1 Stability of a plane jet

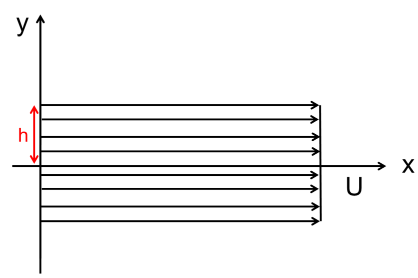
One considers a two-dimensional jet in a cartesian coordinate system $(0xy)$ (U points in the x direction, V points in the y direction). In absence of perturbations, its width is constant and equals $2h$. The flow field is given by

$$U(x, y) = 0 \quad \text{when} \quad |y| > h \tag{1}$$

$$U(x, y) = U \quad \text{when} \quad |y| < h \tag{2}$$

and

$$V(x, y) = 0 \tag{3}$$



The pressure is assumed to be constant and uniform, equal to P_∞. Further assumptions are

- The fluid is inviscid and incompressible of density ρ,
- The flow is two-dimensional,
- The flow is assumed to be irrotational, except on the interface.

1. What is the potential $\Phi(x, y)$ of the base flow?
2. One considers a perturbed flow. Explain why it is necessary to consider three different regions where all physical quantities are continuous, labeled from positive y towards negative y 1, 0 and 2.

$$\phi_1 \quad = \Phi_1 + \quad \epsilon\phi_1' \tag{4}$$

$$\phi_0 \quad = \Phi_0 + \quad \epsilon\phi_0' \tag{5}$$

$$
\begin{aligned}
\phi_2 &= \Phi_2 + \epsilon\phi_2' && (6)\\
p_1 &= P_\infty + \epsilon p_1' && (7)\\
p_0 &= P_\infty + \epsilon p_0' && (8)\\
p_2 &= P_\infty + \epsilon p_2' && (9)\\
\eta_1 &= h + \epsilon\eta_1' && (10)\\
\eta_2 &= -h + \epsilon\eta_2' && (11)
\end{aligned}
$$

where $\epsilon \ll 1$.

Write the equation satisfied by the velocity potential perturbation in each domain.

3. Kinematic conditions

   Write the impermeability conditions on the interfaces η_1 et η_2 at order ϵ and obtain two relations between certain derivatives of $\phi_1'(h)$, $\phi_0'(h)$ and η_1'. Write two similar relations between the derivatives of $\phi_2'(-h)$, $\phi_0'(-h)$ and η_2'. One of the four relations is for instance

$$\frac{\partial \eta_1'}{\partial t} = \frac{\partial \phi_1'}{\partial y}(h) \tag{12}$$

4. Do the above conditions ensure continuity of tangential velocities at the interfaces?

5. Dynamic conditions

   Write the dynamic interface conditions in h and $-h$, and express them using the velocity potentials. Show for instance that

$$\frac{\partial \phi_0'}{\partial t}(h) + U\frac{\partial \phi_0'}{\partial x}(h) = \frac{\partial \phi_1'}{\partial t}(h). \tag{13}$$

   What famous fluid mechanics theorem do you have used?

6. Normal mode decomposition

   We consider normal mode perturbations of real wavenumber k and complex frequency ω

$$\eta_1'(x,t) = E_1 \exp(i(kx - \omega t)) \tag{14}$$

$$\phi_1'(x,y,t) = \hat{\phi}_1(y) \exp(i(kx - \omega t)) \tag{15}$$

   and similar relations for all useful quantities.

Determine the solution for the perturbation potential in each subdomain. In subdomain 0, a useful advice is to consider the sum of an even and odd functions.

7. Even perturbations

   In this question, we only consider even perturbations

   $$\hat{\phi}_0(-y) = \hat{\phi}_0(y) \tag{16}$$

   and

   $$\hat{\phi}_1(y) = \hat{\phi}_2(-y), \quad \text{for} \quad y > 0. \tag{17}$$

   Determine the dispersion relation and plot the real and imaginary parts of ω as a function of k.

   What is the most unstable wavenumber? Is there a cut-off wavenumber? Why?

8. Odd perturbations

   In this question, we only consider odd perturbations

   $$\hat{\phi}_0(-y) = -\hat{\phi}_0(y) \tag{18}$$

   and

   $$\hat{\phi}_1(y) = -\hat{\phi}_2(-y), \quad \text{pour} \quad y > 0. \tag{19}$$

   Determine the dispersion relation and plot the real and imaginary parts of ω as a function of k.

   What is the most unstable wavenumber? Is there a cut-off wavenumber? Why?

9. What is, according to this analysis, the most unstable perturbation, even or odd? The even perturbations in ϕ correspond to a so-called varicose mode such that the transverse motion of the two shear layers is in phase opposition, $v(x,y) = -v(x,-y)$. In contrast, the odd perturbations in ϕ correspond to a so called sinuous mode such that the transverse motion of the two shear layers in in phase, $v(x,y) = v(x,-y)$.

10. What is the physical mechanism underlying this instability? What name is often given to it?

11. In this question, fluid 0 is assumed to be immiscible with the surrounding fluid in domains 1 and 2, though it has the same density. We denote by γ the surface tension. Without determining the dispersion relation, do you expect surface tension to have a stabilizing influence or a destabilizing one?

12. Thanks to the matlab codes provided, explore the spatial instability properties of the sinuous and varicose modes. Do they have equal spatial growth-rates? In light of the Gaster approximation, can you interpret (no rigorous proof, just a plausible explanation) the cross-over between the most spatially amplified mode as the forcing frequency increases?