Thin-film Rayleigh–Plateau instability around a cylinder

ME-466 Instability

1 Problem formulation

We are interested in the stability of a layer of viscous liquid with dynamic viscosity η [Pa s] and surface tension γ [Pa m] of typical thickness H [m], coating a cylinder of radius $R \gg H$ [m] (figure 1). We neglect inertial effects and gravity. We further assume axisymmetry and no azimuthal velocity.

The system is governed by the incompressible Stokes equations with no-penetration and no-slip conditions at the solid surface, no tangential stress and Laplace jump in the normal stress dynamic conditions on the liquid–air interface.

$$\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*} + \frac{\partial u_z^*}{\partial z^*} = 0, \quad (1)$$

$$-\frac{\partial p^*}{\partial z^*} + \eta\left(\frac{\partial^2 u_z^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u_z^*}{\partial r^*} + \frac{\partial^2 u_z^*}{\partial z^{*2}}\right) = 0, \quad (2)$$

$$-\frac{\partial p^*}{\partial r^*} + \eta\left(\frac{\partial^2 u_r^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u_r^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}} - \frac{u_r^*}{r^{*2}}\right) = 0, \quad (3)$$

$$u_r^*(r^* = R) = u_z^*(r^* = R) = 0, \quad (4)$$

$$\left(1 - \left(\frac{\partial h^*}{\partial z^*}\right)^2\right)\left(\frac{\partial u_z^*}{\partial r^*} + \frac{\partial u_r^*}{\partial z^*}\right) + 2\frac{\partial h^*}{\partial z^*}\left(\frac{\partial u_r^*}{\partial r^*} - \frac{\partial u_z^*}{\partial z^*}\right)\Bigg|_{r^*=R+h^*} = 0, \quad (5)$$

$$-p^* + \frac{2\eta}{1+\left(\frac{\partial h^*}{\partial z^*}\right)^2}\left(\left(\frac{\partial h^*}{\partial z^*}\right)^2\frac{\partial u_z^*}{\partial z^*} - \frac{\partial h^*}{\partial z^*}\left(\frac{\partial u_z^*}{\partial r^*} + \frac{\partial u_r^*}{\partial z^*}\right) + \frac{\partial u_r^*}{\partial r^*}\right)\Bigg|_{r^*=R+h^*} = -\gamma\mathcal{C}^*, \quad (6)$$

where the asterisks $*$ indicate dimensional variables, and the total curvature of the free surface reads

$$\mathcal{C}^* = \frac{1}{(R+h^*)\sqrt{1+\left(\frac{\partial h^*}{\partial z^*}\right)^2}} - \frac{\frac{\partial^2 h^*}{\partial z^{*2}}}{\left(1+\left(\frac{\partial h^*}{\partial z^*}\right)^2\right)^{3/2}}. \quad (7)$$

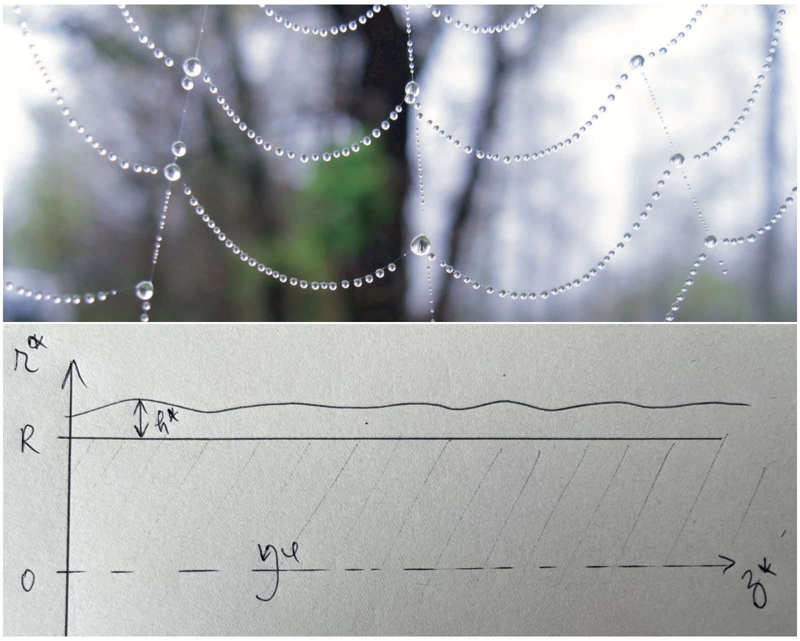

Figure 1: (top panel) Spider web with dew pearls (reprinted from Duprat [2009]). (bottom panel) Cartoon of the problem at hand.

2 Thin-film (lubrication) equation

We start by non-dimensionalising the system of equations,

$$\begin{aligned} z^* = Rz\,, \quad r^* = Rr\,, \quad \mathrm{d}r^* = H\mathrm{d}r, \\ u_z^* = Uu_z^*\,, \quad u_r^* = U_r u_r\,, \quad p^* = Pp\,, \quad h^* = Hh. \end{aligned} \tag{8}$$

To make use of the separation of scales between the film thickness and the cylinder's radius, we introduce two length scales, R and $H = \delta R$, with $\delta \ll 1$, and two velocity scales, U and U_r. It is crucial to notice that while r^* scales as R, the variations along r^* happen on an interval of the scale of H.

Notice that the axial dimension is also rescaled with R. This is because of the anticipation that the characteristic length of the Raleigh–Plateau instability is set by the liquid column's radius.

Introduce the rescaled variables in the full governing equations. Determine the scales by dominant balance. Keep only the first order in δ.

1. Show that
$$p = \left(\delta^{-1} - h - \frac{\partial^2}{\partial z^2}\right).$$

2. Show that
$$u_z = \frac{1}{2}\frac{\partial p}{\partial z}(r-1)(r-1-2h).$$

3. Either through a mass balance argument or by writing the kinematic boundary condition, show that
$$\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial z} = -\frac{\partial}{\partial z}\left[\frac{h^3}{3}\left(\frac{\partial h}{\partial z} + \frac{\partial^3 h}{\partial z^3}\right)\right]. \tag{9}$$

What is the meaning of q?

3 Linear instability analysis

1. Linearise equation (9) around $h_0 = 1$. Decompose the perturbation into normal modes. Find the dimensionless dispersion relation.

2. Re-dimensionalise and show that

$$\omega^*(k^*) = \mathrm{i}\frac{\gamma H^3 k^{*2}(1 - R^2 k^{*2})}{3\eta R^2}.$$

3. Comment.

References

C. Duprat. *Instabilités d'un film liquide en écoulement sur une fibre verticale.* PhD thesis, Université Pierre et Marie Curie - Paris VI, 2009.