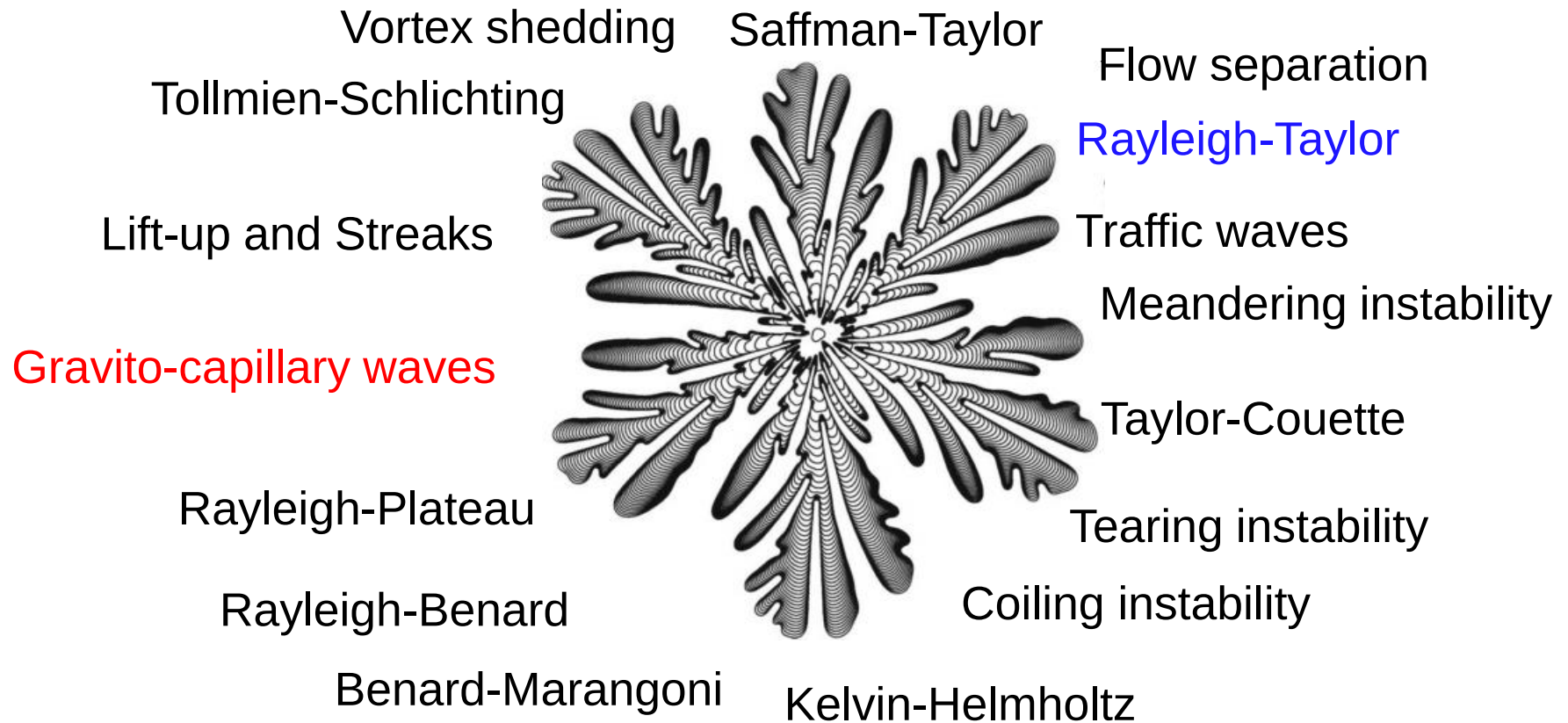
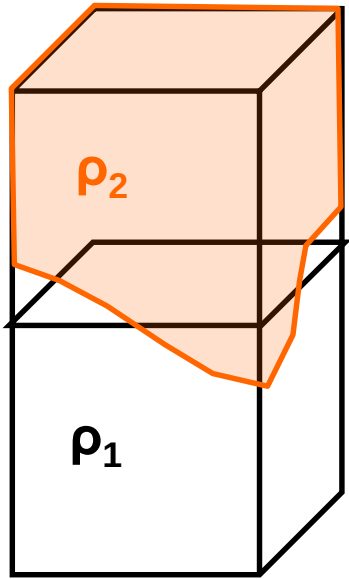


Most flows are unstable...

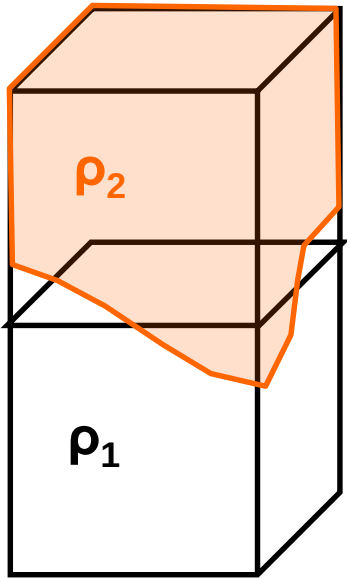


Dispersion relation

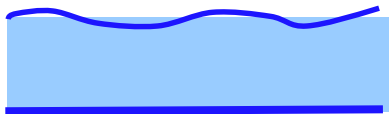


$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

Dispersion relation for water waves



$$\omega^2 = \frac{-kg(\cancel{\rho_2} - \rho_1) + \gamma k^3}{\rho_1 + \cancel{\rho_2}}$$



$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$

Dispersion relation

$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$

Capillary wavenumber: $k_c = \sqrt{\rho g / \gamma}$

Length scale: $\tilde{k} = k / k_c$

Time scale: $\tilde{\omega} = \omega / \sqrt{g k_c}$

One single non-dimensional parameter $\tilde{H} = H k_c$

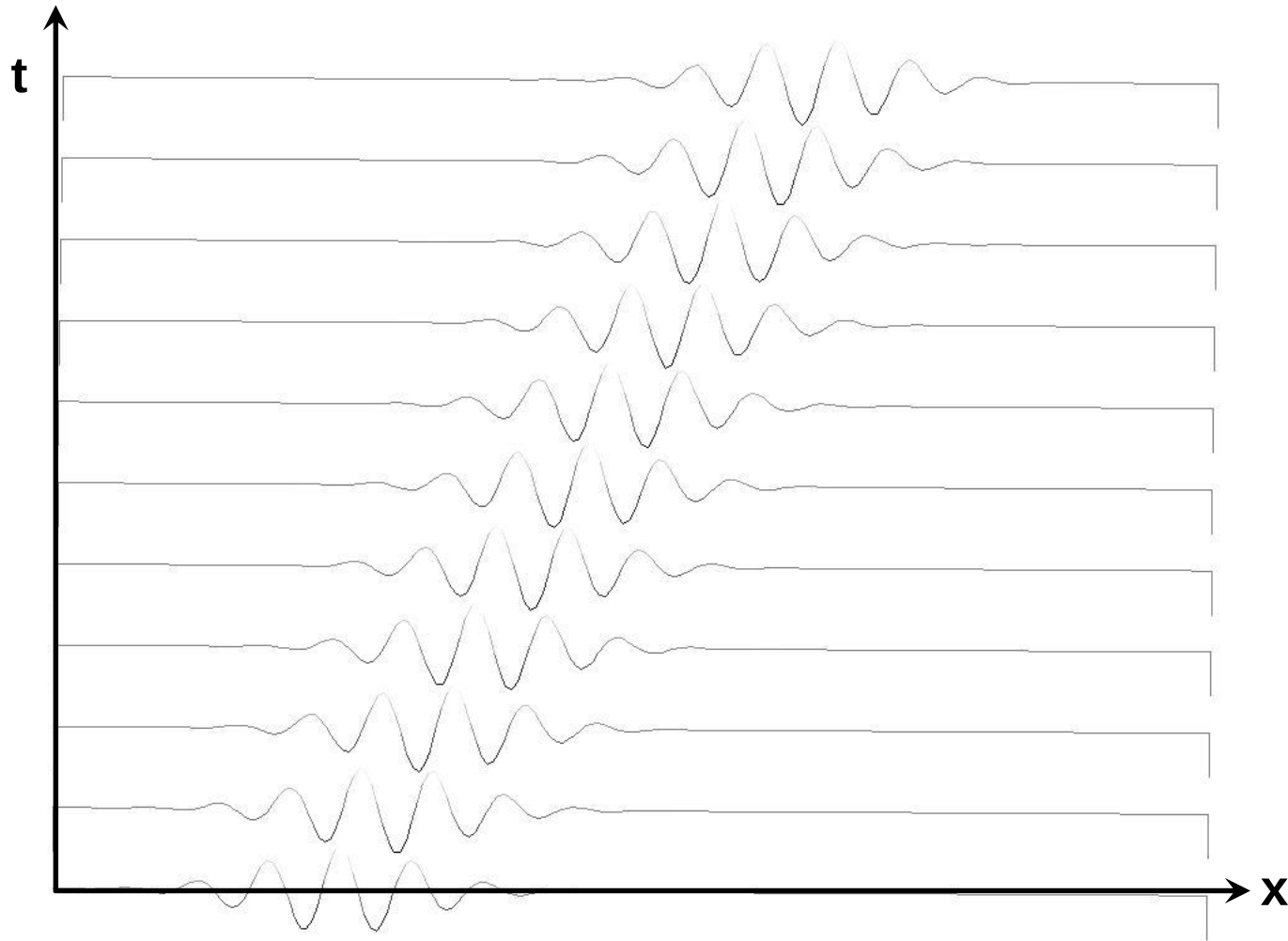
$$\tilde{\omega}^2 = \tanh(\tilde{k} \tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

Dispersion relation

$$\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\pm \tilde{k} \sqrt{\tilde{H}}$	$\pm \tilde{k}^2 \sqrt{\tilde{H}}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\pm \sqrt{\tilde{k}}$	$\pm \tilde{k} \sqrt{\tilde{k}}$

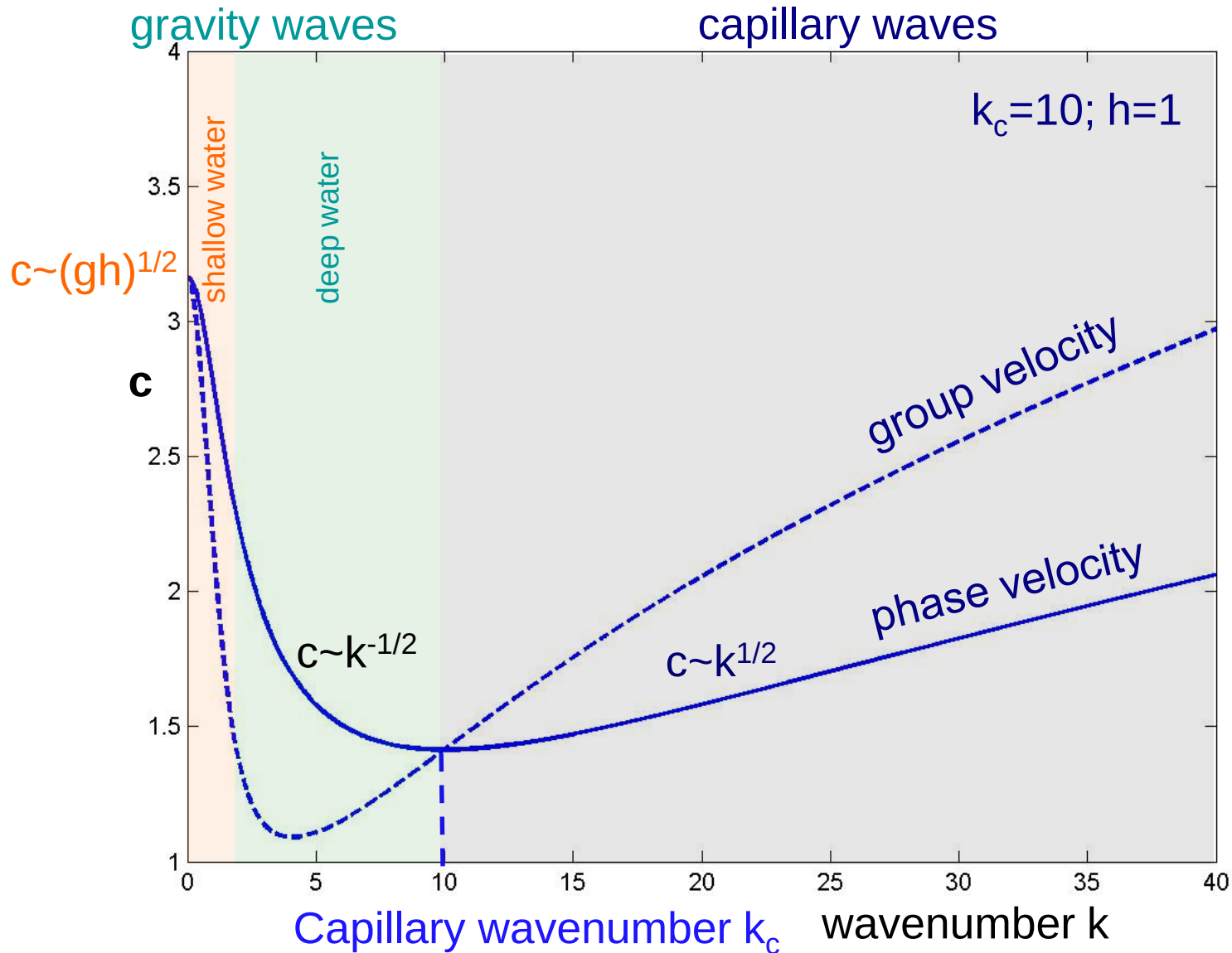
Difference between group velocity $v = d\omega/dk$
and phase velocity $c = \omega/k$



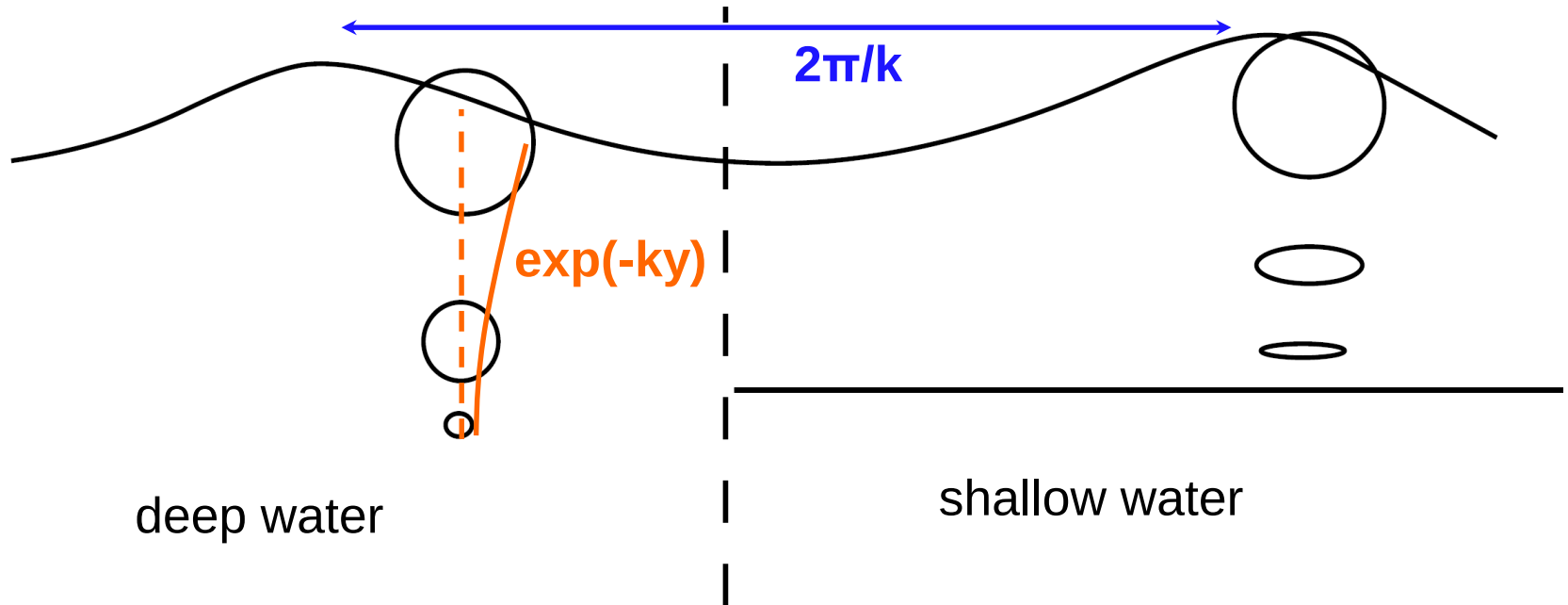
Dispersion relation

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\omega_{\text{shallow/gravity}} \sim \pm k \sqrt{gH}$ $c_{\text{shallow/gravity}} \sim \pm \sqrt{gH}$ $v_{\text{shallow/gravity}} \sim \pm \sqrt{gH}$	$\omega_{\text{shallow/capillary}} \sim \pm k^2 \sqrt{\gamma H / \rho}$ $c_{\text{shallow/capillary}} \sim \pm k \sqrt{\gamma H / \rho}$ $v_{\text{shallow/capillary}} \sim \pm 2k \sqrt{\gamma H / \rho}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$ $c_{\text{deep/gravity}} \sim \pm \sqrt{\frac{g}{k}}$ $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$	$\omega_{\text{deep/capillary}} \sim \pm k^{3/2} \sqrt{\gamma / \rho}$ $c_{\text{deep/capillary}} \sim \pm k^{1/2} \sqrt{\gamma / \rho}$ $v_{\text{deep/capillary}} \sim \pm 3/2 k^{1/2} \sqrt{\gamma / \rho}$

Dispersion relation



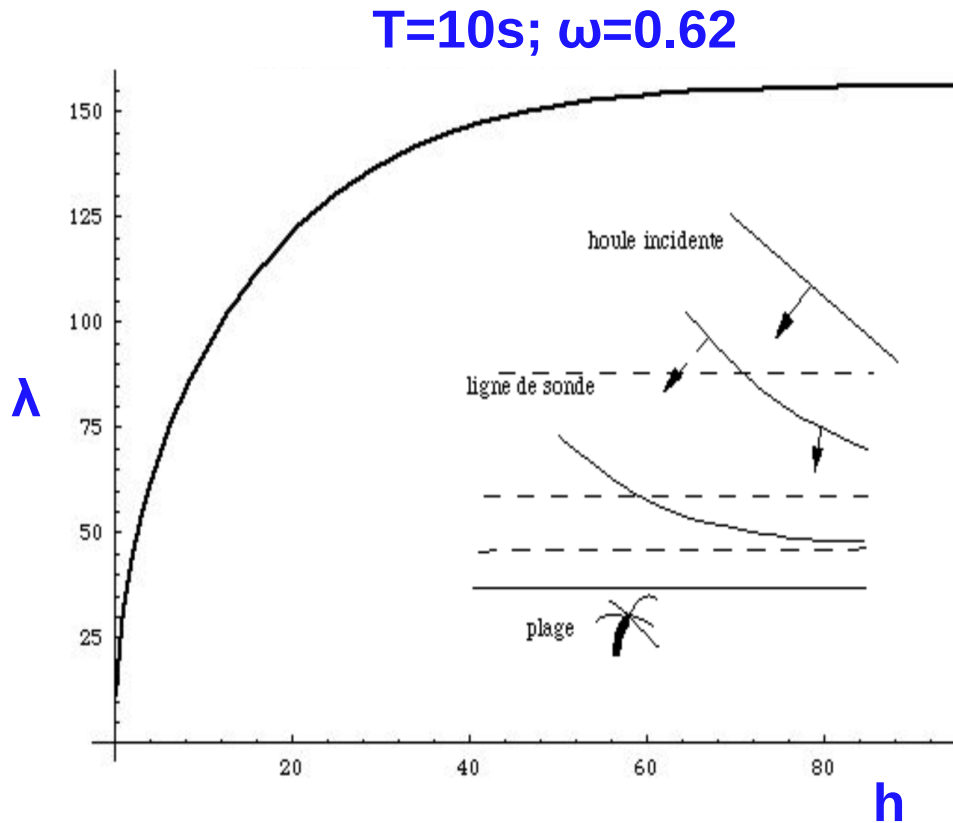
Trajectories below waves



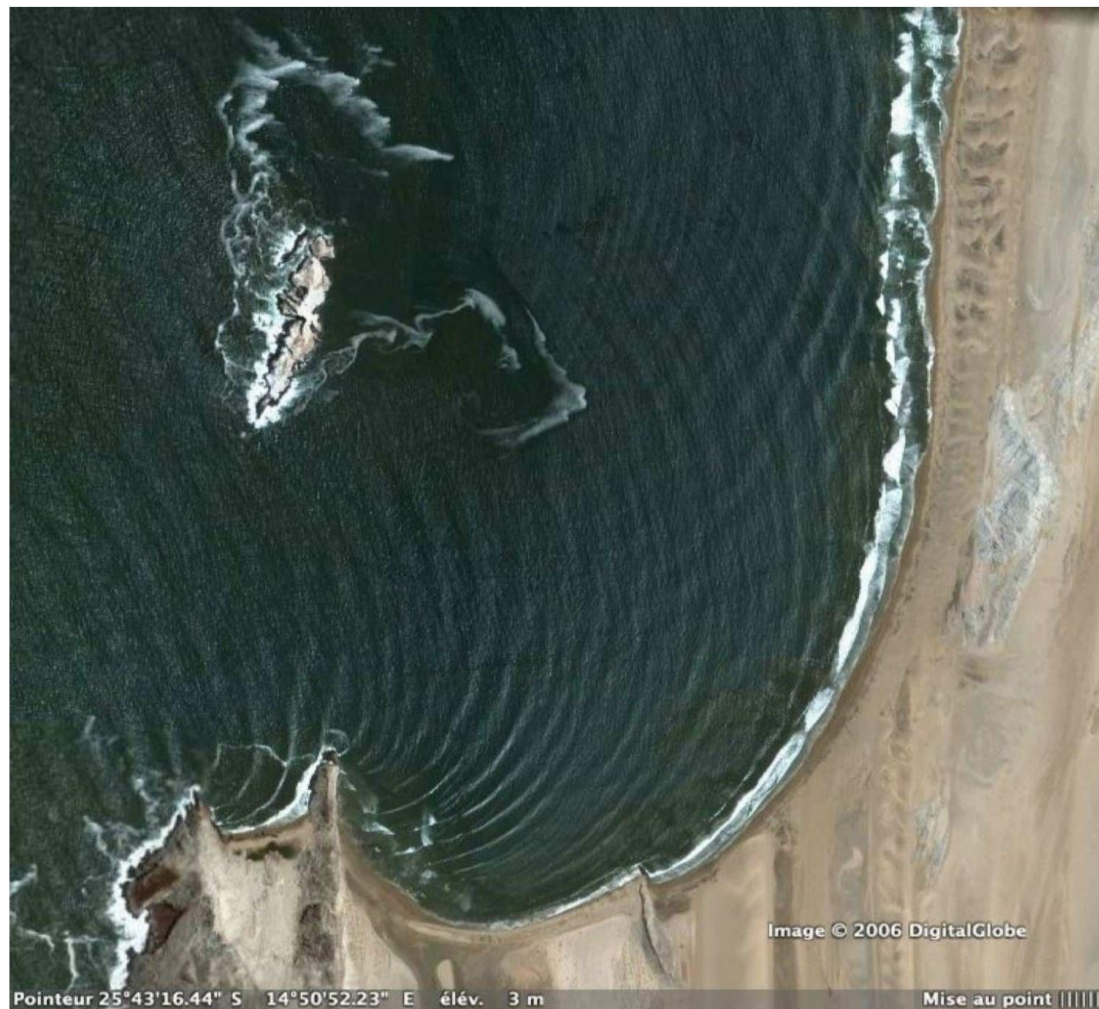
Why are the waves parallel to the shore?

$$c \sim (gh)^{1/2}$$

$$\lambda \sim T(gh)^{1/2}$$

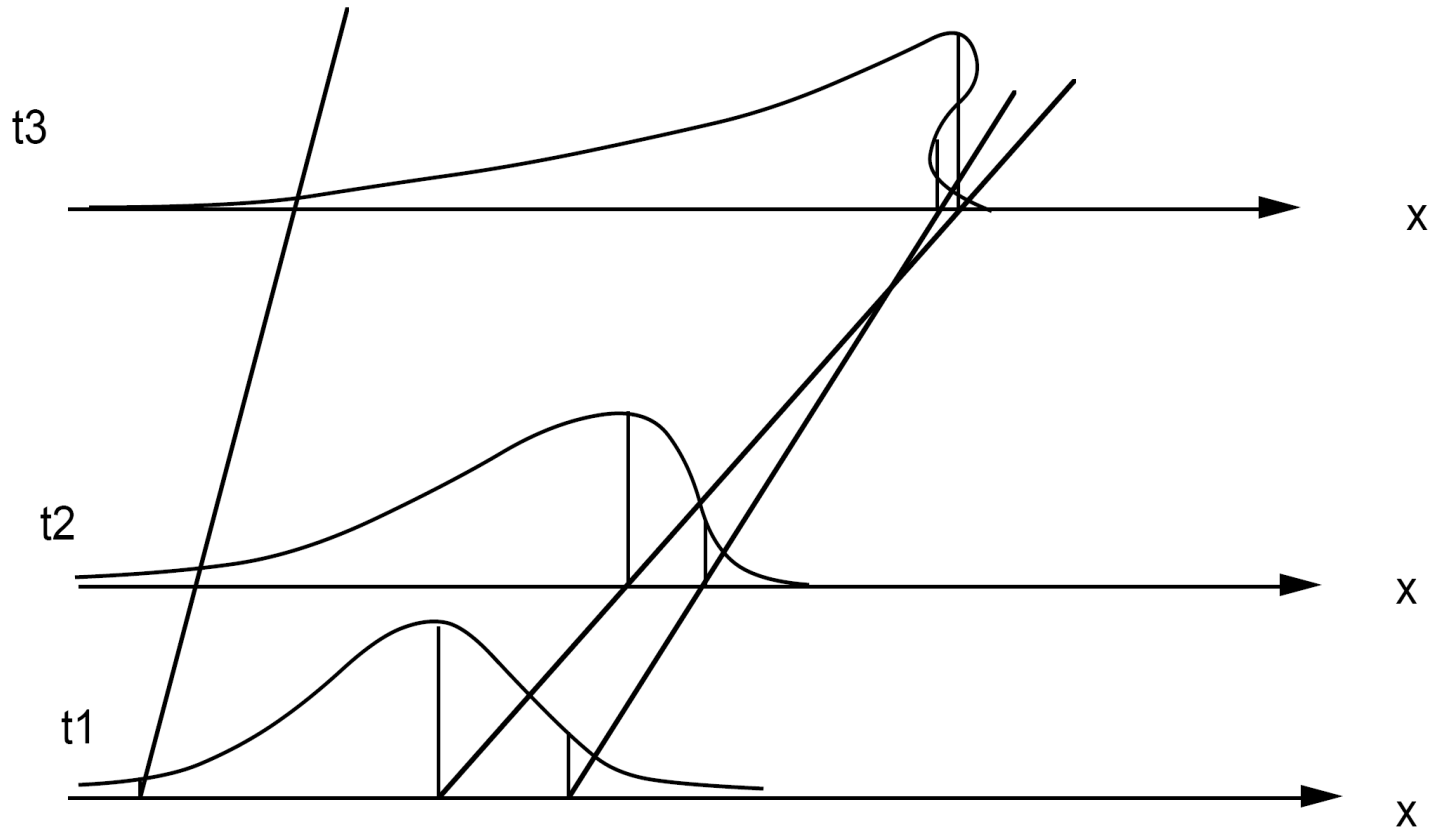


Refraction and diffraction of waves



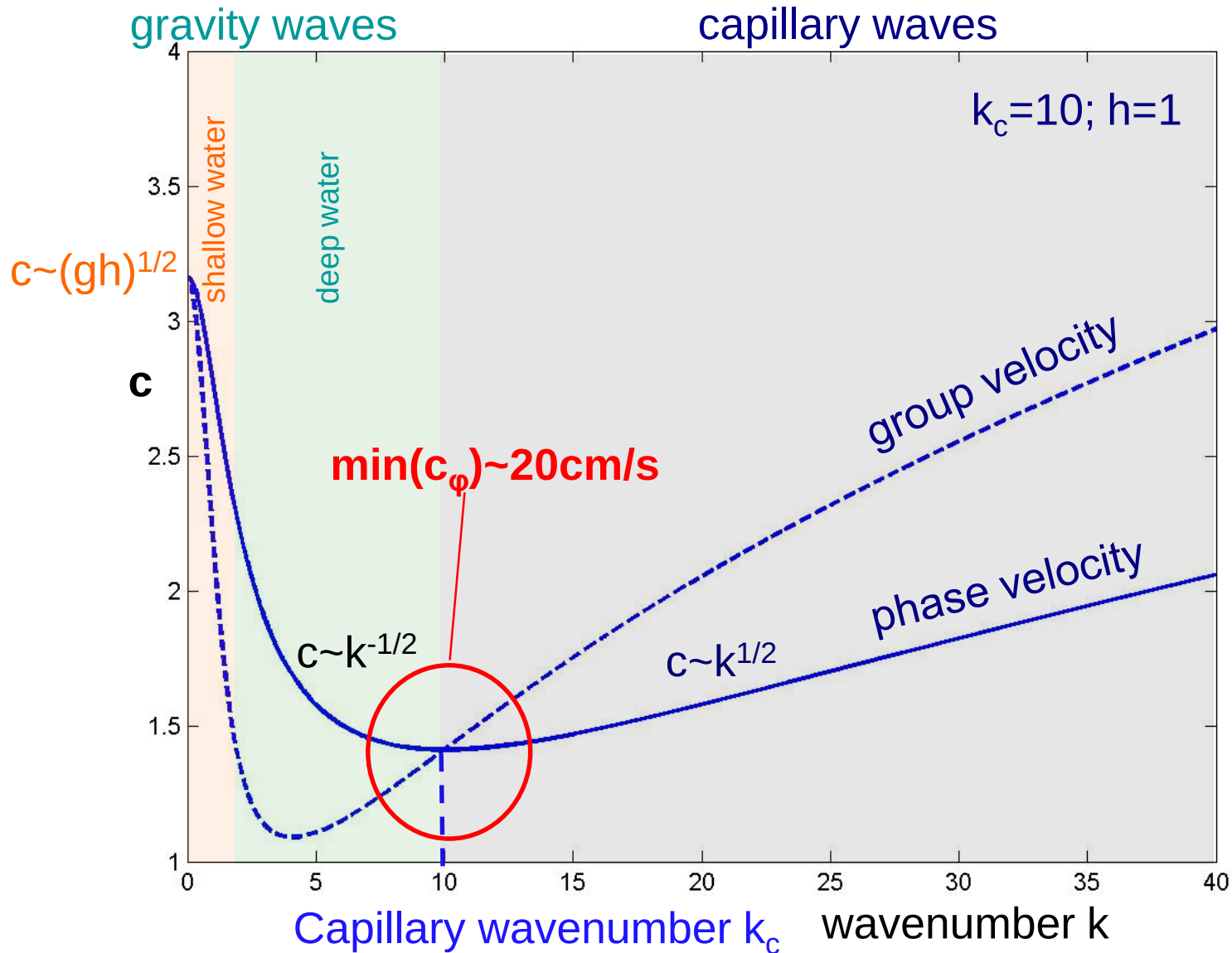
Satellite view of the shore of Namibia

Nonlinear waves/wave breaking



The velocity increases with amplitude

Dispersion relation

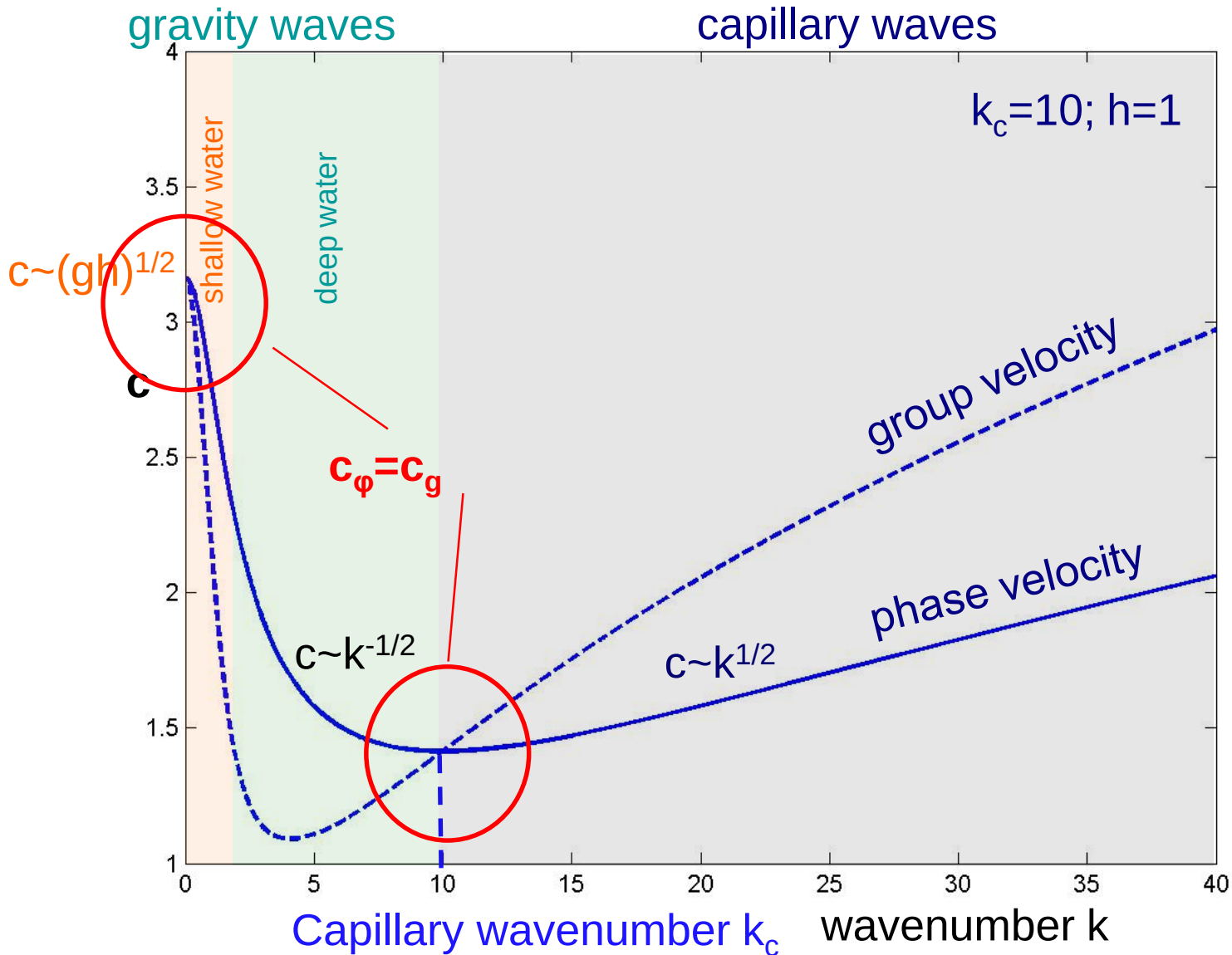


Conditions for wave pattern formation?

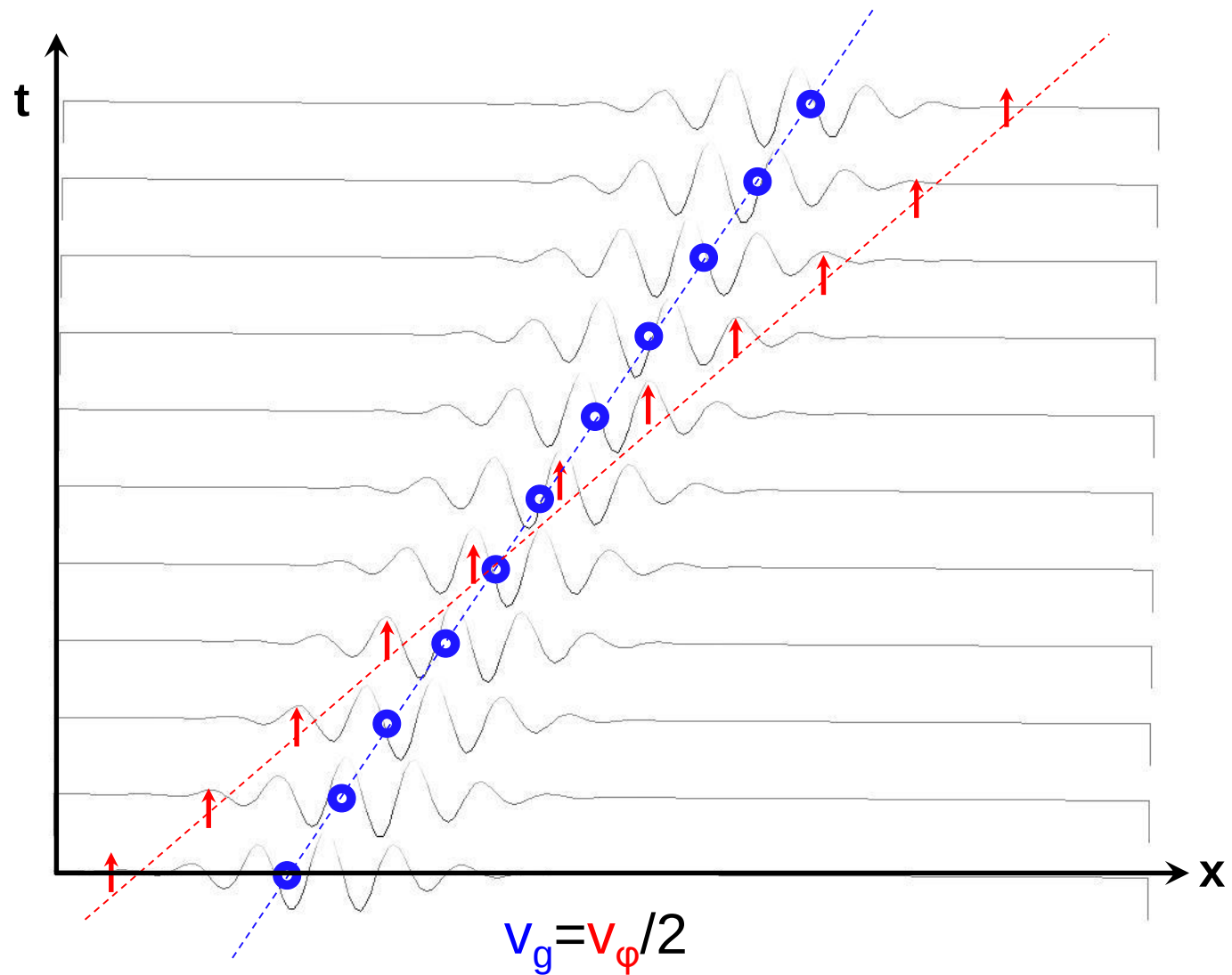


$$V_{\text{duck}} \lesseqgtr c_{\text{min}} \quad ?$$

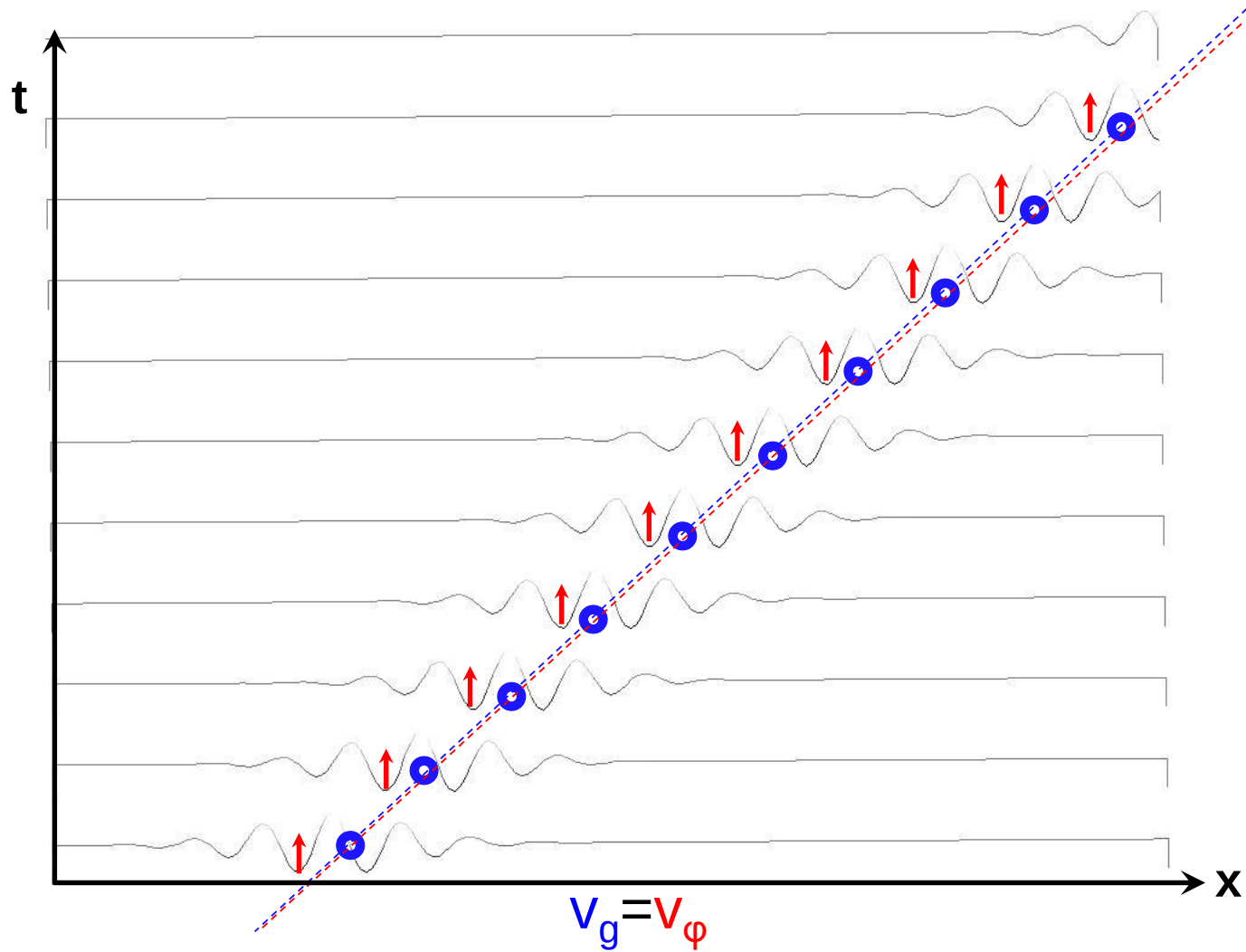
Dispersion relation



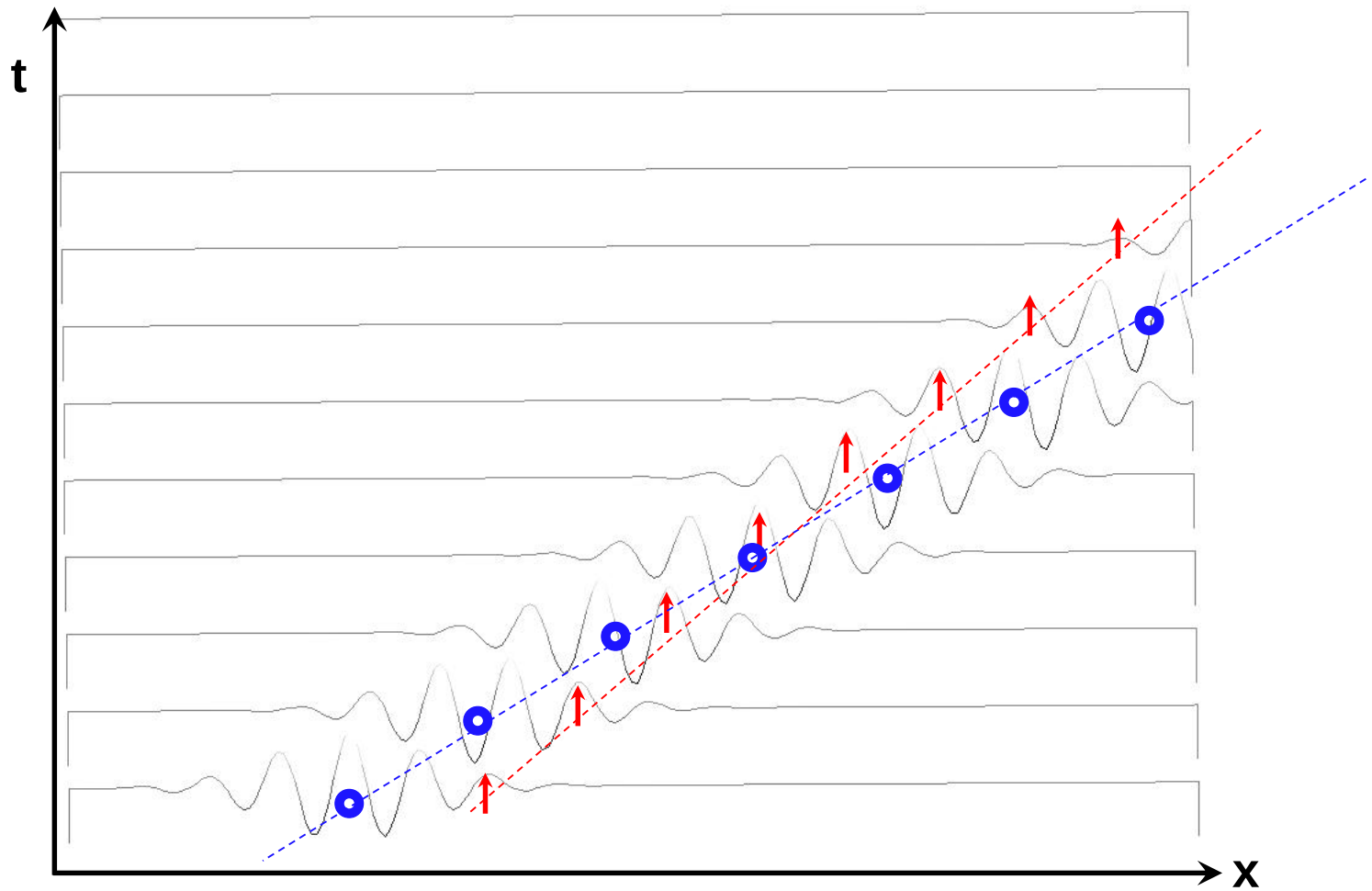
Spatio-temporel diagram



Spatio-temporal diagram



Spatio-temporal diagram



$$v_g = \frac{3}{2} v_\phi$$

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k, \omega) e^{i(kx - \omega t)} dk d\omega$$

$$\hat{u}(k, \omega) = (2\pi)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x, t) e^{-i(kx - \omega t)} dx dt$$

Direct Fourier Transform

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k, \omega) e^{i(kx - \omega t)} dk d\omega$$

Use dispersion relation $\omega(k)$!

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

Carrier/enveloppe

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Enveloppe :

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k - k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis at time=0

Fourier transform:

$$u(x, 0) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx)} dk + \text{c.c.}$$

$\hat{u}(k)$ is given by Fourier transform at time $t=0$

Envelope :

$$A(x, 0) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x} dk + \text{c.c.}$$

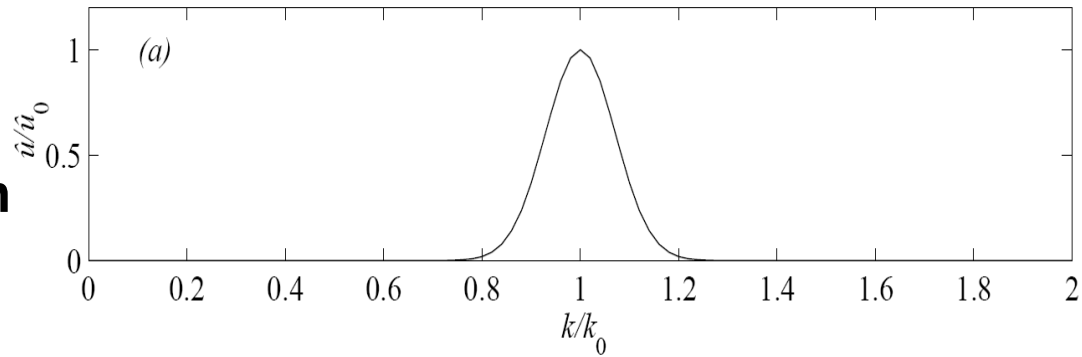
Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

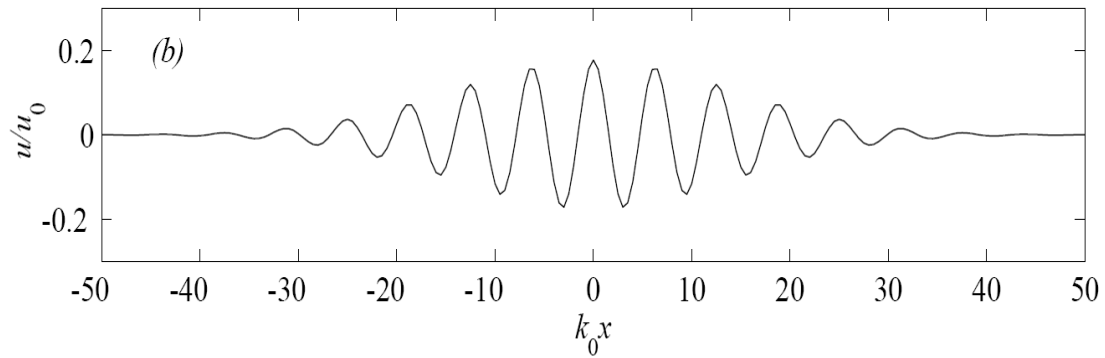
Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum

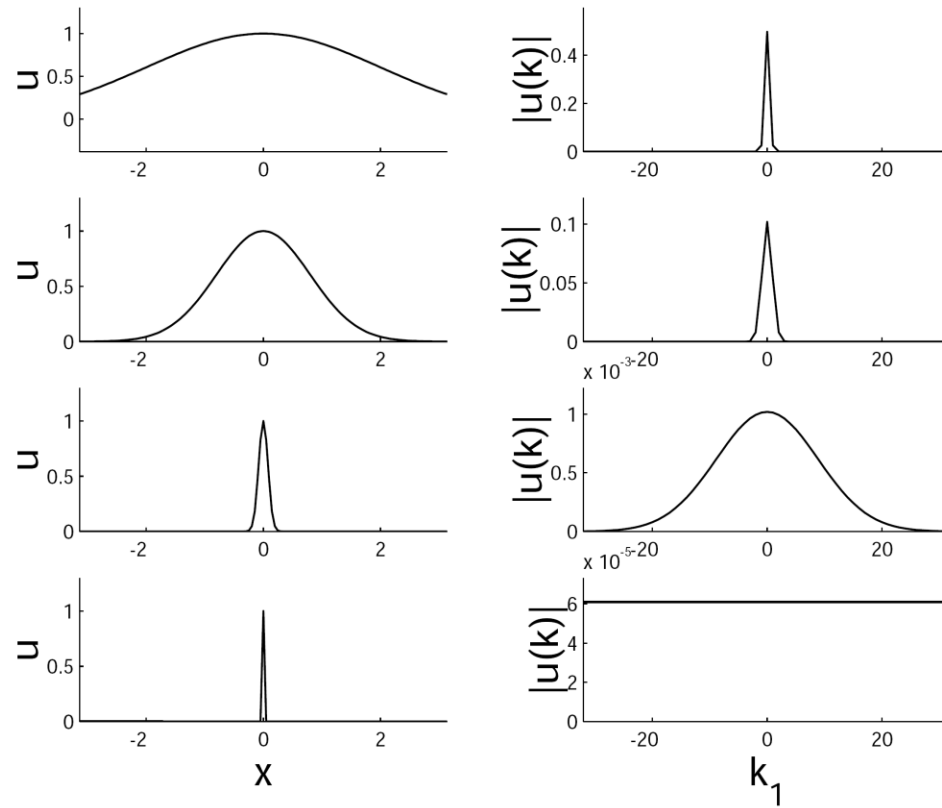
spectrum



wave

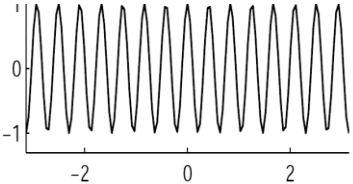
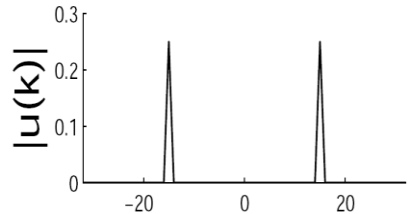
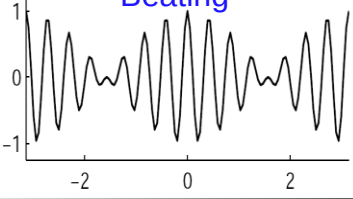
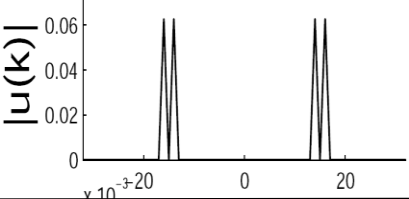
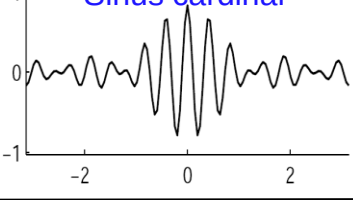
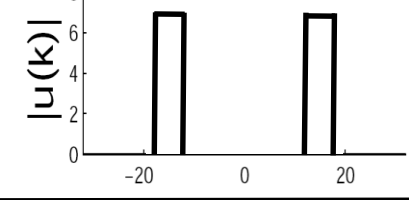
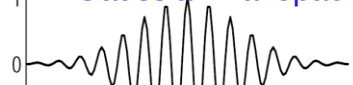


Gaussian wavepackets



$u_0(x)$	$\hat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Waves and spectra

$2 \cos(k_0 x)$	Monochromatic wave  	$\delta(k_1 + k_0) + \delta(k_1 - k_0)$
$\cos[(k_0 - \kappa)x] + \cos[(k_0 + \kappa)x]$ $= 2 \cos(k_0 x) \cos(\kappa x)$	Beating  	$\delta(k_1 + k_0 + \kappa) + \delta(k_1 - k_0 - \kappa)$ $+ \delta(k_1 + k_0 - \kappa) + \delta(k_1 - k_0 + \kappa)$
$\int_{k_0 - \kappa}^{k_0 + \kappa} \cos(k_1 x) dk_1$ $= 4 \kappa \cos(k_0 x) \frac{\sin(\kappa x)}{\kappa x}$	Sinus cardinal  	$= 1 \text{ pour } k_1 \pm k_0 \leq \kappa$ $= 0 \text{ sinon}$
$2 \kappa \sqrt{2\pi} \cos(k_0 x) \exp\left(-\frac{\kappa^2 x^2}{2}\right)$	Gaussian wavepacket  	$\exp\left[-\frac{(k_1 + k_0)^2}{2 \kappa^2}\right] + \exp\left[-\frac{(k_1 - k_0)^2}{2 \kappa^2}\right]$

Spectral analysis

Initial envelope :

$$A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:

$$\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$$

Spectral analysis

Initial envelope :

$$A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:

$$\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$$

Evolution of envelope :

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis

Initial envelope :

$$A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:

$$\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$$

Evolution of envelope :

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition group velocity

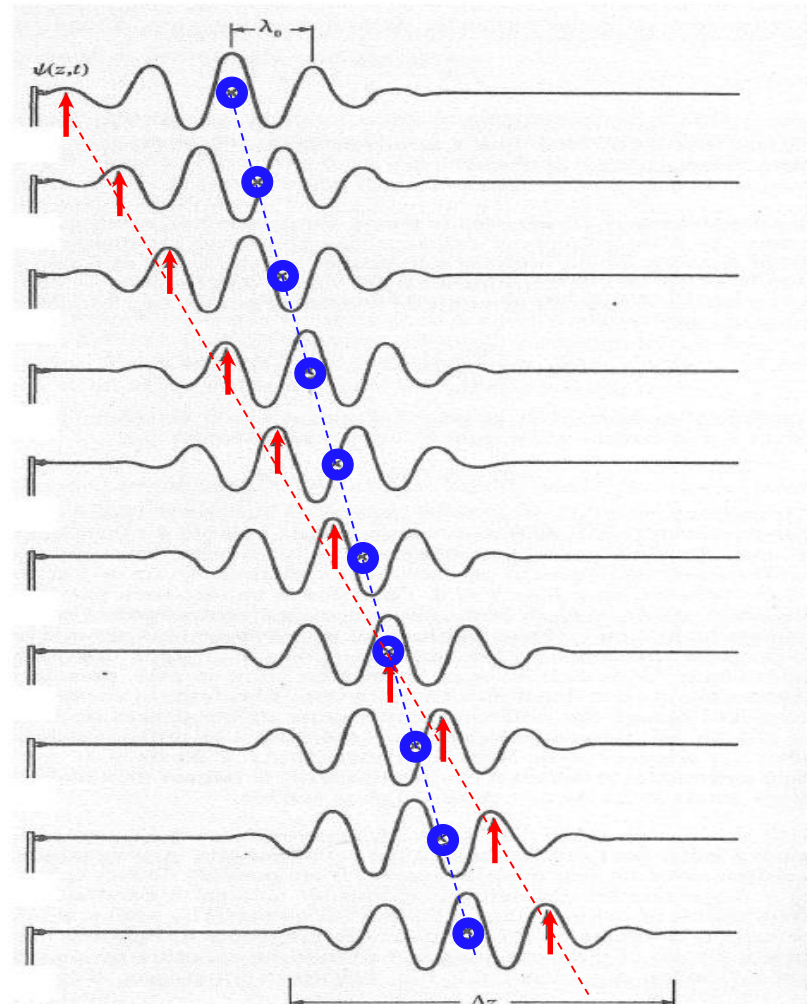
$$\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$$

Spectral analysis

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$A(x, t) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{(x - c_g t)^2}{4\sigma^2}}$$

Group velocity



Wavepacket

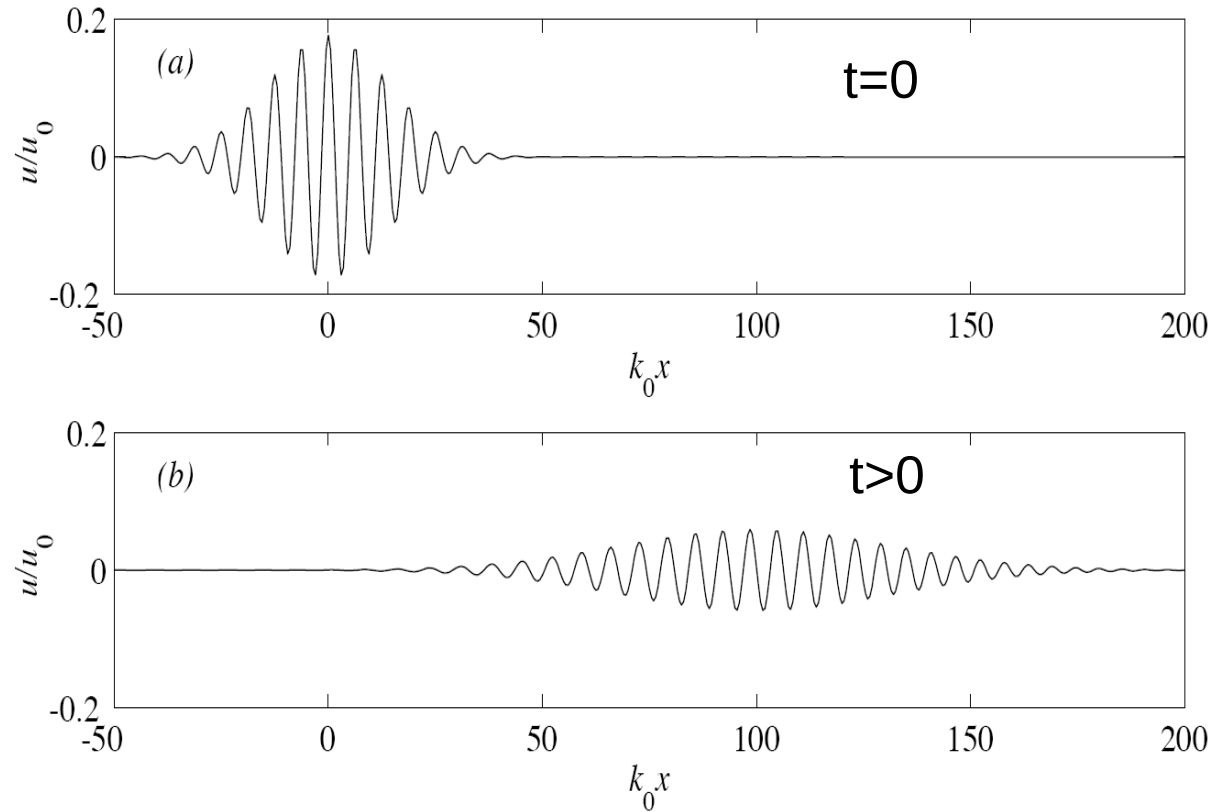
Spectral analysis

Higher order
development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$
$$c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

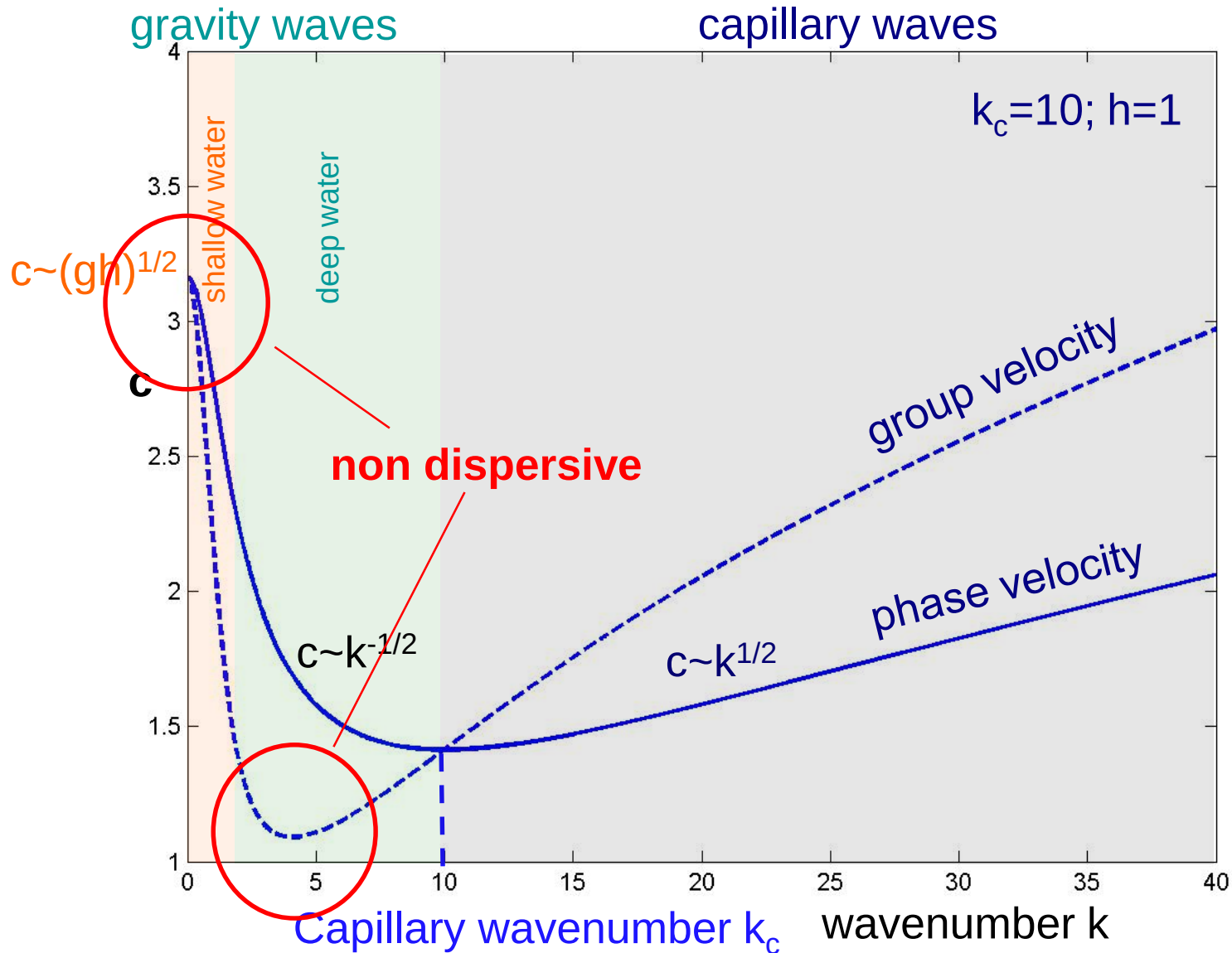
$$A(x, t) = \frac{u_0}{2} \sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}i\omega_0''t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}i\omega_0''t)}\right)$$

Wave packet dispersion

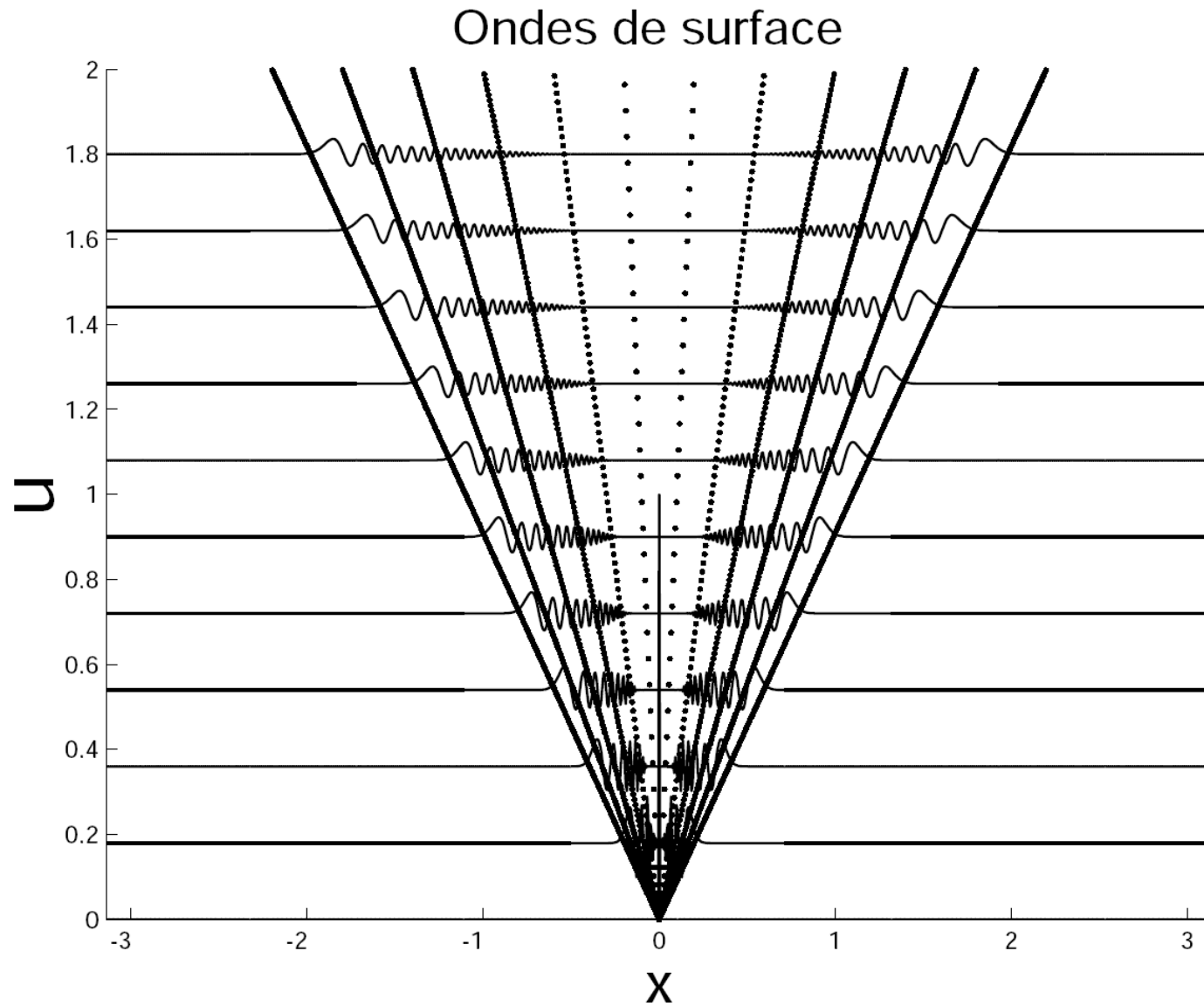


Onde correspondant à l'enveloppe $\hat{u}(k)$ pour $\sigma^{-1}k_0 = 0,1$ et $\omega_0'' = 4c_g/k_0$: (a), instant initial $t = 0$; (b), $c_g t = 100/k_0$.

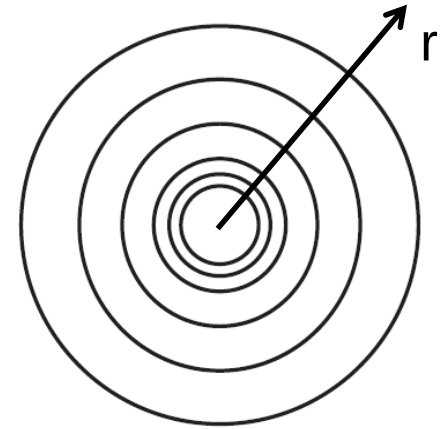
Relation de dispersion



Dispersion



Dispersion



Waves with k reach r at time $t=r/v(k)$

For deep gravity waves: $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$

$$k = gt^2/4r^2$$

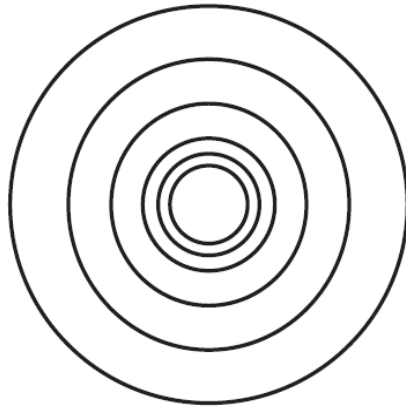
Since $\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$

$$\omega = gt/2r$$

⇒ The frequency increases with time

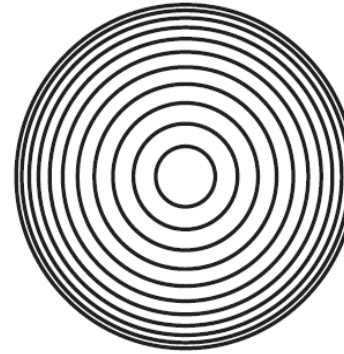
Rings in water

Gravity
waves

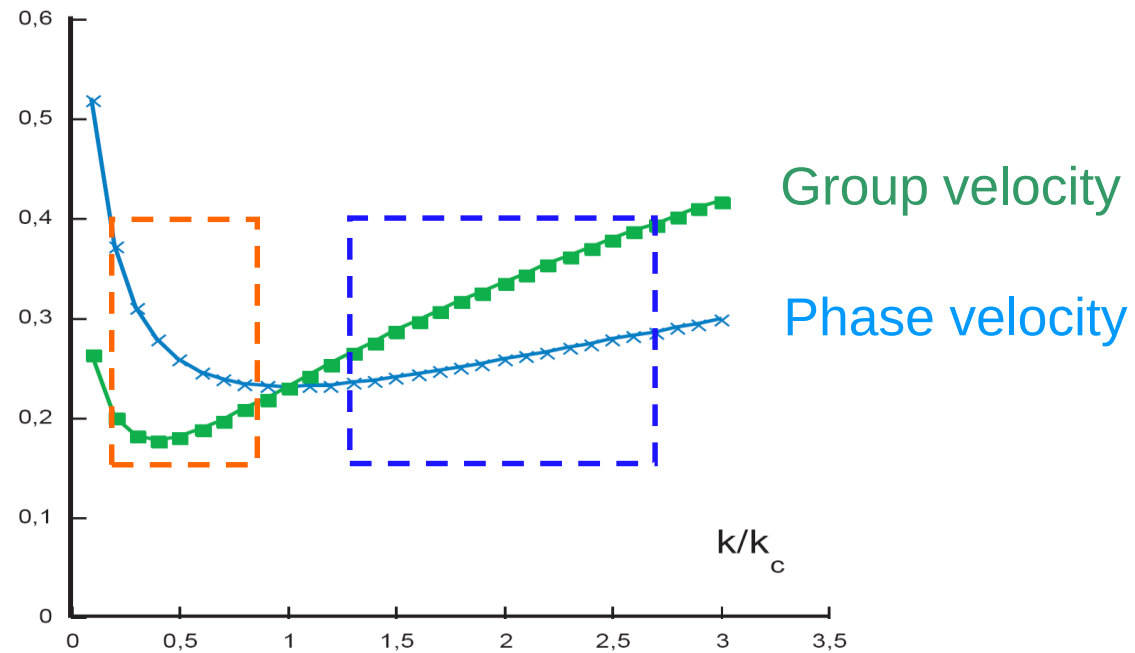


stone $> l_c$

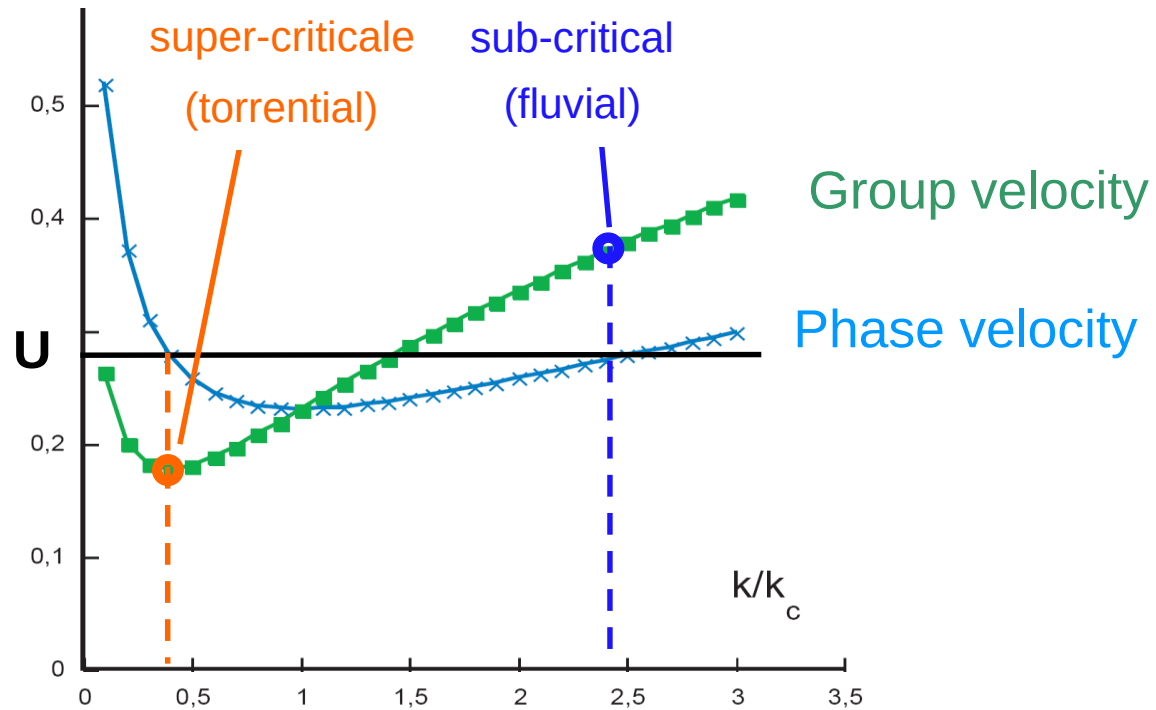
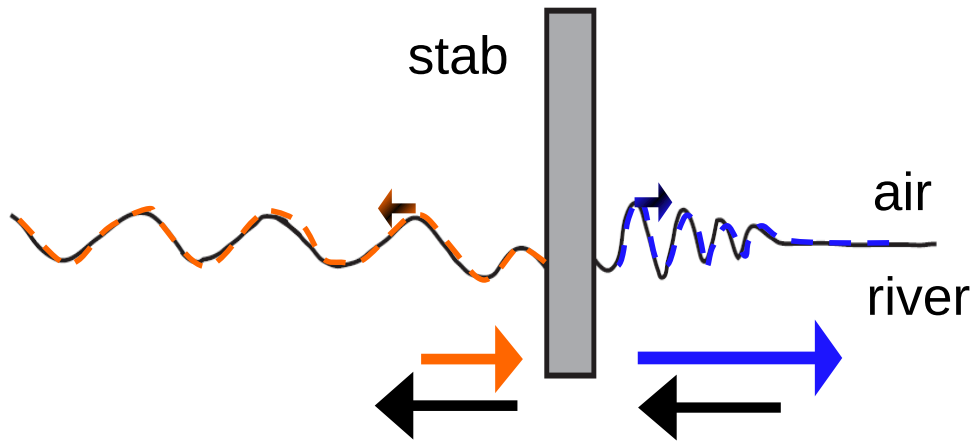
Capillary
waves



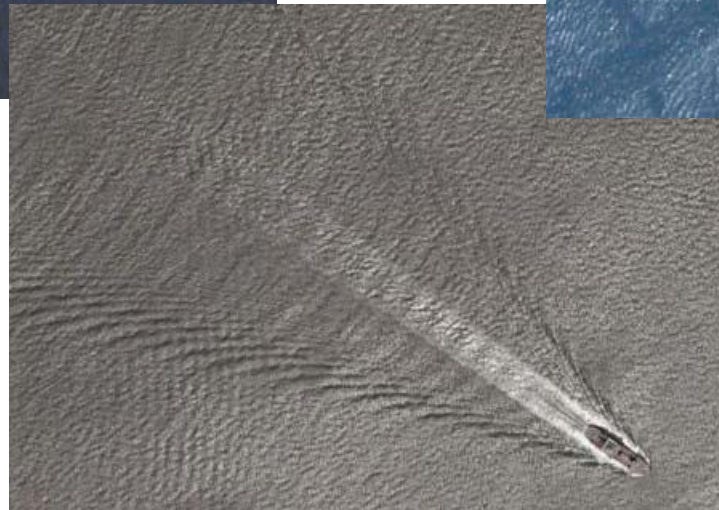
droplet $< l_c$



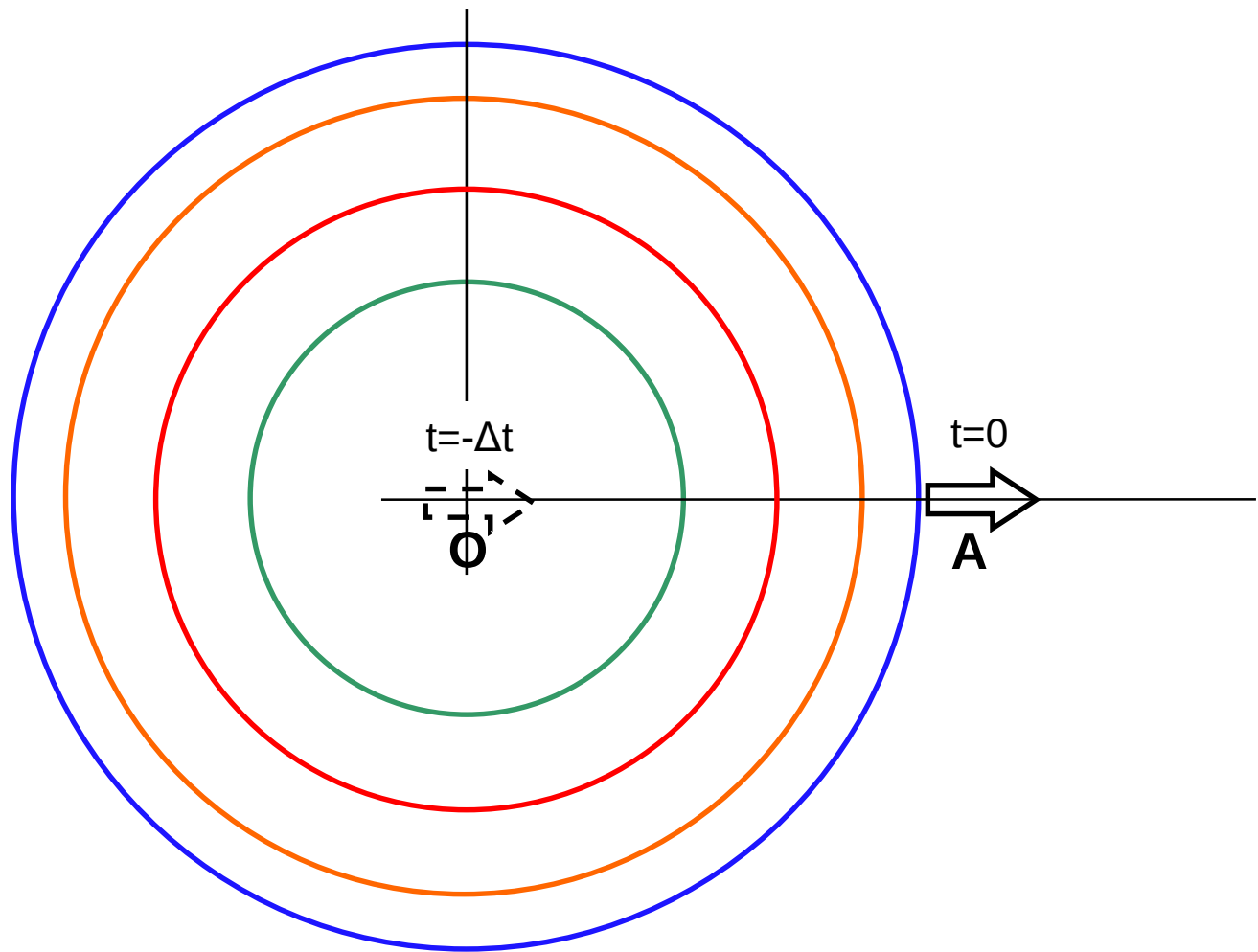
Waves created by obstacle in a river



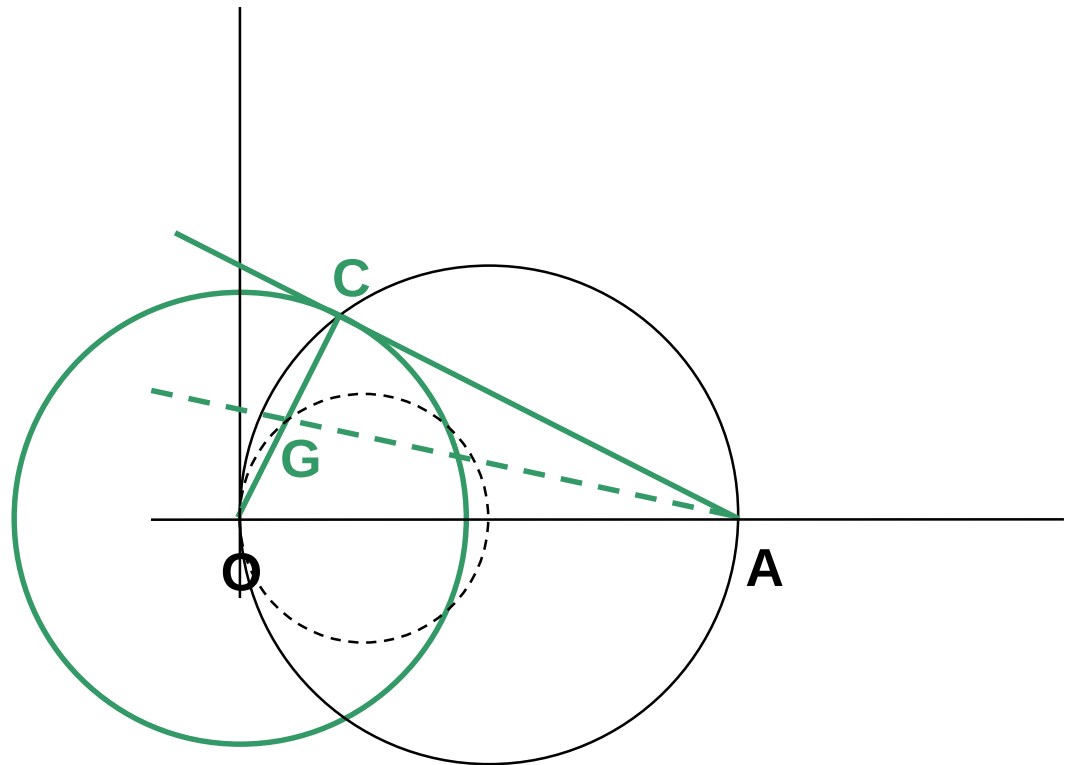
Kelvin's wake



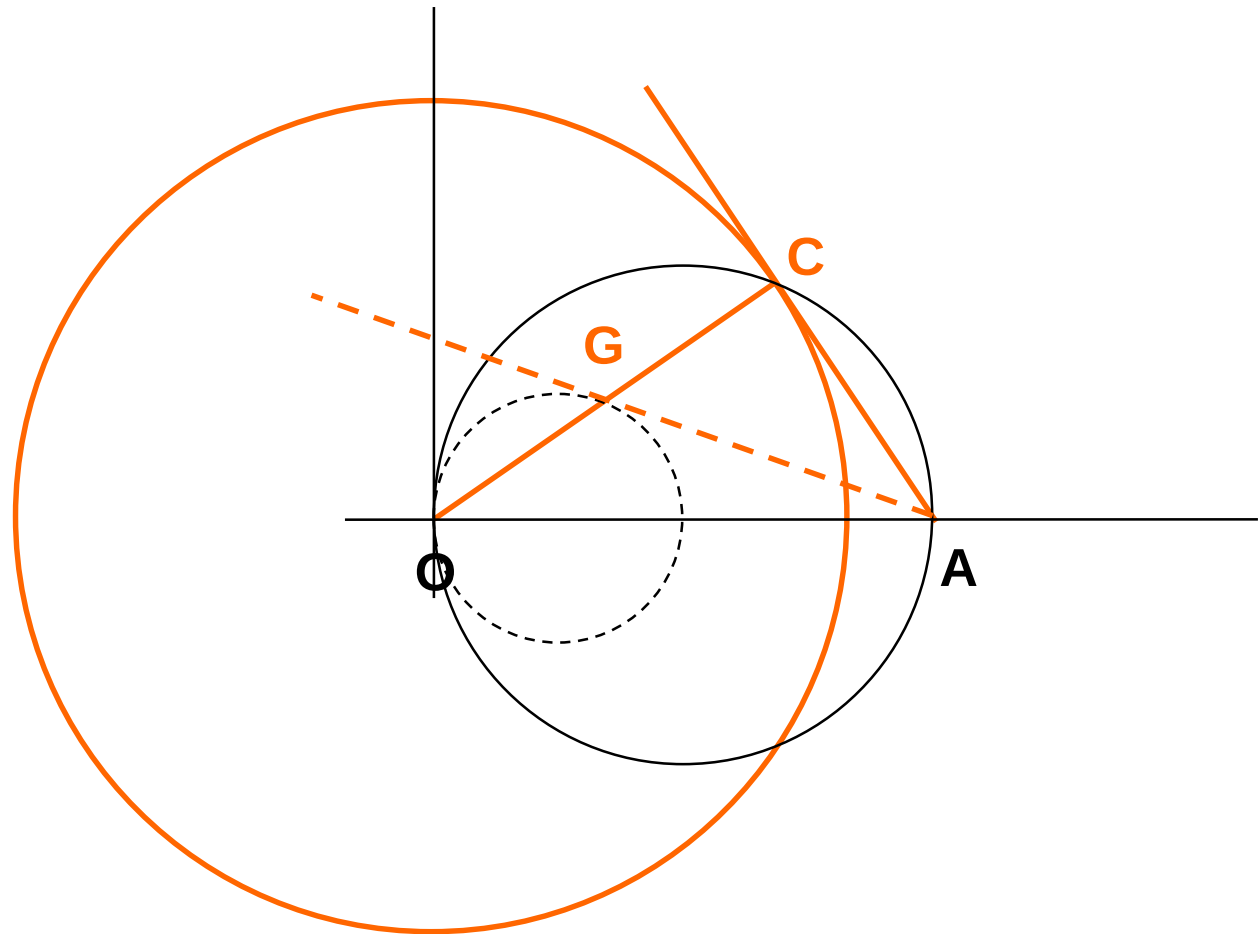
Kelvin's wake



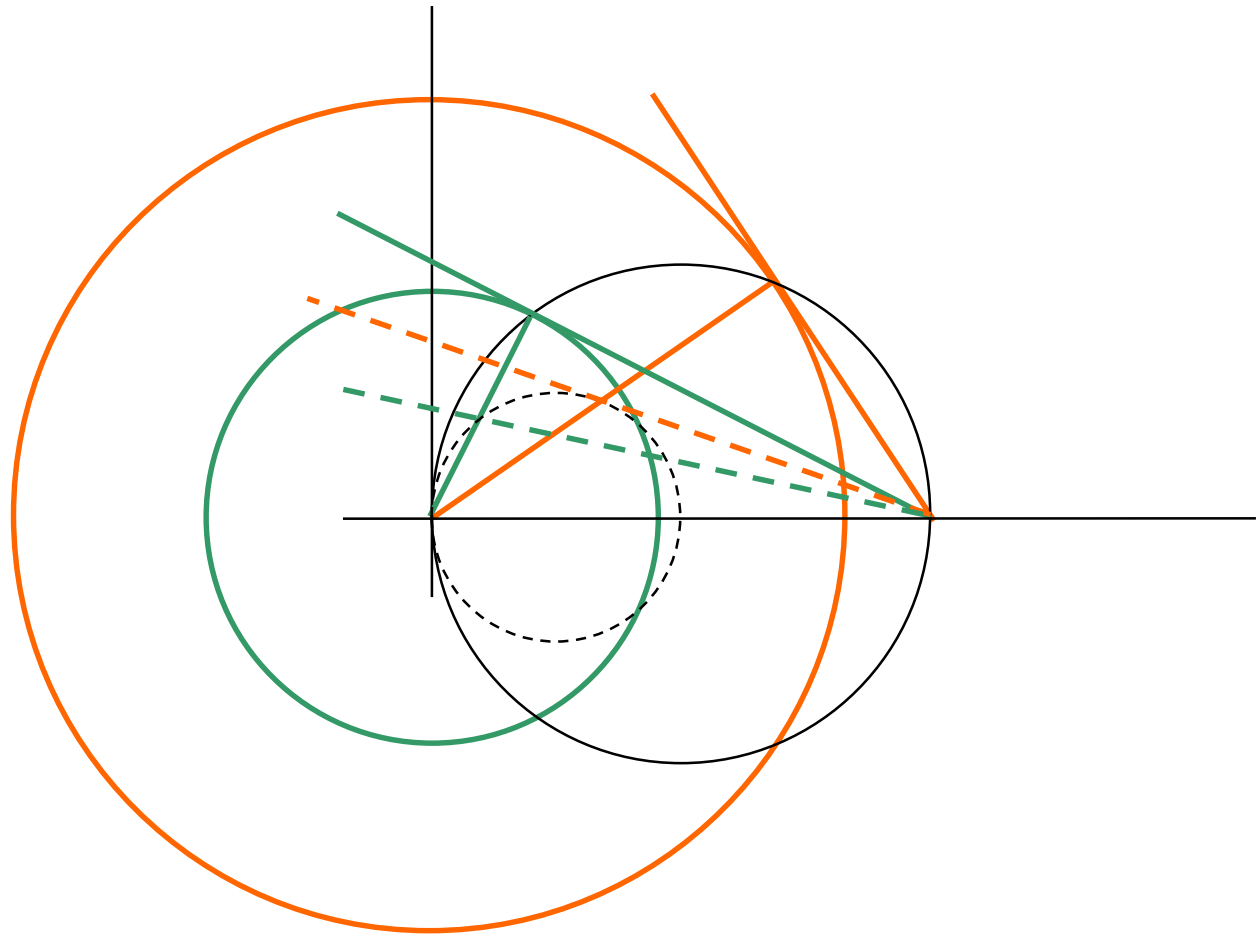
Gravity waves created by a ship



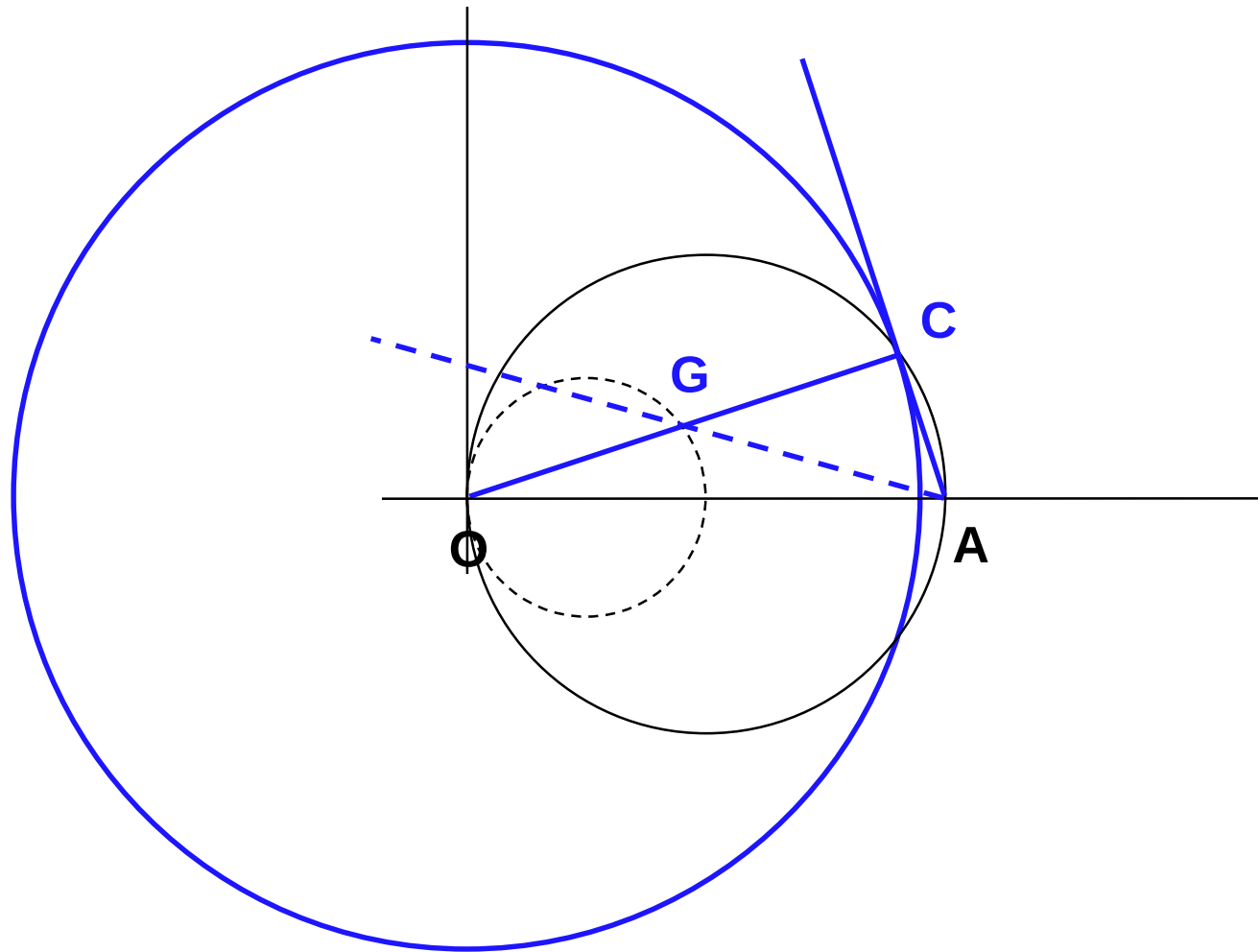
Gravity waves created by a ship



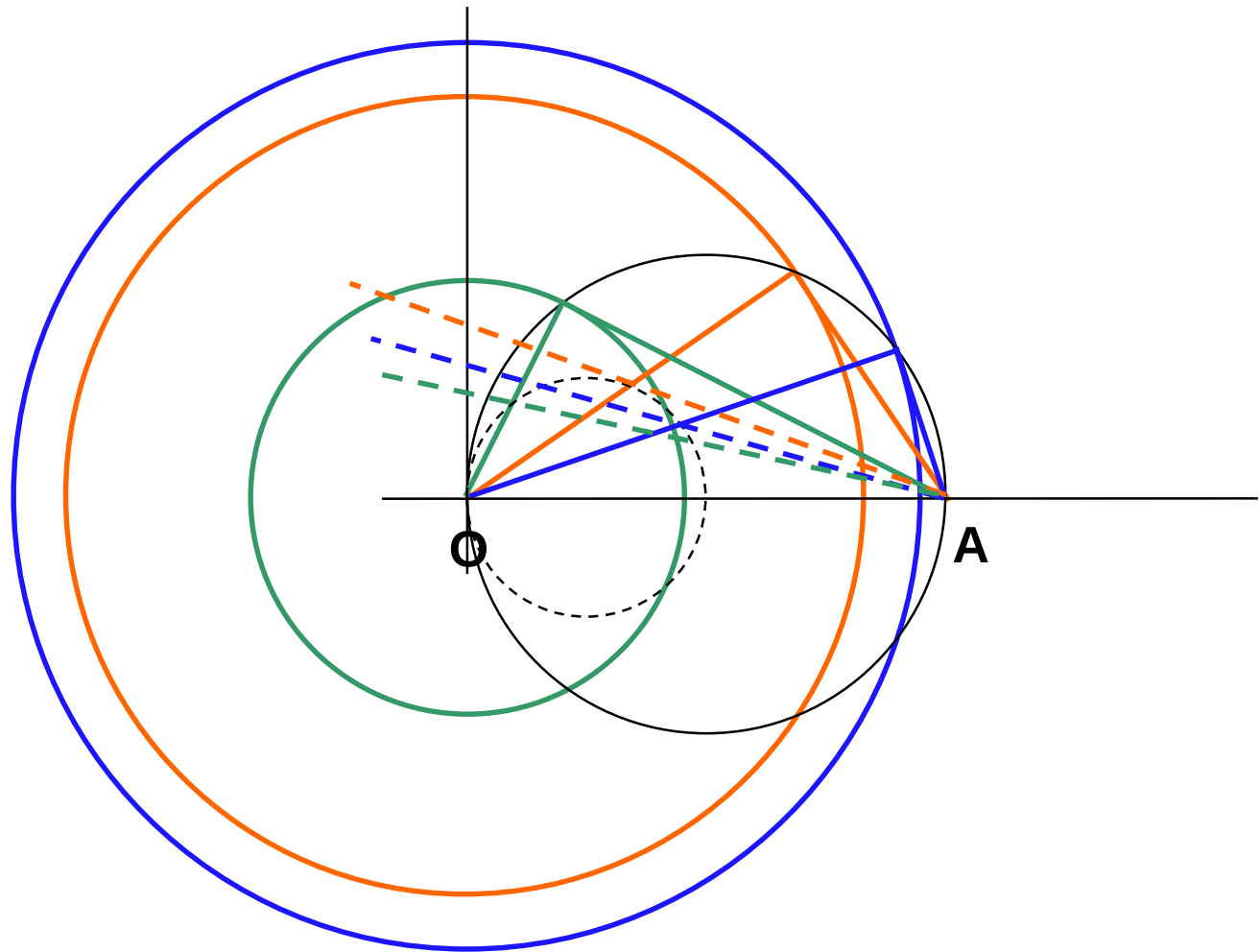
Gravity waves created by a ship



Gravity waves created by a ship



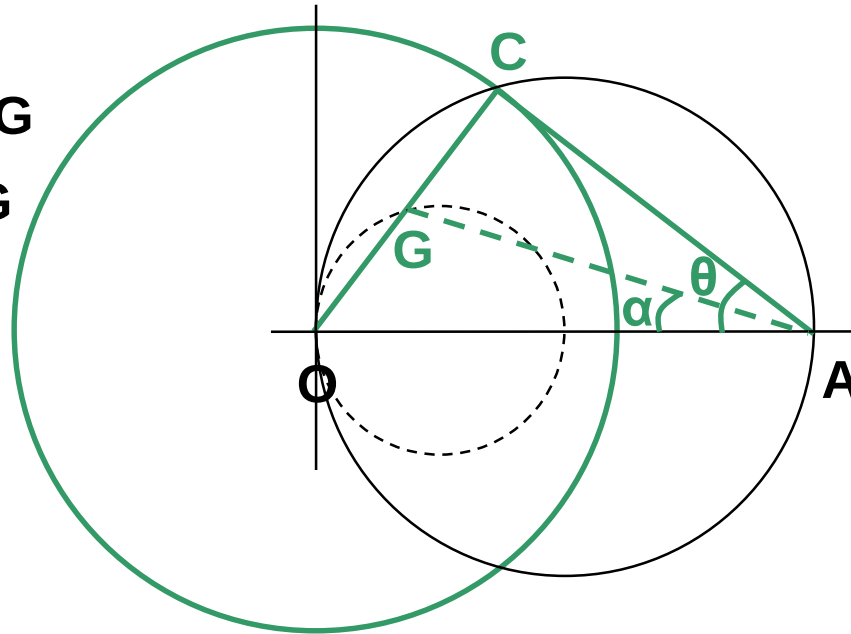
Gravity waves created by a ship



Gravity waves created by a ship

$$\sin(\alpha)/OG = \cos(\theta)/AG$$

$$\sin(\theta - \alpha)AG = GC = OG$$

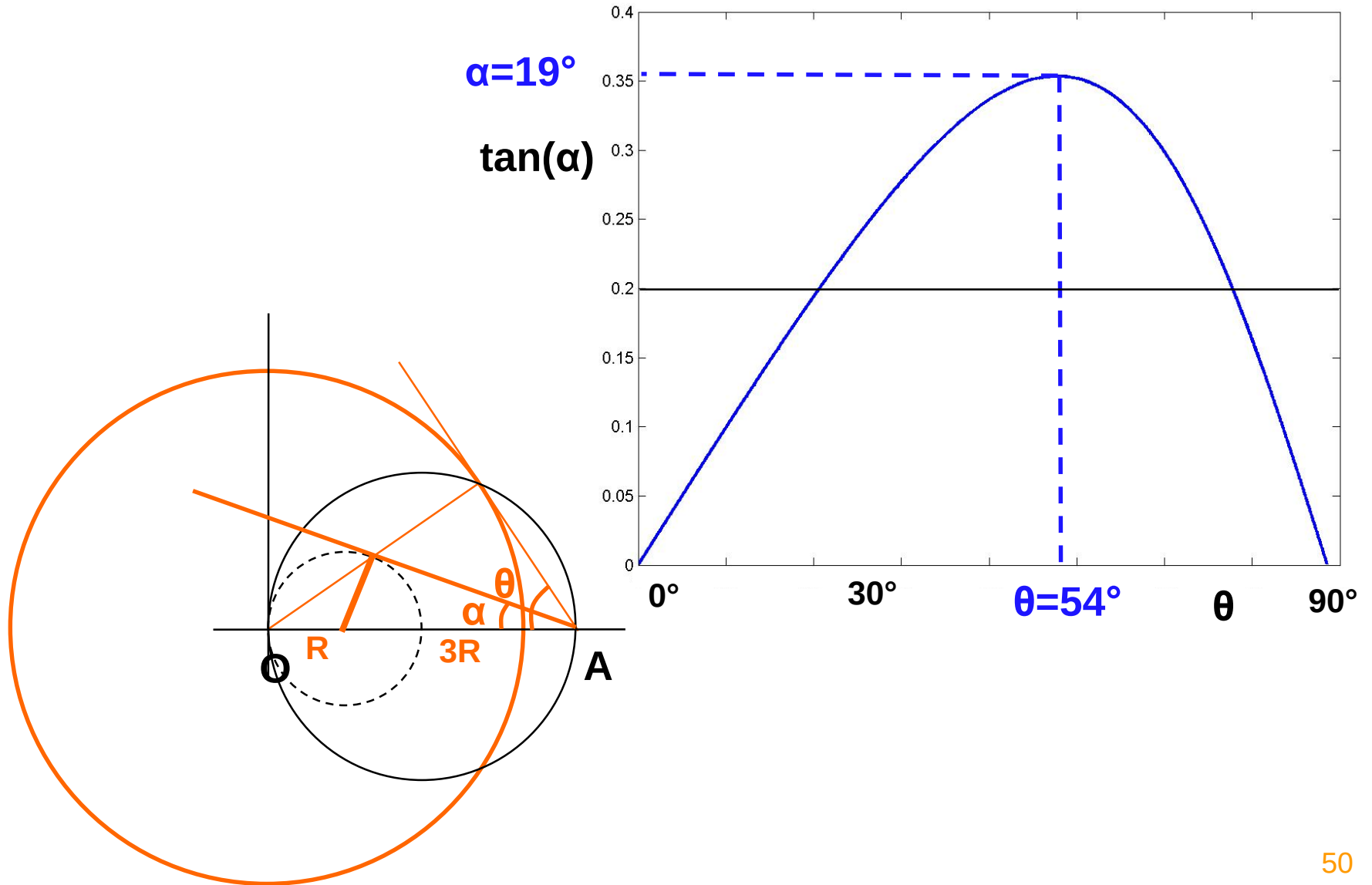


$$\Rightarrow \sin(\alpha) = \cos(\theta) \sin(\theta - \alpha)$$

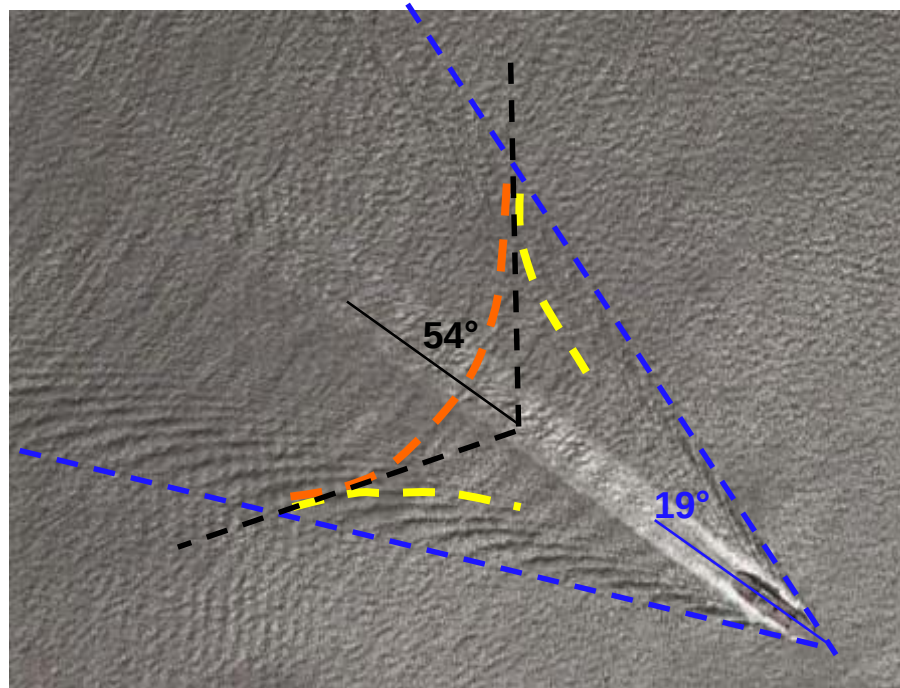
$$\Rightarrow \sin(\alpha) = \cos(\theta) (\sin(\theta) \cos(\alpha) + \cos(\theta) \sin(\alpha))$$

$$\Rightarrow \tan(\alpha) = \cos(\theta) \sin(\theta) / (1 + \cos^2(\theta))$$

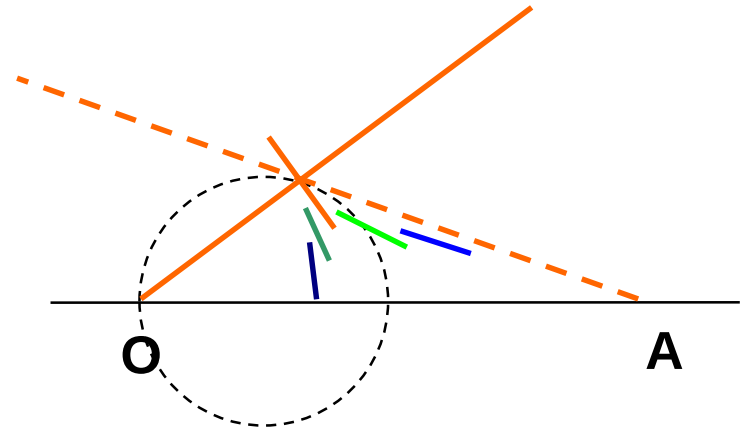
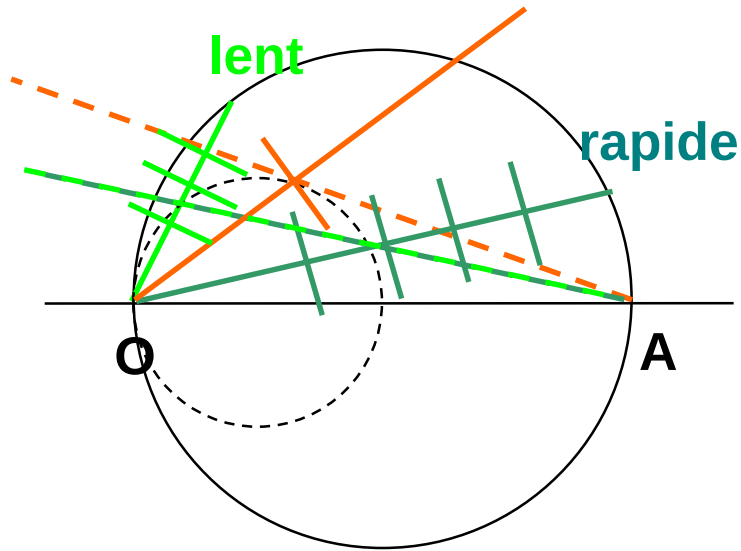
Gravity waves created by a ship



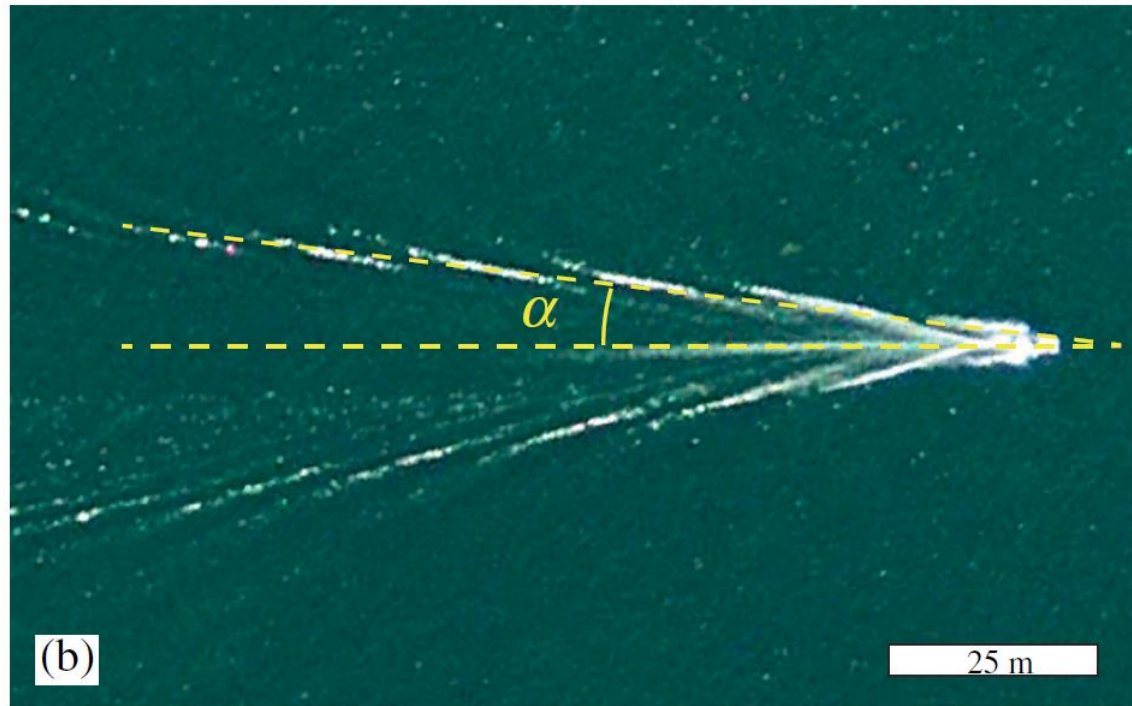
Gravity waves created by a ship



Gravity waves created by a ship

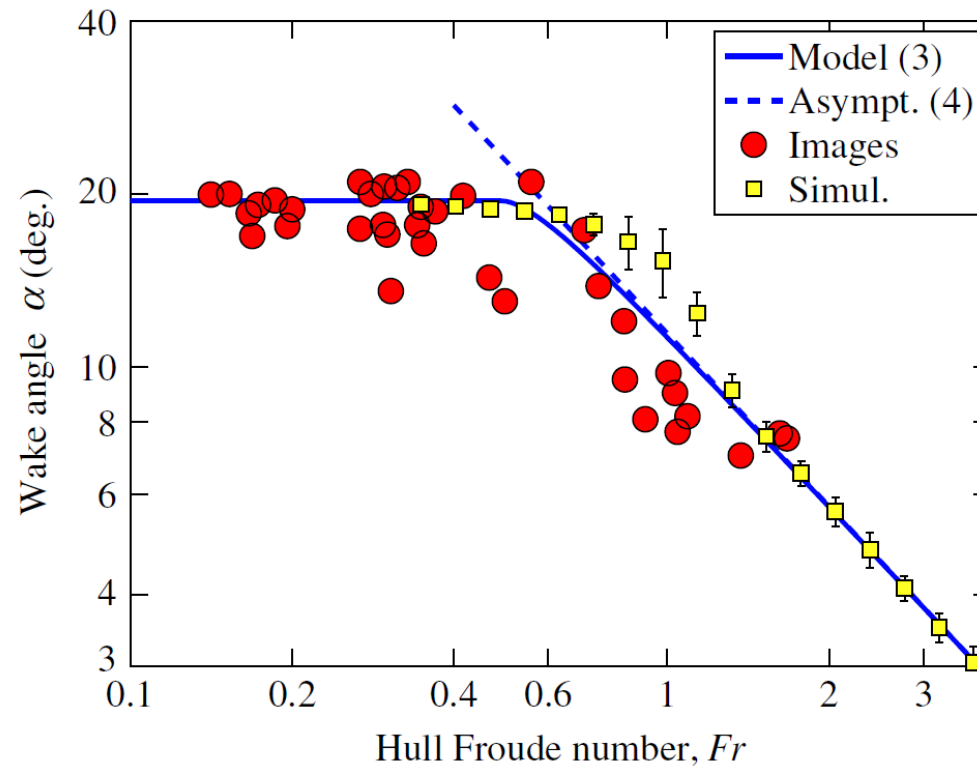


But observations show



Moisy and Rabaud 2013

But observations show



Moisy and Rabaud 2013