 can now be solved in each of the three distinct regions in which $U(y)$ is defined (see fig. 1). Write the corresponding solutions. For $y < -\delta_\omega/2$ and for $y > \delta_\omega/2$, take into account the boundary conditions at $|y| \rightarrow \infty$.

Question 3. In the previous question, you should have ended up with four complex constants that should be determined through boundary conditions at $y = \pm\delta_\omega/2$, where the derivative of $U(y)$ is discontinuous. Two boundary conditions should be imposed at each interface. One is given by the kinematic condition at the interface or, equivalently, by the continuity of the vertical displacement of fluid particles. Show that for a discontinuity at $y = y_0$, this condition leads to

$$\left. \frac{\phi(y)}{U(y) - \omega/k} \right|_{y=y_0^+} = \left. \frac{\phi(y)}{U(y) - \omega/k} \right|_{y=y_0^-}. \quad (6)$$

The other boundary condition can be obtained from equation (1a) after requiring the continuity of the pressure at the interface. Show that this requirement imposes the condition

$$\left[\left(U(y) - \frac{\omega}{k} \right) \frac{d\phi}{dy} - \frac{dU}{dy} \phi(y) \right]_{y=y_0^+} = \left[\left(U(y) - \frac{\omega}{k} \right) \frac{d\phi}{dy} - \frac{dU}{dy} \phi(y) \right]_{y=y_0^-}. \quad (7)$$

Question 4. Write the conditions at the two interfaces $y = \pm\delta_\omega/2$ that (6) and (7) impose on the constants found on question 2. You should have four equations and four constants. Reduce the system to two equations and two undetermined constants. (Hint: The equations imposed by condition (6) allow to easily eliminate the constants of the $|y| > \delta_\omega/2$ solutions). Now you must have a system of equations that can be written in the form

$$M\mathbf{a} = 0 \quad (8)$$

where M is 2×2 matrix and $\mathbf{a}$ is a vector with the undetermined constants. Equation (8) admits non trivial solutions for $\mathbf{a}$ only if M has zero determinant. Compute the dispersion relation, i.e. show that the condition such that $\det(M) = 0$ is

$$4(k\delta_\omega)^2 \left(\frac{\omega}{k} - \bar{U} \right)^2 - [(k\delta_\omega - 1)^2 - e^{-2k\delta_\omega}] \Delta U^2 = 0 \quad (9)$$

where $\bar{U} \equiv (U_1 + U_2)/2$ is the average velocity and $\Delta U \equiv U_1 - U_2$ is the velocity difference.

Question 4. Find ω as a function of k . Is it unstable? Plot $\omega(k)$. Is there a maximum of growth rate for some k ? Is there a minimum wavelength below which the perturbations are no longer amplified?