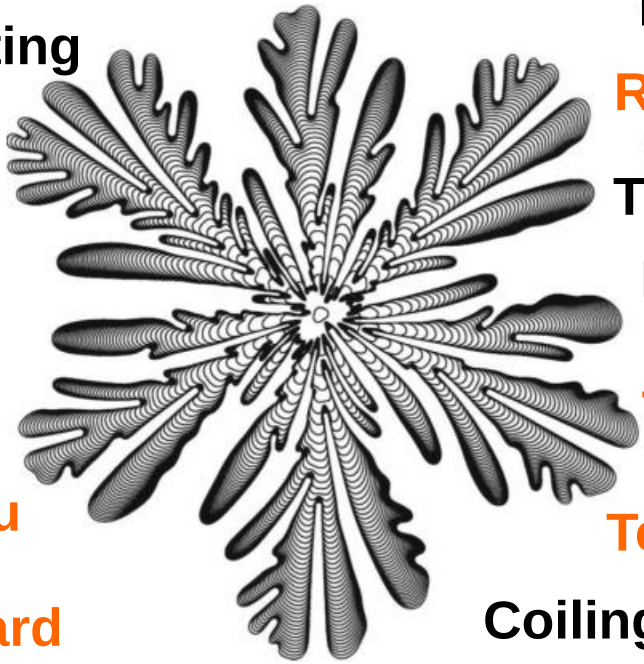
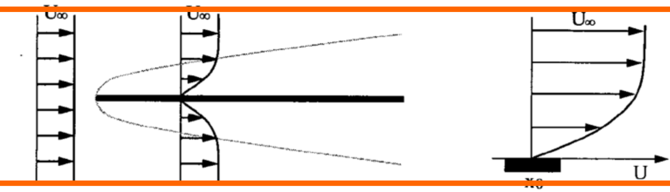
Most flows are unstable...



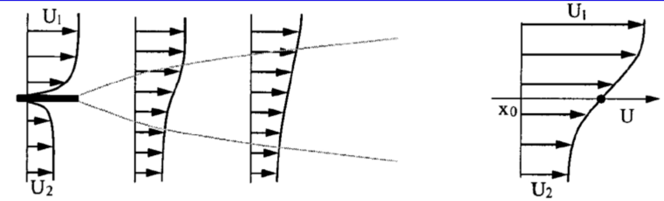
Vortex shedding
Saffman-Taylor
Flow separation
Tollmien-Schlichting
Rayleigh-Taylor
Lift-up and Streaks
Traffic waves
Meandering instability
Gravito-capillary waves
Taylor-Couette
Rayleigh-Plateau
Tearing instability
Rayleigh-Benard
Coiling instability
Benard-Marangoni
Kelvin-Helmholtz

SPATIALLY DEVELOPING SHEAR FLOWS

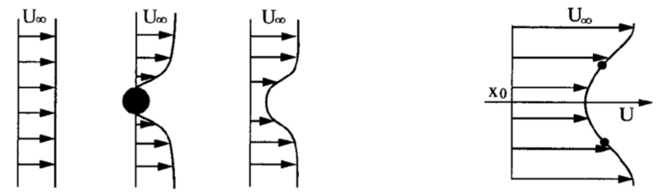
Flat plate boundary layer



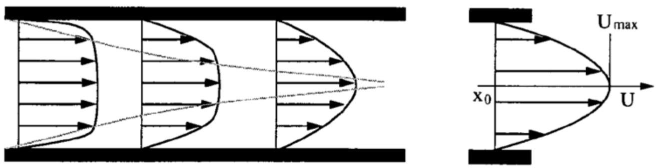
Mixing layer



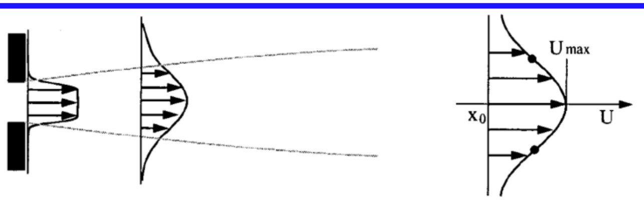
Cylinder wake



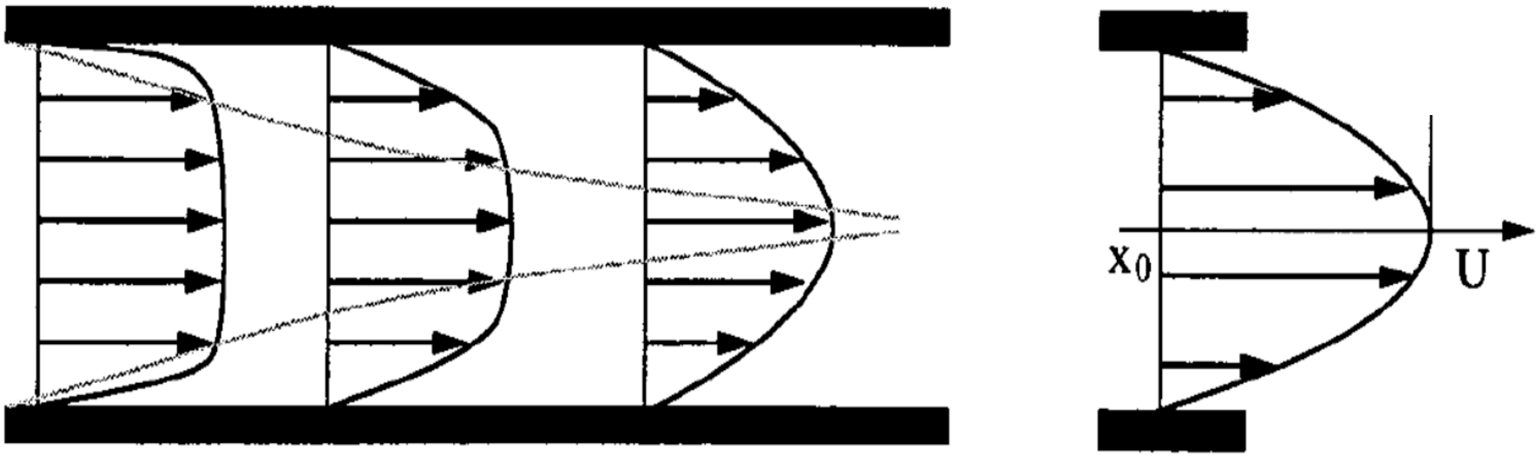
Plane channel flow

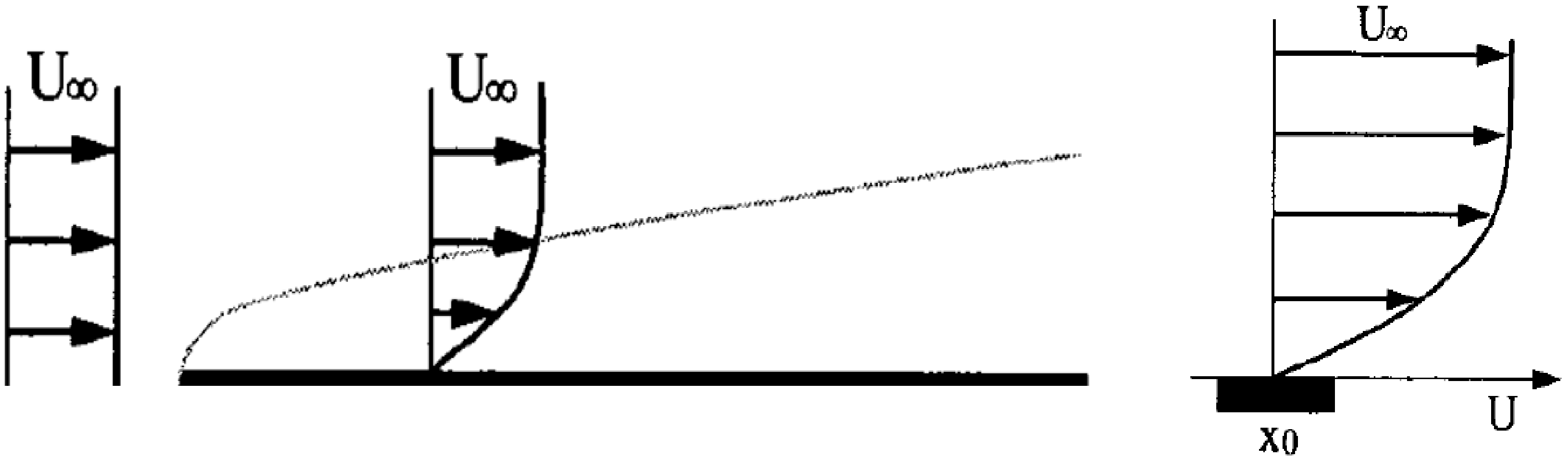


2D jet



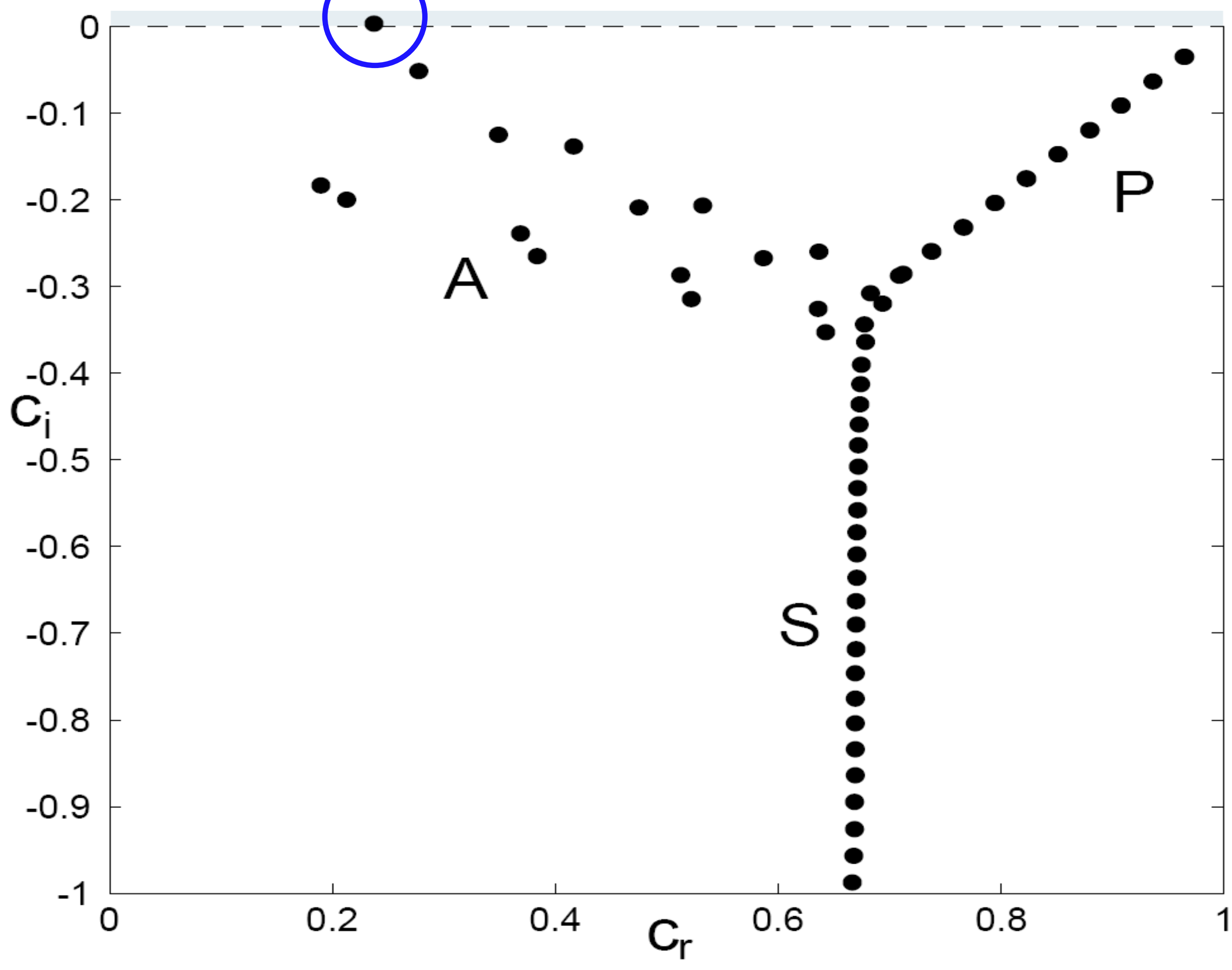
What about stable flows (no inflexion point)?





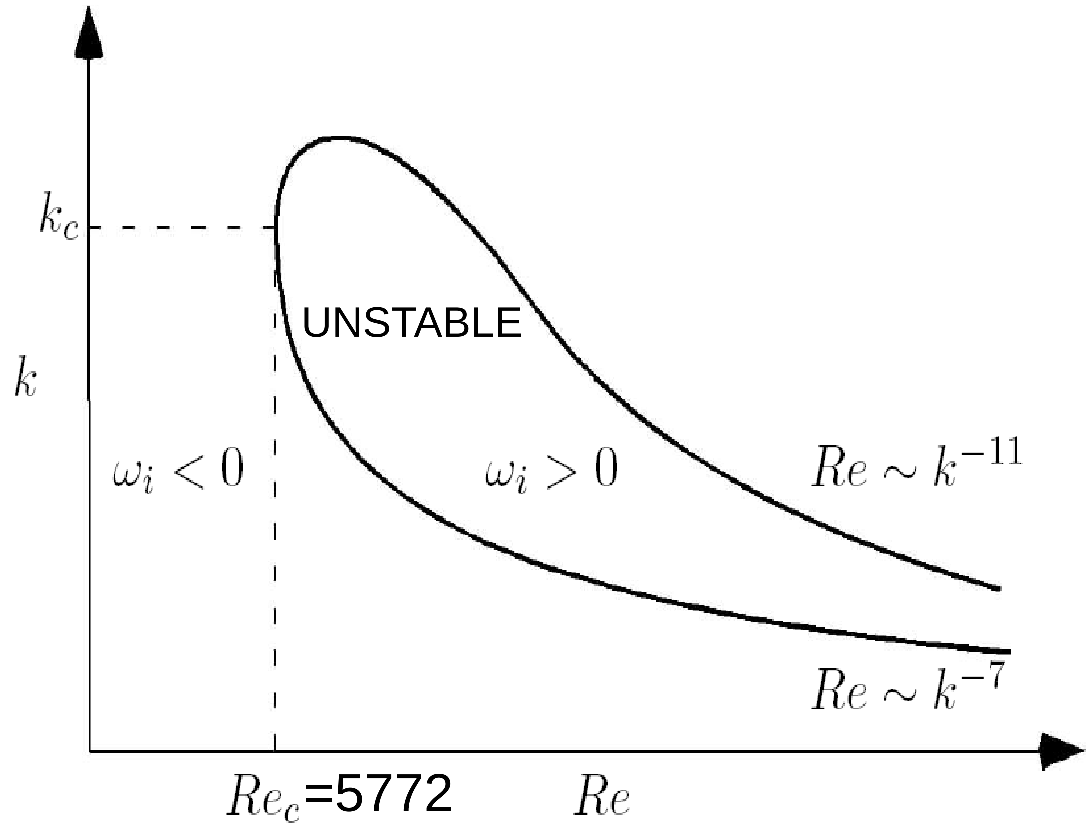
Plane poiseuille flow; Re=10000; α=1, β=0

No inflection point but unstable!



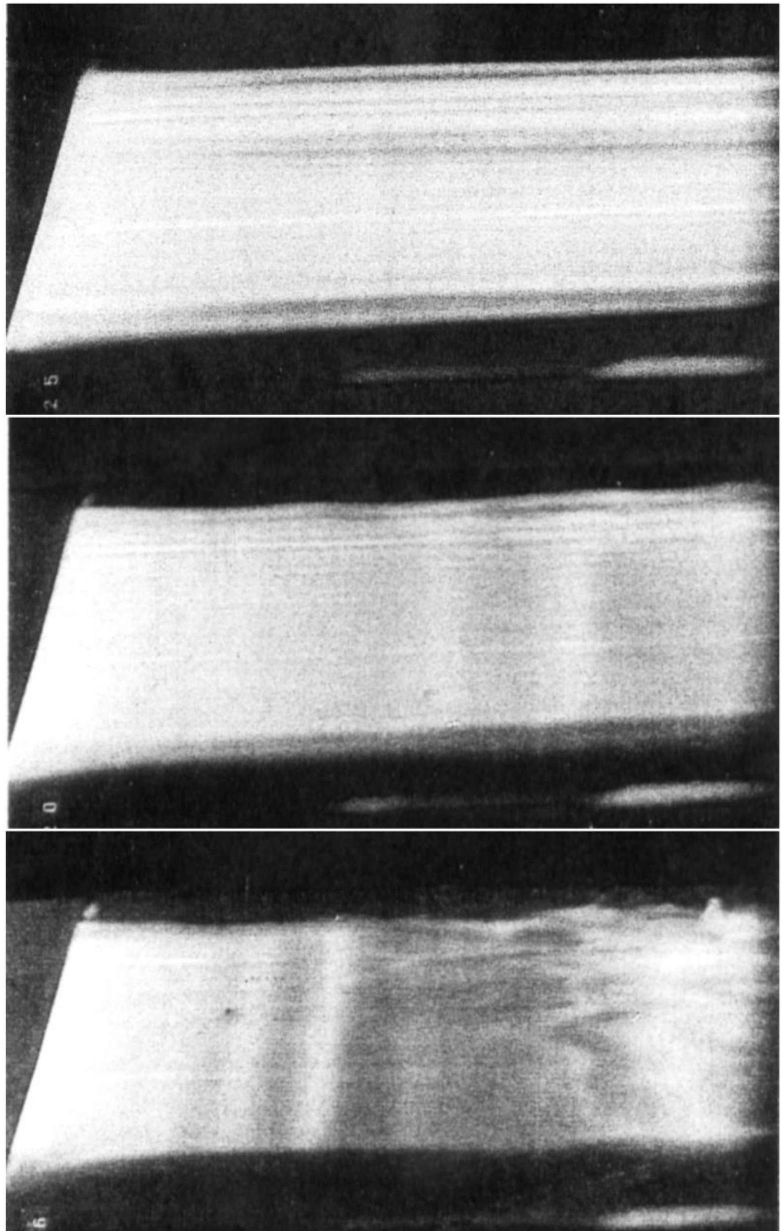
$c=\omega/k$

Neutral curve for plane poiseuille flow



– *Allure de la courbe de stabilité marginale dans le plan $Re - k$ pour l'écoulement de Poiseuille plan.*

Boundary layer at increasing Reynolds number



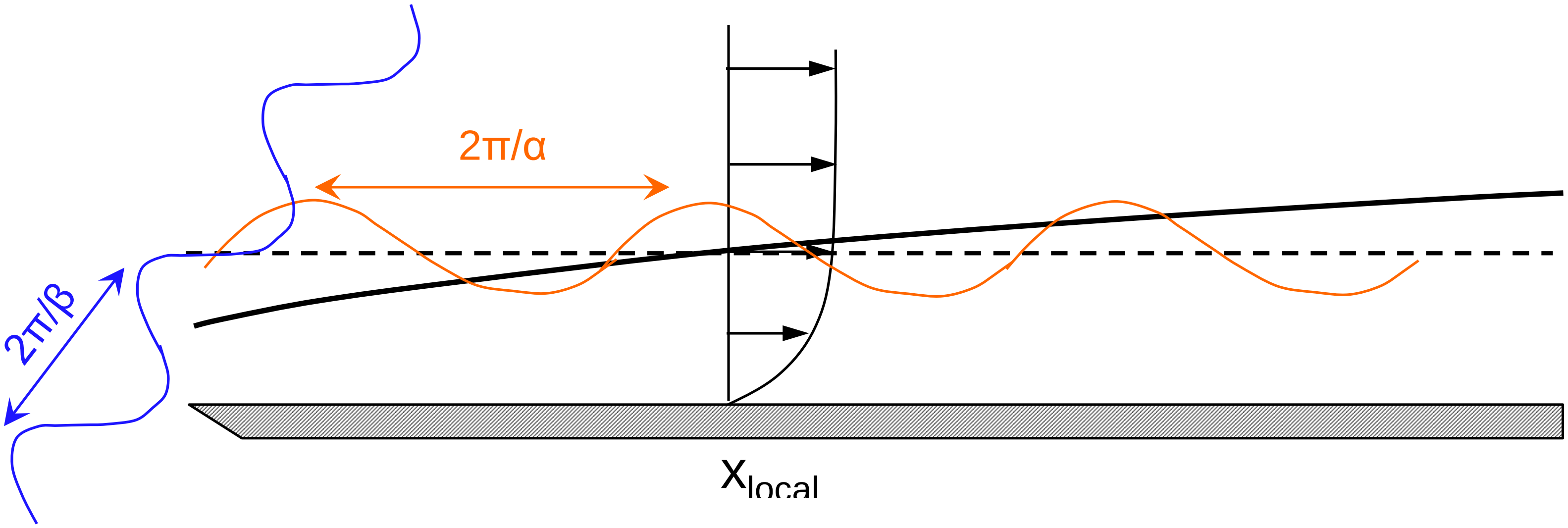
Boundary layers are destabilized by viscosity
Reynolds Orr equation

Perturbation kinetic energy: $e_c = \frac{1}{2}(u^2 + v^2 + w^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \langle e_c \rangle \mathrm{d}y = \int_{y_1}^{y_2} \partial_y \overline{U} \tau_{xy} \mathrm{d}y - \frac{1}{Re}\int_{y_1}^{y_2} \langle \boldsymbol{\omega}.\boldsymbol{\omega} \rangle \mathrm{d}y$$

Production term Dissipation term

Local parallel flow approximation



$$(\mathbf{u}, p) = (\mathbf{u}(y), p(y))\, \mathrm{e}^{\sigma t + i(\alpha x + \beta z)}$$

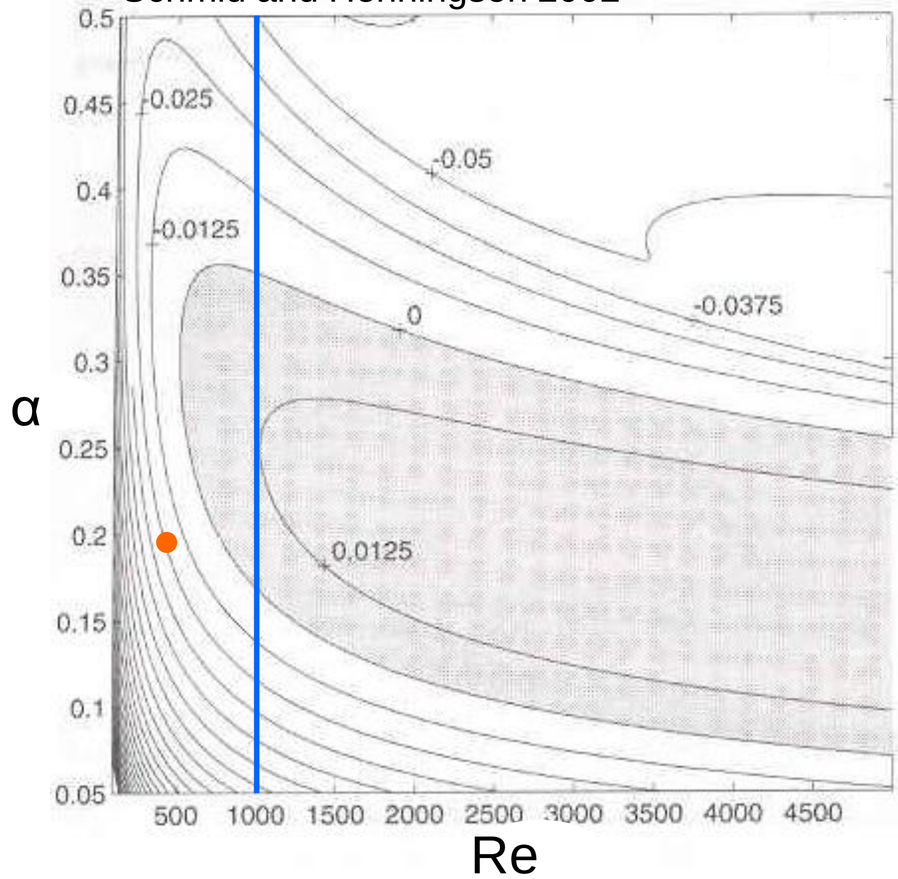
⇨Orr-Sommerfeld-Squire equation

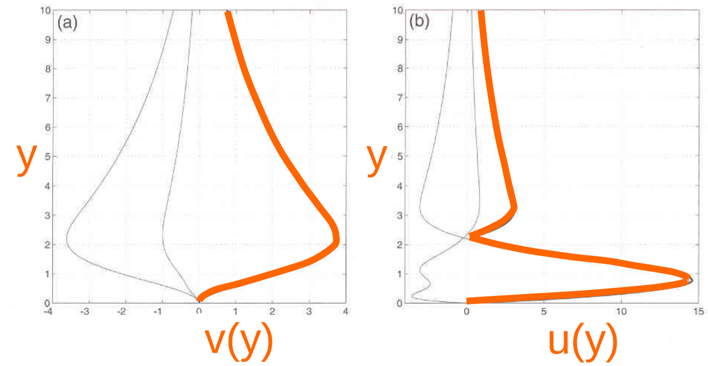
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}(y)\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{U}(y) = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$

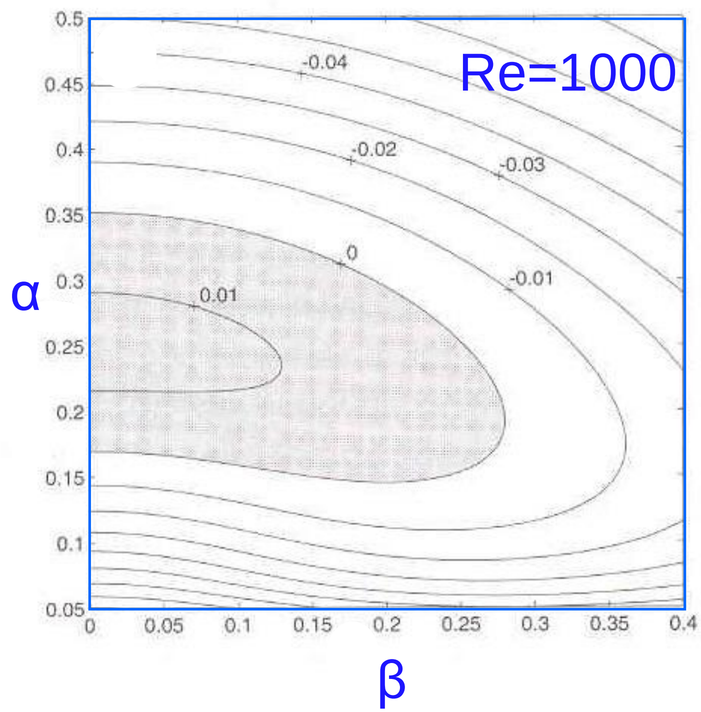
$$\nabla.\mathbf{u} = 0$$

Tollmien Schlichting waves

Schmid and Henningson 2002



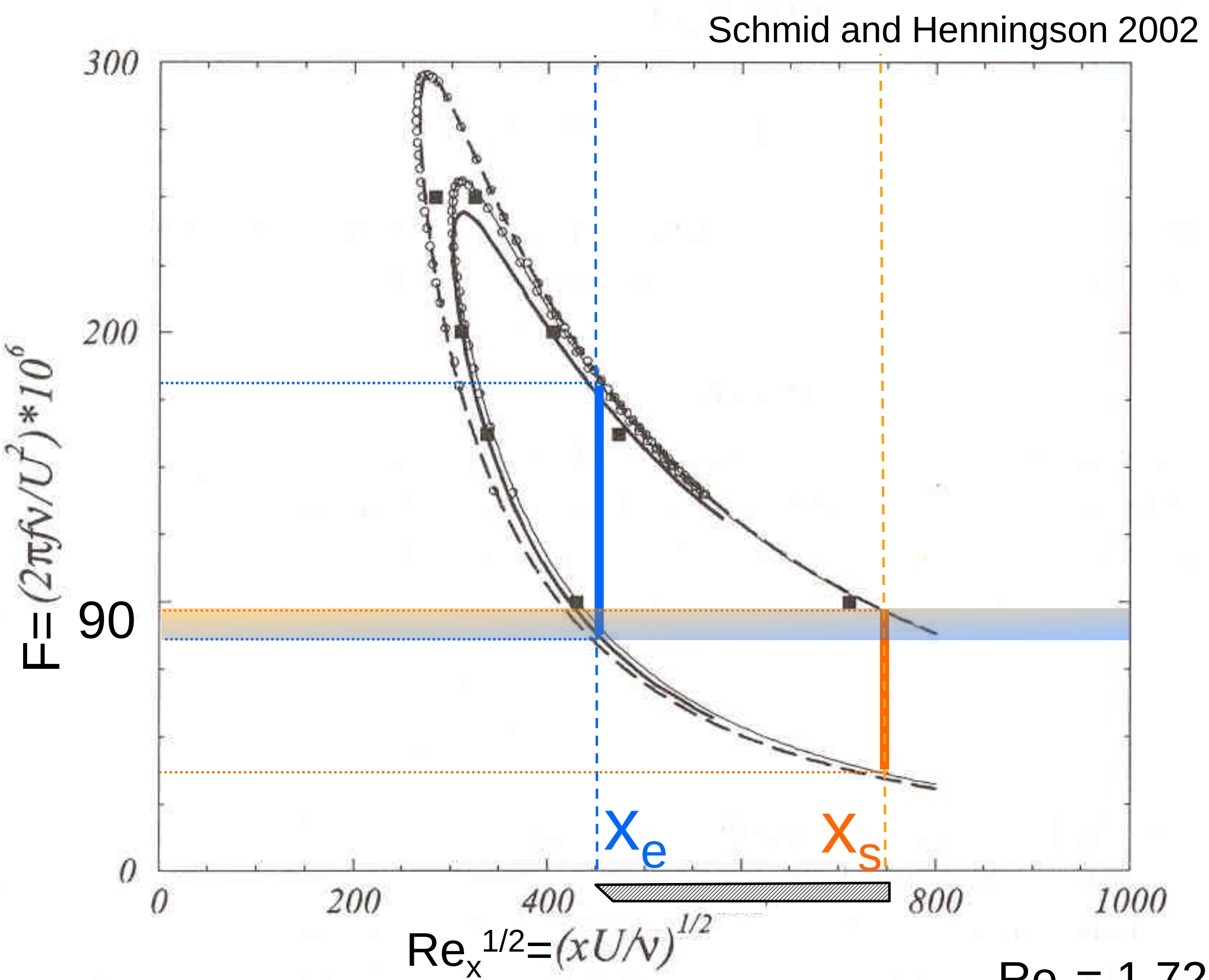




α: axial wavenumber

β: transverse wavenumber

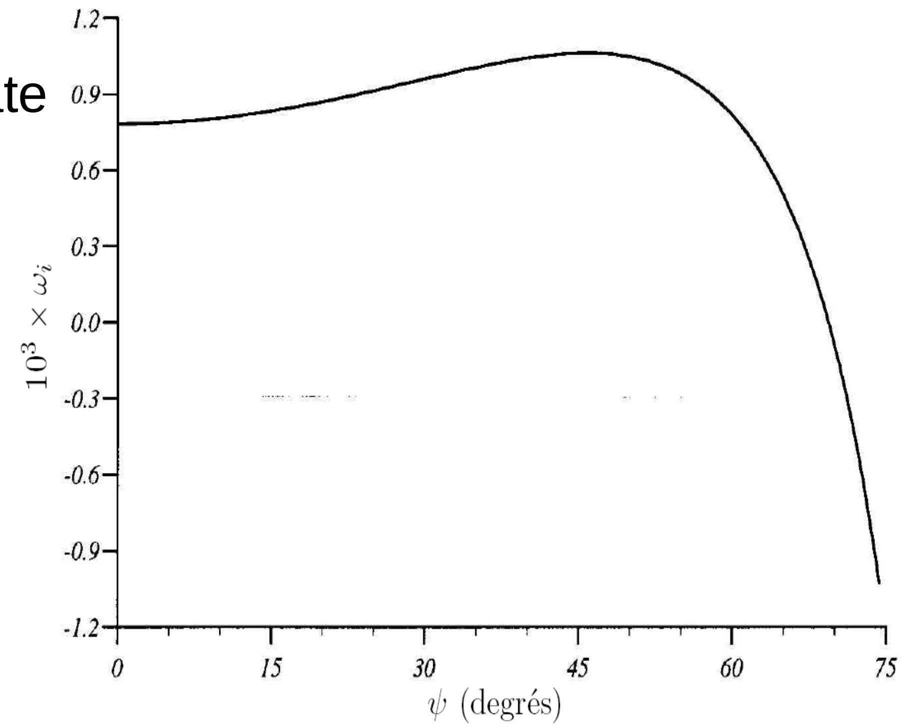
Neutral curve



$Re_\delta = 1.72 Re_x^{1/2}$

Boundary layer at Re_δ =1500

Growth-rate

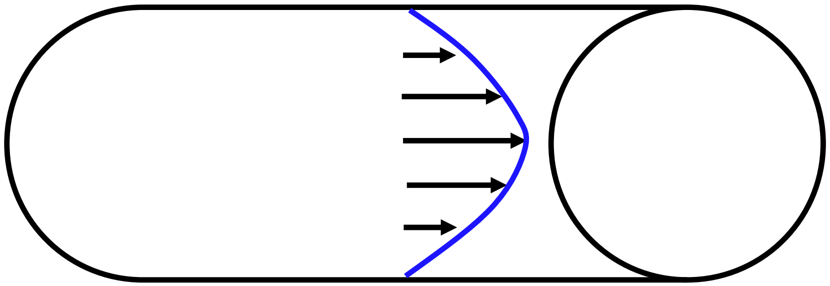


ψ= tan(β/α)

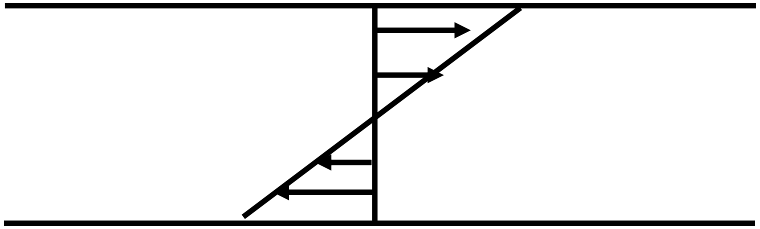
Taux de croissance temporel (s^{-1}) de l'instabilité d'une couche limite, en fonction de l'angle ψ de la perturbation avec la direction de l'écoulement de base. $R_{\delta 1} = 1500$, $\omega\nu/U_\infty^2 = 0,3 \times 10^{-4}$ (calcul G. Casalis, ONERA).

The most unstable perturbation is oblique!

Hagen Poiseuille



Couette



Flow type	Re_{exp}	Re_{lin}
Pipe flow	≈ 2000	∞
Plane Poiseuille flow	≈ 1000	5772
Plane Couette flow	≈ 360	∞

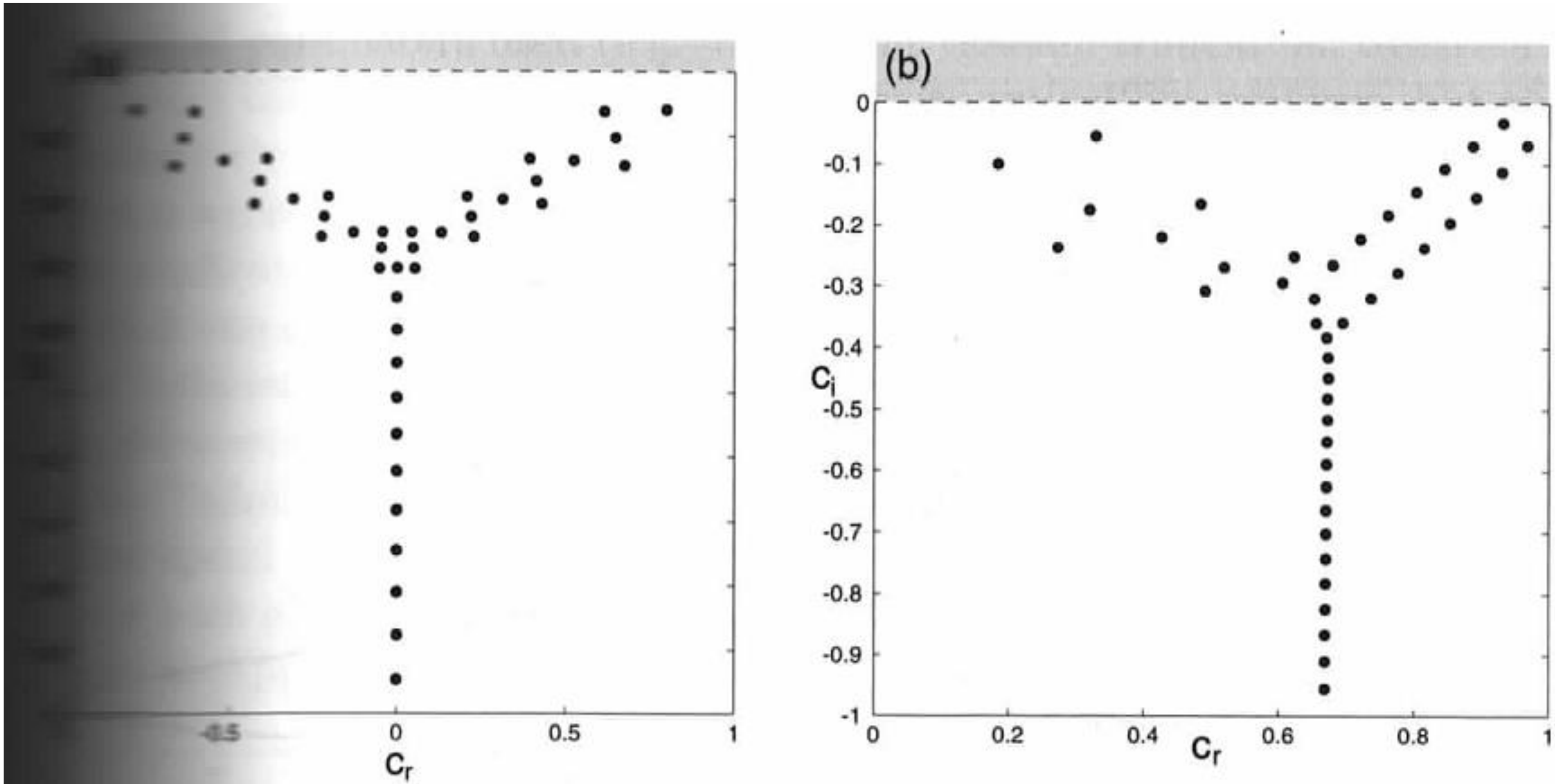


FIGURE 3.3. Spectrum of plane Couette and pipe flow. (a) Plane Couette flow for $\alpha = 1, \beta = 1, \mathrm{Re} = 1000$; (b) Pipe flow for $\alpha = 1, n = 1, \mathrm{Re} = 5000$.

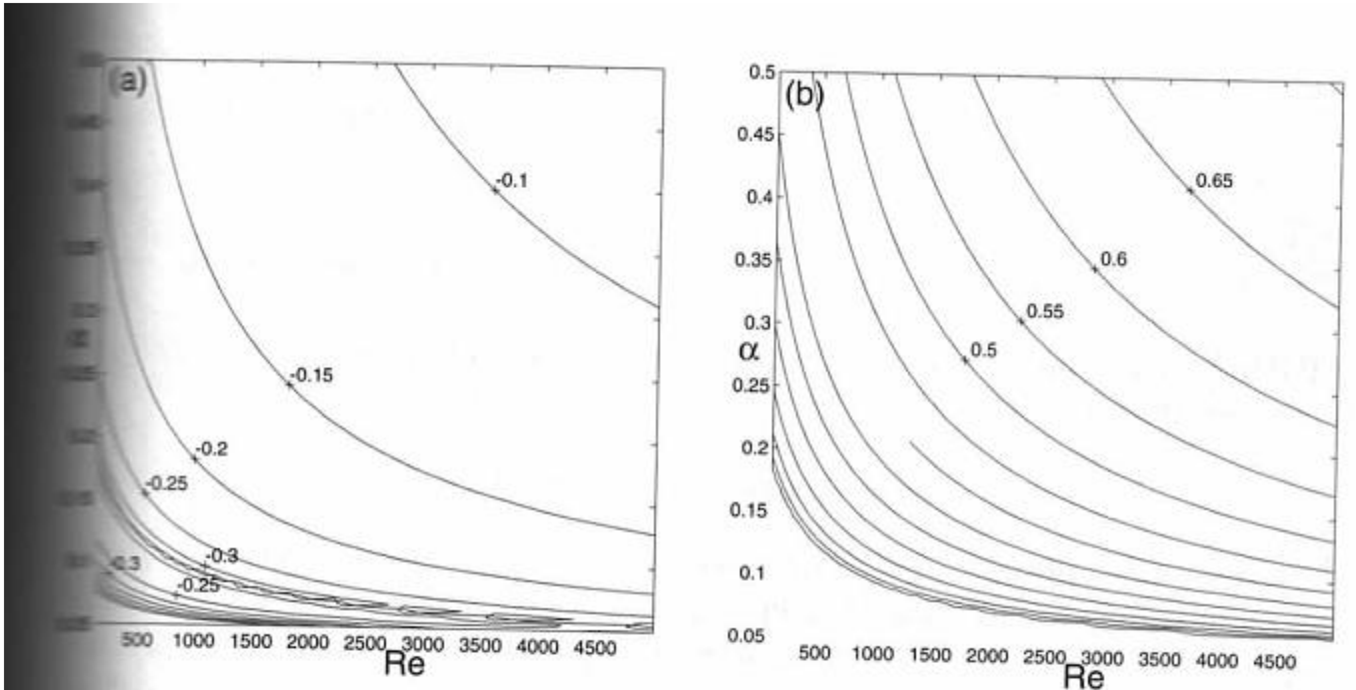


FIGURE 3.11. Contours of constant growth rate (a) and phase speed (b) for plane Couette flow.

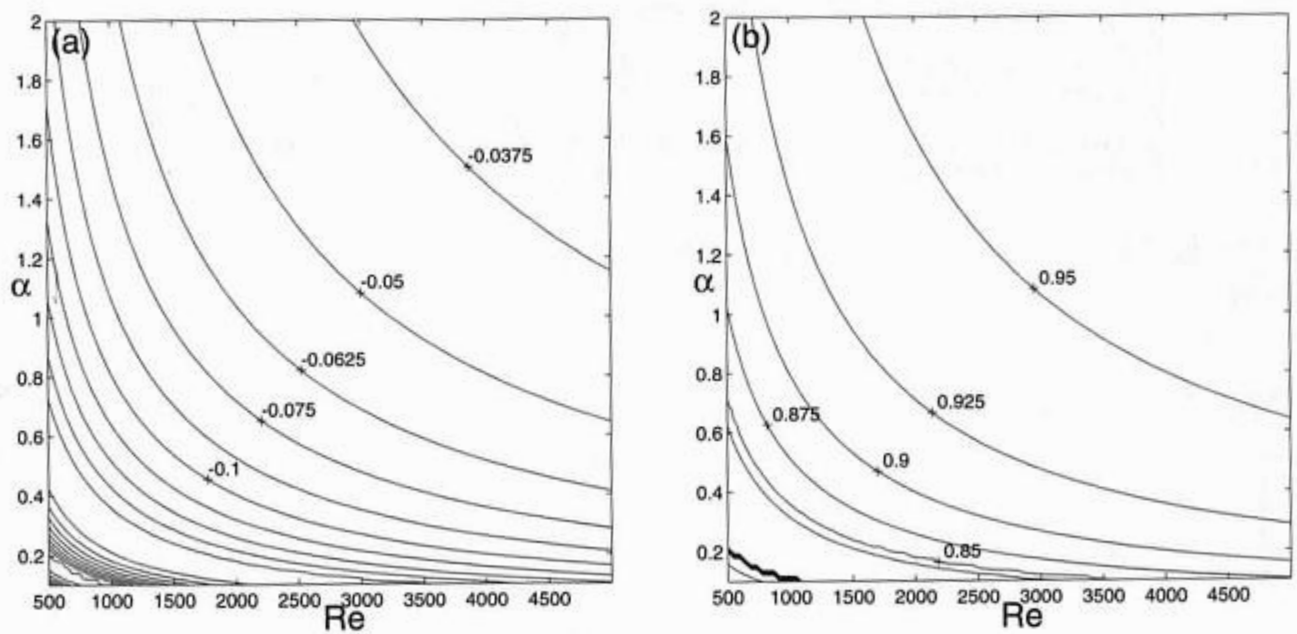
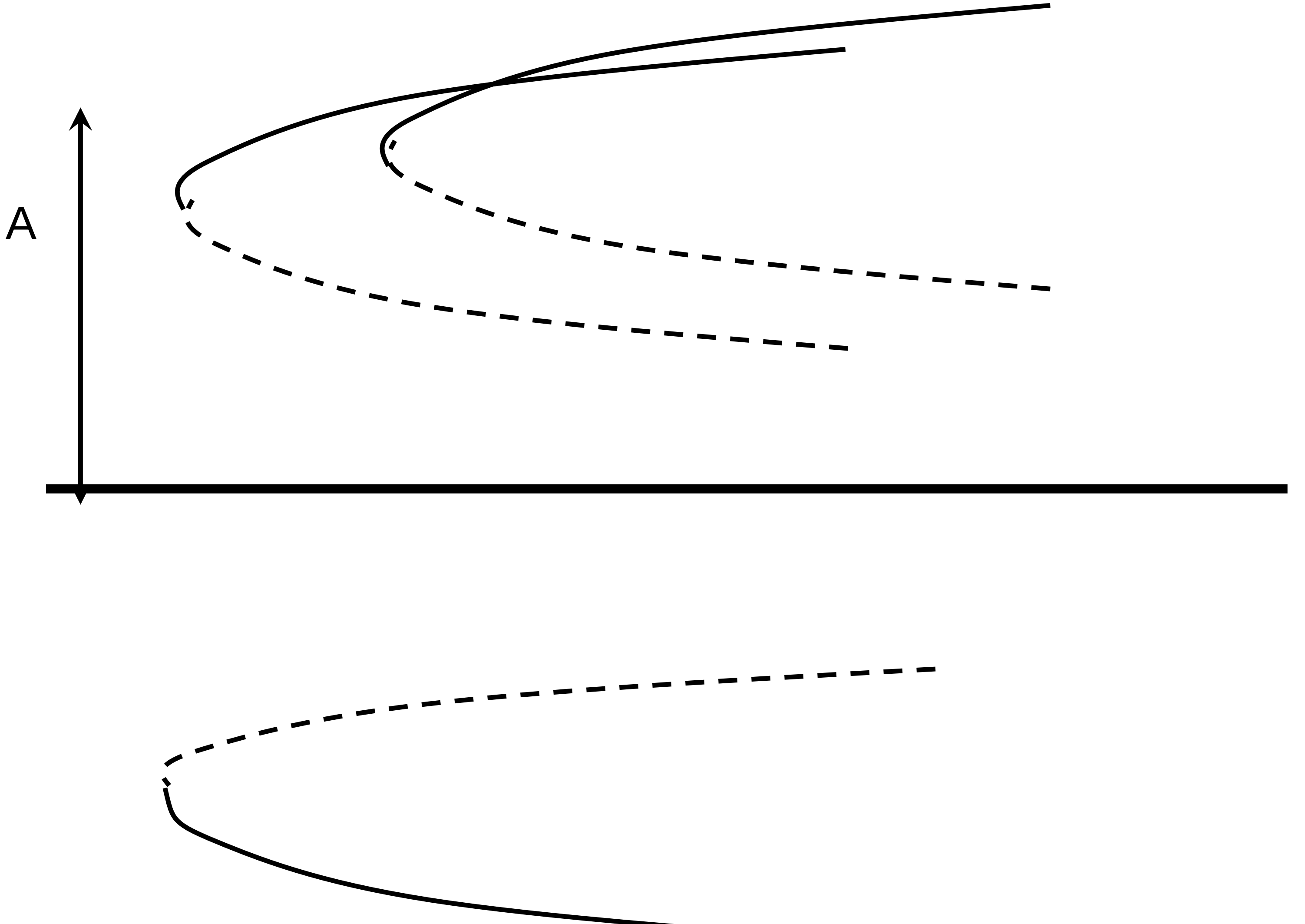


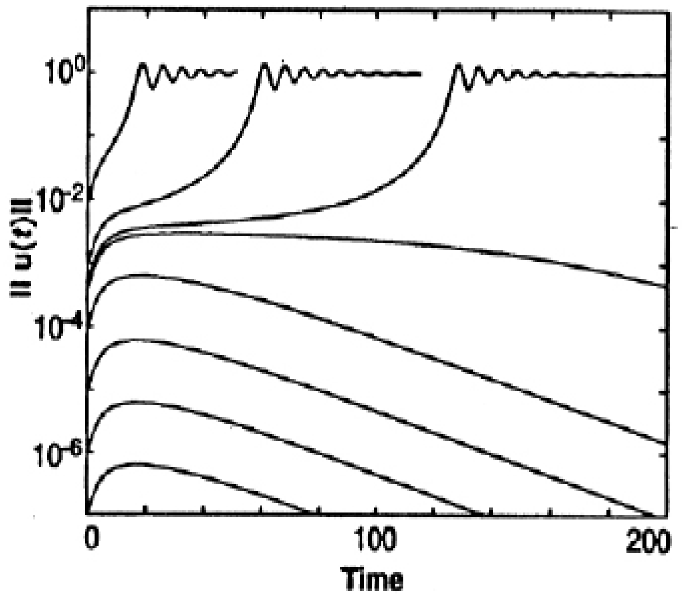
FIGURE 3.12. Contours of constant growth rate (a) and phase speed (b) for pipe Poiseuille flow.

What about nonlinearities?



Edge states

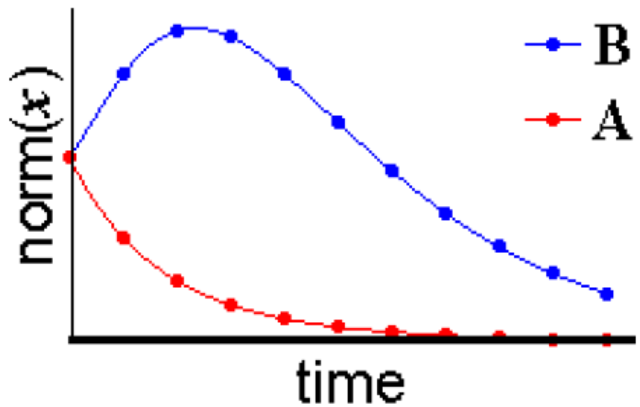
Transient growth and bye pass transition

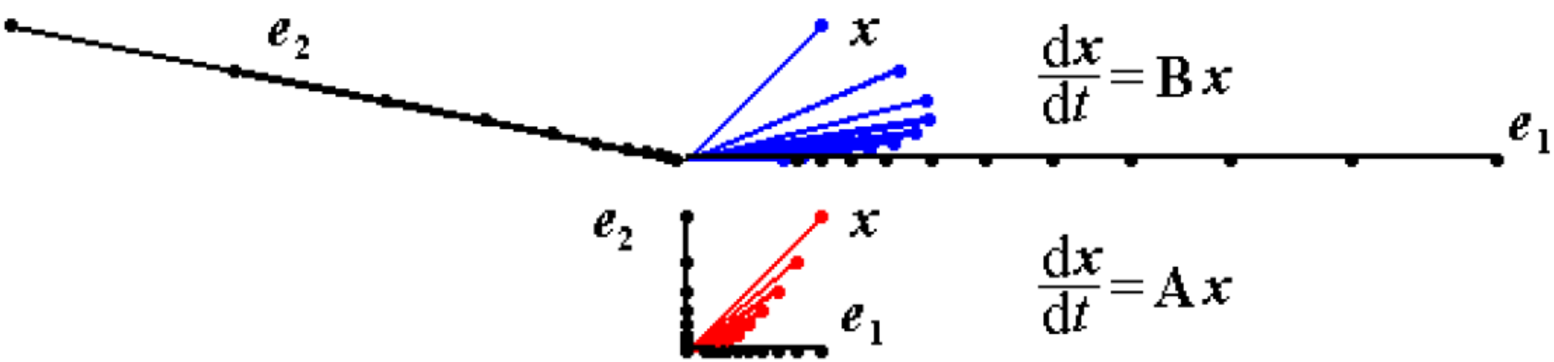


Conditional stability – subcritical bifurcation
Trefethen 1993

$$\mathbf{B} = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$





Definition of optimal gain of a linear system

Linear system $$\frac{dq}{dt} = \mathbf{L}q$$

Initial condition $$q(t=0) = q_0$$

Optimal gain $$G(t) = \max_{q_0} \frac{||q||}{||q_0||} \;?$$

norm

Definition of optimal gain of a linear system

$$\frac{dq}{dt} = \mathbf{L}q$$

$$q(t=0) = q_0$$

Matrix exponential

$$q(t) = \exp(\mathbf{L}t)q_0$$

Definition of optimal gain of a linear system

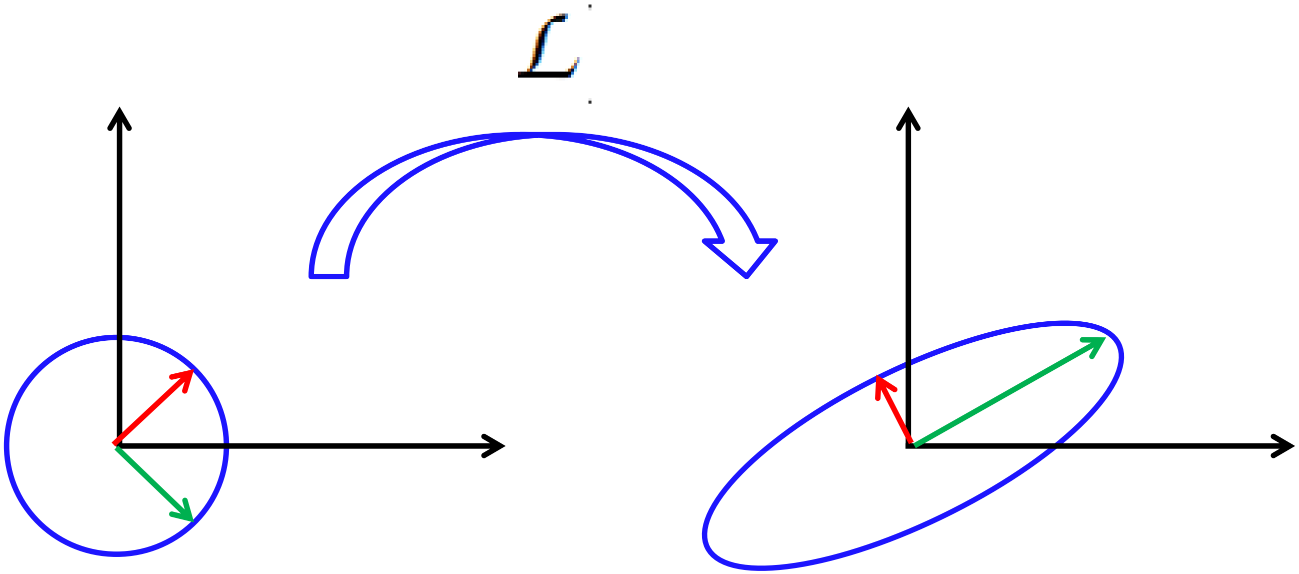
Optimal gain $$G(t) = \max_{q_0} \frac{||q||}{||q_0||}$$

$$= \max_{q_0} \frac{||\exp(t\mathcal{L})q_0||}{||q_0||}$$

$$= ||\exp(t\mathcal{L})||$$

How to compute a matrix 2-norm?

Optimal gain $G(t) = ||\exp(t\mathcal{L})||$



unitary sphere

rotated ellipsoid

Optimal gain is associated to a norm

Kinetic energy written in v,η form

iαu=Dv-iβw and η=iβu-iαw

$$E(t) = \frac{1}{2k^2} \int_\Omega \left[|\mathcal{D}v|^2 + k^2 |v|^2 + |\eta|^2 \right] \, d\Omega$$

Optimal gain is associated to a norm

Kinetic energy written in v,η form

iαu=Dv-iβw and η=iβu-iαw

$$
\begin{aligned}
E(t) &= \frac{1}{2k^2}\int_\Omega \left[|\mathcal{D}v|^2 + k^2|v|^2 + |\eta|^2\right] \, d\Omega \\
&= \|q\|^2 = \frac{1}{2k^2}\int_\Omega \begin{pmatrix} v \\ \eta \end{pmatrix}^H \begin{pmatrix} -\mathcal{D}^2 + k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} \, d\Omega \\
&= \frac{1}{2k^2}\int_\Omega q^H M q \, d\Omega
\end{aligned}
$$

energy matrix

Definition of optimal gain of a linear system

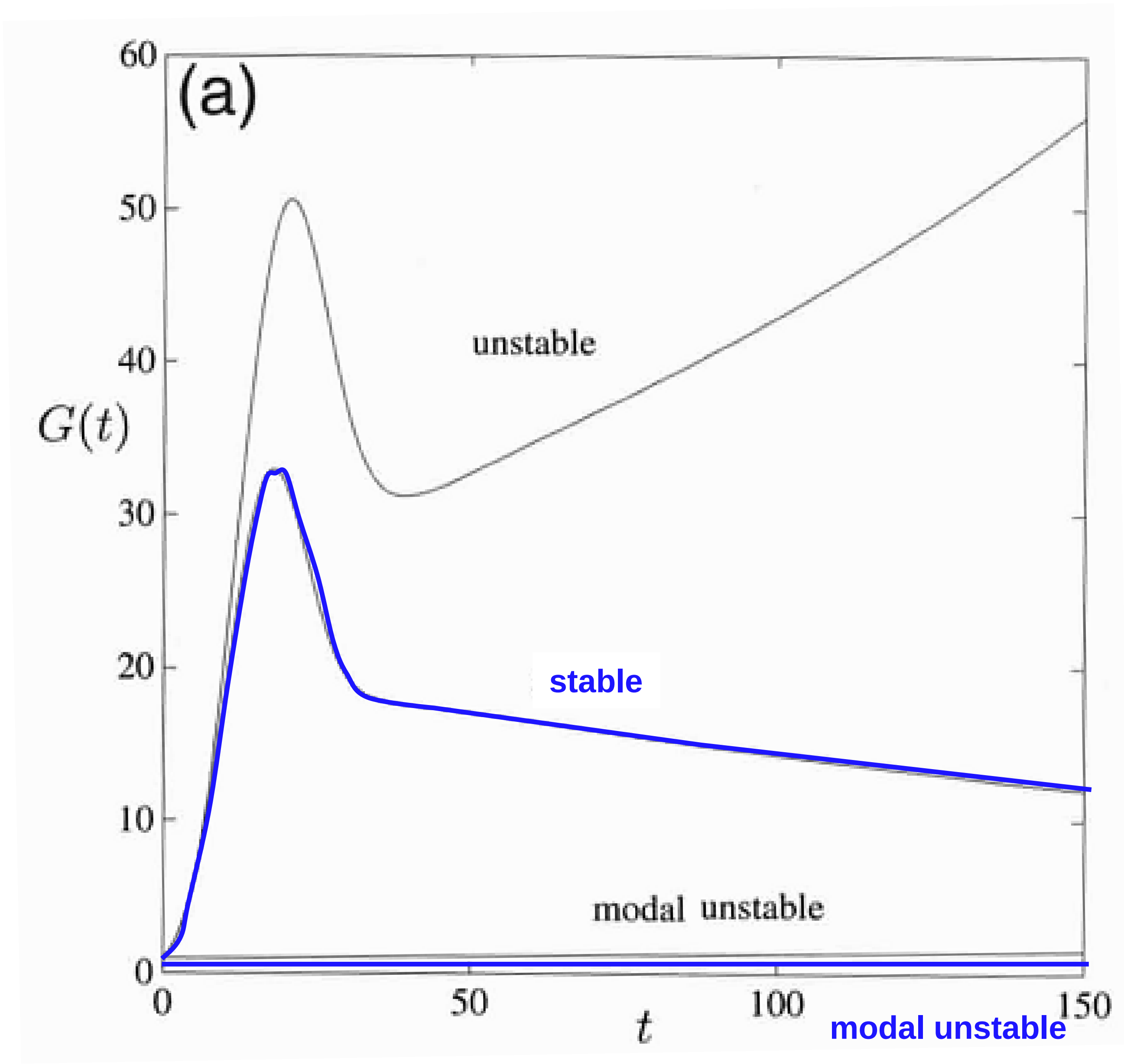
Cholevski dec. $M = F^H F$

$$||q||^2 = \frac{1}{2k^2}\int_\Omega q^H F^H F q \, d\Omega = \frac{1}{2k^2}\int_\Omega (Fq)^H F q \, d\Omega$$

Optimal gain

$$\begin{aligned} G(t) &= \max_{q_0} \frac{||q||_E}{||q_0||_E} = \max_{q_0} \frac{||Fq||_2}{||Fq_0||_2} = \max_{q_0} \frac{||F\exp(tL)q_0||_2}{||Fq_0||_2} \\ &= \max_{q_0} \frac{||F\exp(tL)F^{-1}Fq_0||_2}{||Fq_0||_2} = ||F\exp(tL)F^{-1}||_2 \end{aligned}$$

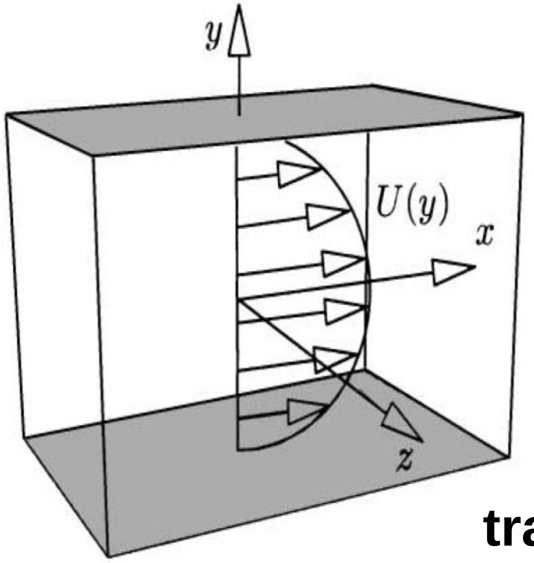
$=q'_0$



(a)
60
50
40
30
20
10
0
G(t)
unstable
stable
modal unstable
0
50
100
150
t
modal unstable

Plane Poiseuille flow

normal

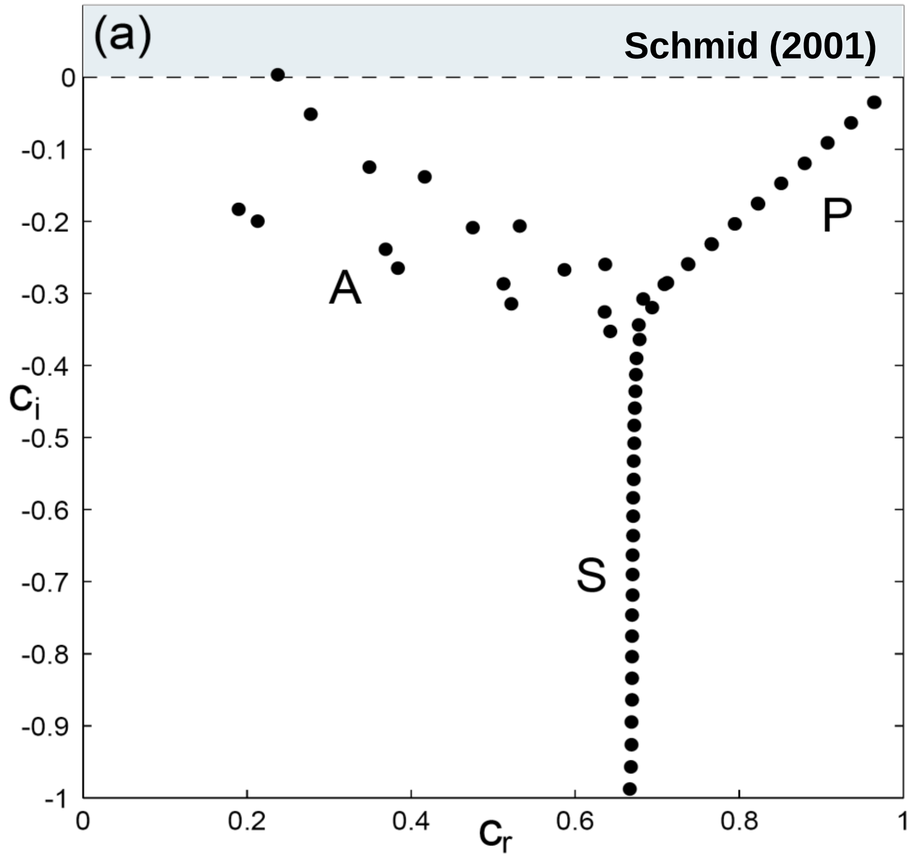


longitudinal, $e^{i\alpha x}$

transverse, $e^{i\beta z}$

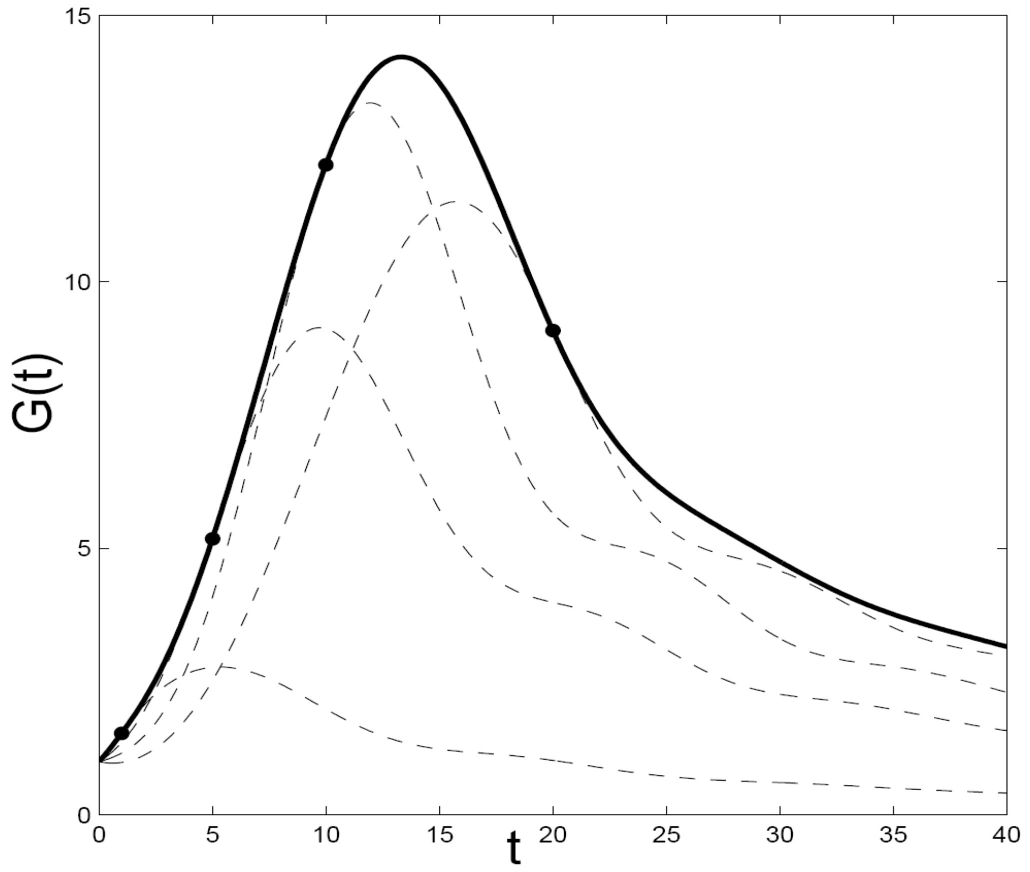
Linearly stable until Re=5772, but the transition is observed experimentally close to Re=1000-2000!

Plane Poiseuille flow - stability



Spectrum for plane Poiseuille flow for $\alpha = 1, \beta = 0, Re = 10000$.

Orr mechanism (2D)

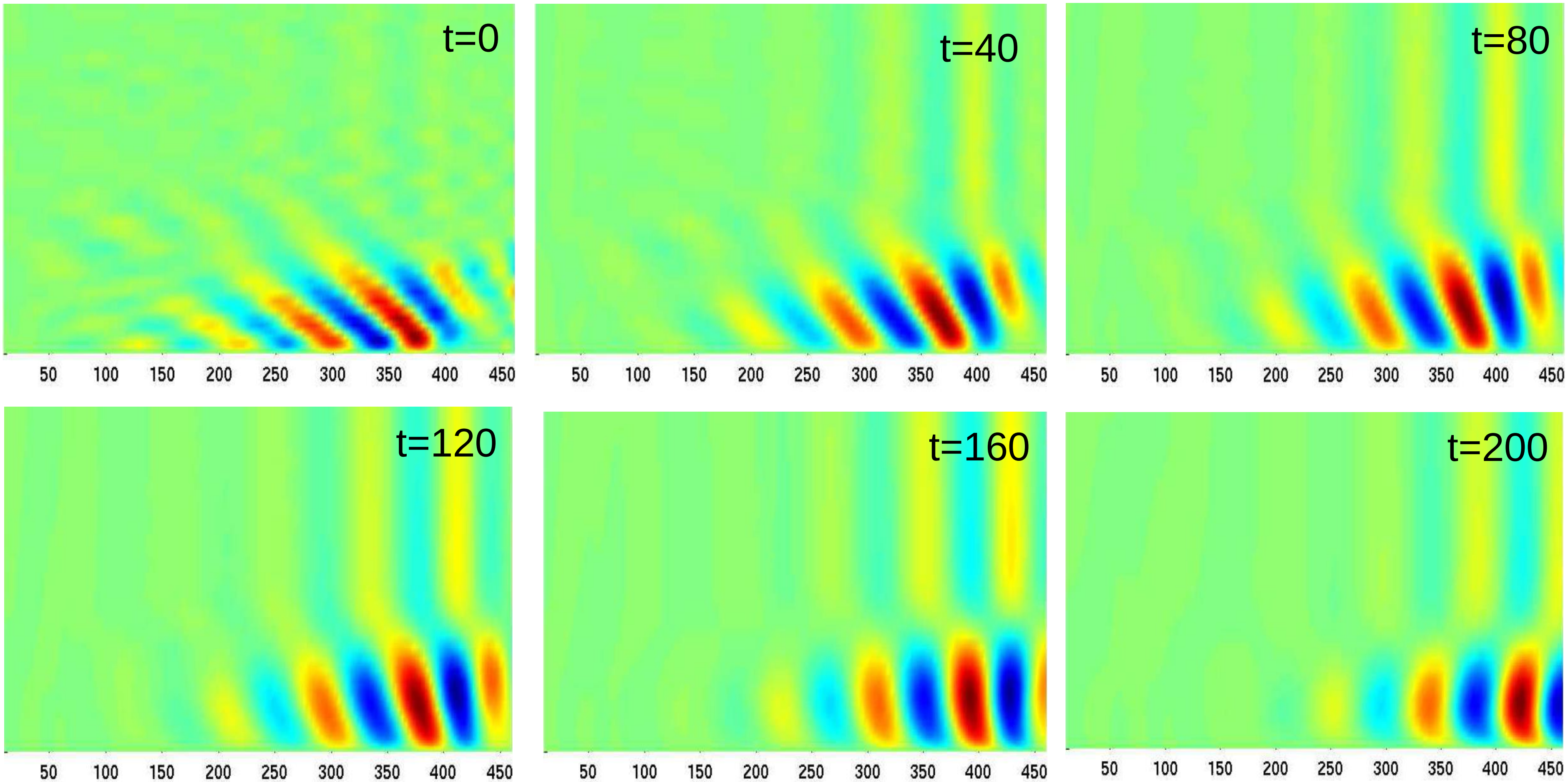


Amplification $G(t)$ for Poiseuille flow with $\mathrm{Re} = 1000, \alpha = 1$ (solid line)

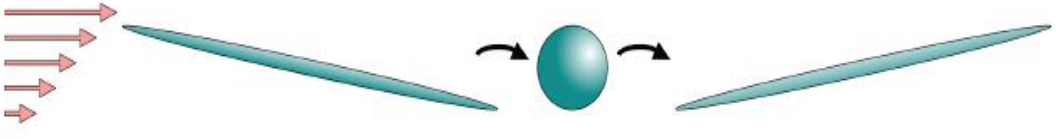
« By-pass transition »

Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

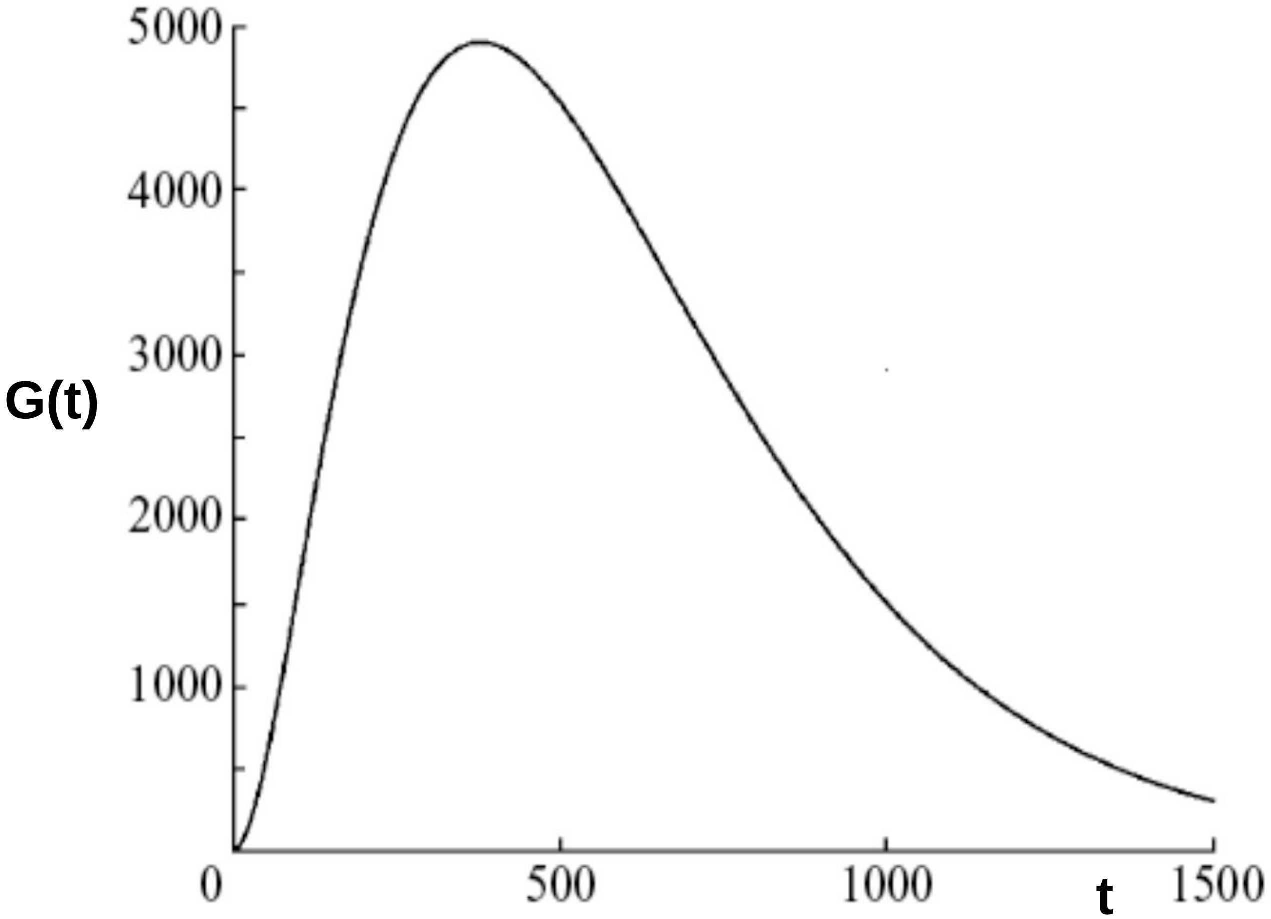
Optimal perturbation (here in boundary layer)



Orr mechanism



Plane Poiseuille flow
Lift-up mechanism (3D)



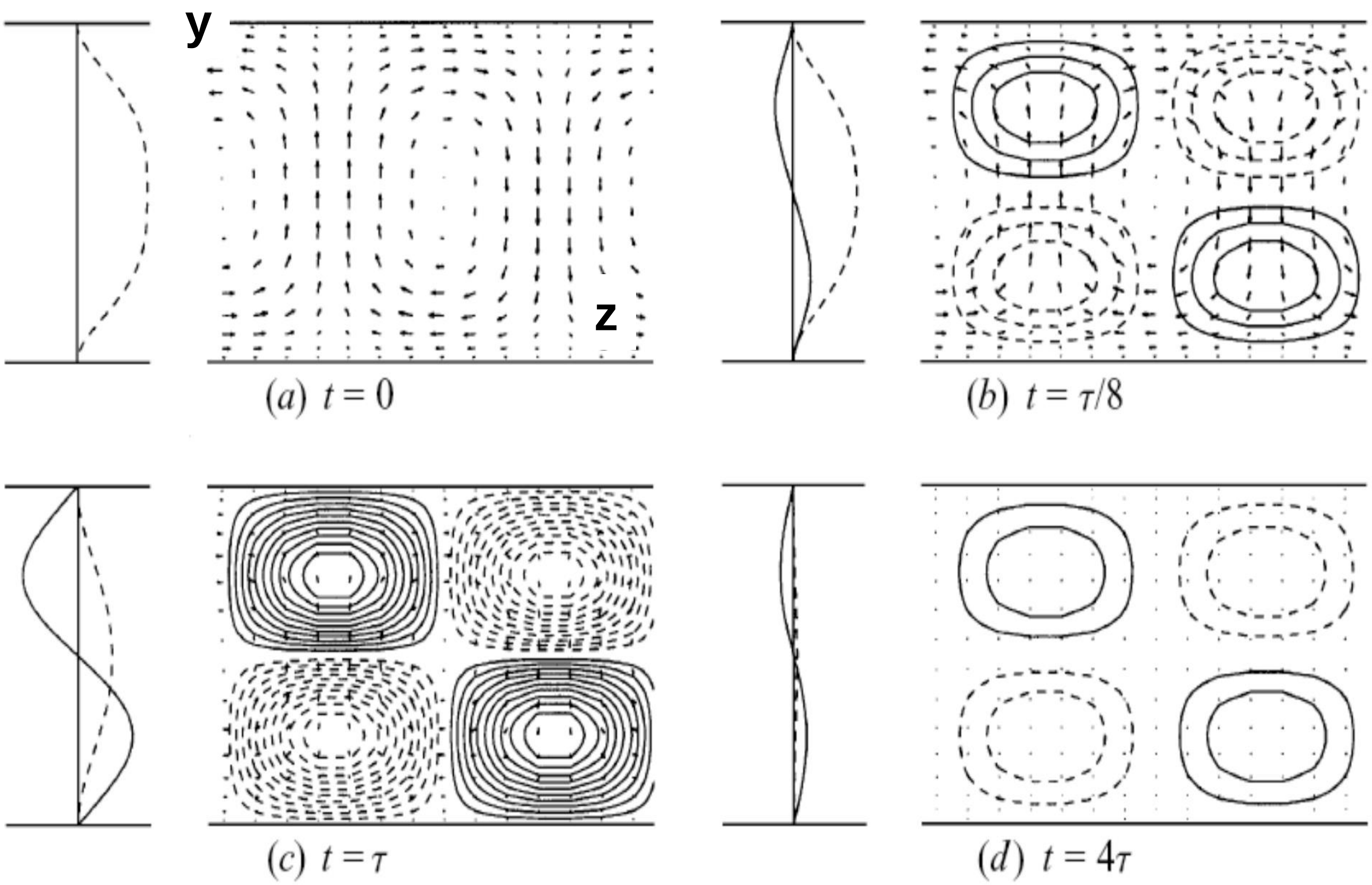
Re=5000, α=0, β=1

« By-pass transition »

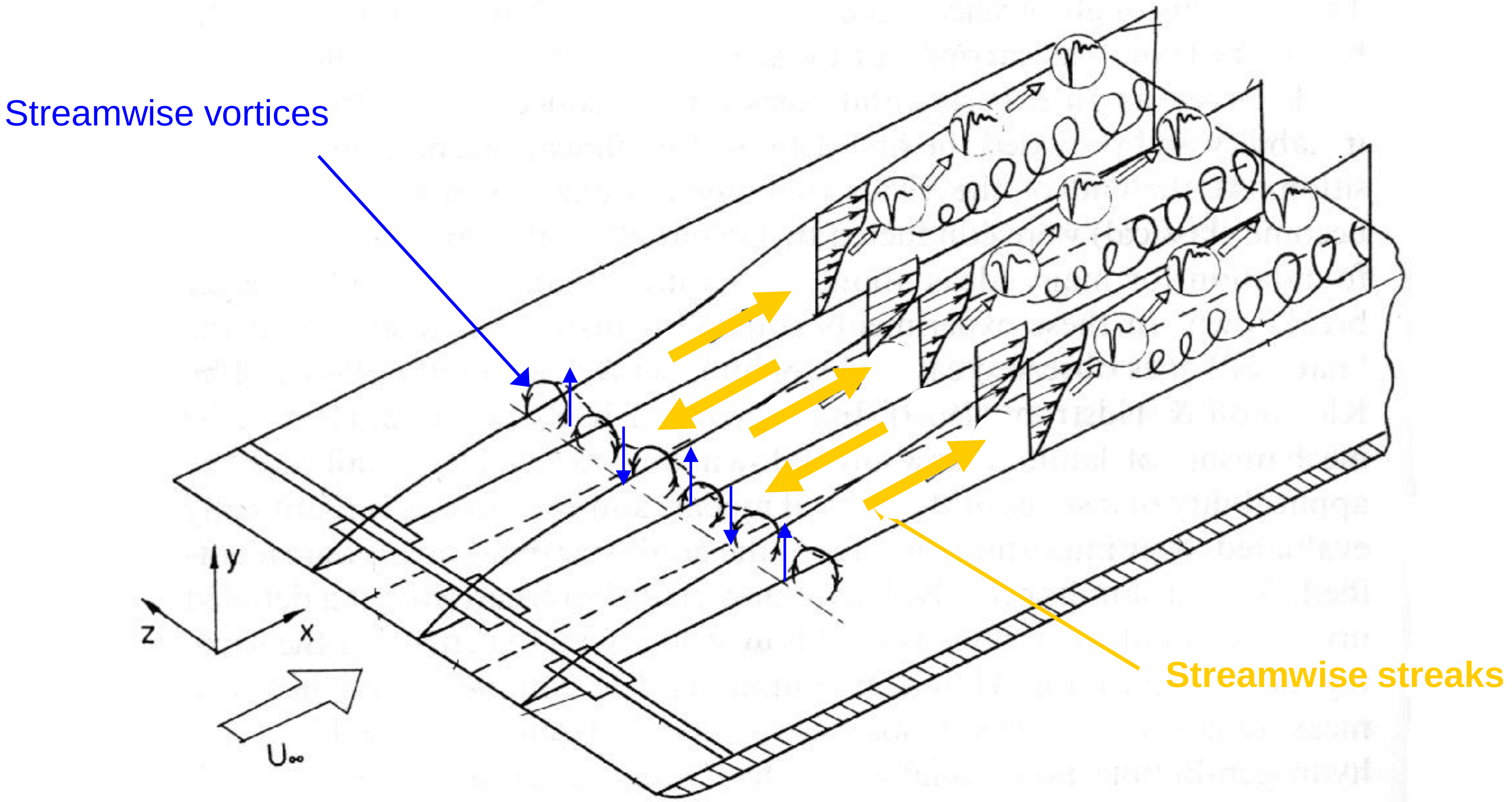
Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

Lift-up mechanisms

Optimal transformation of vortices into streaks



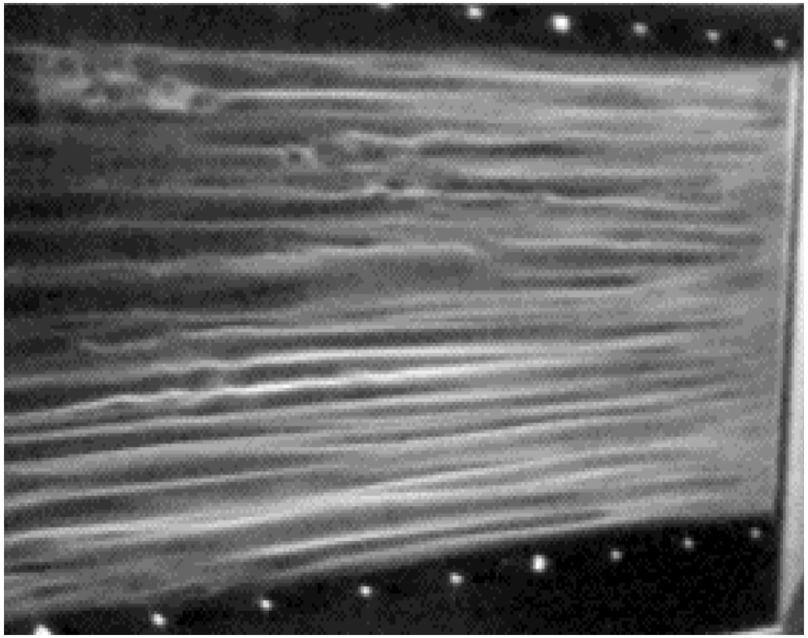
Lift-up mechanism



Schmid & Henningson.(2001)

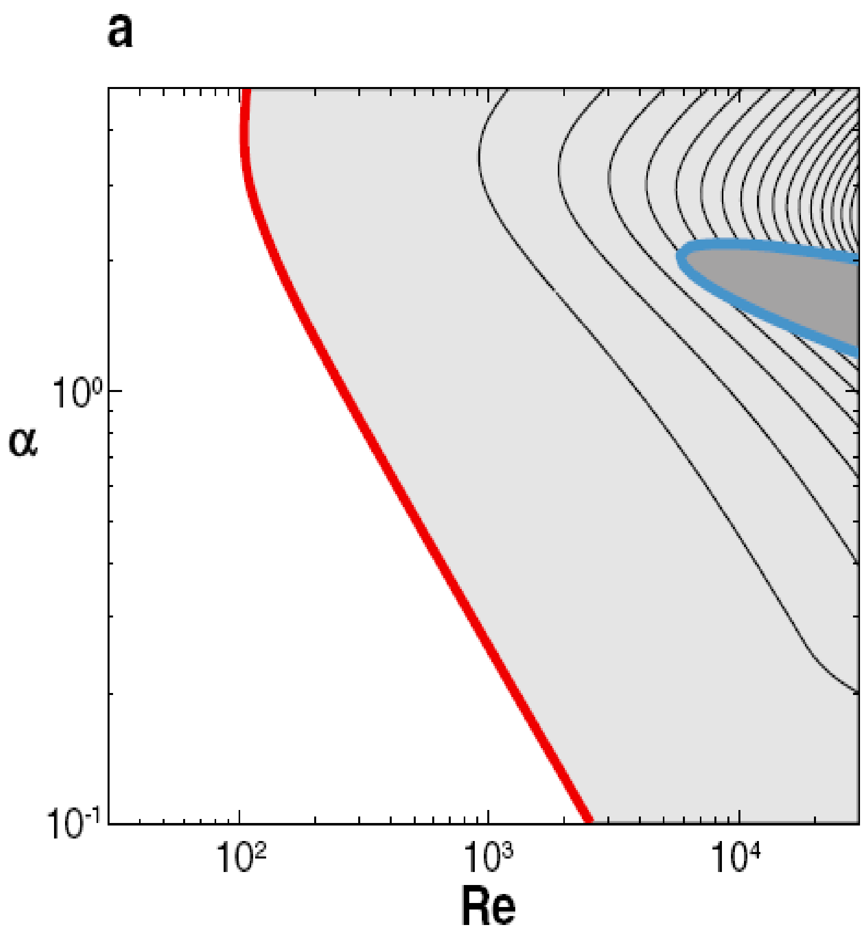
Lift-up mechanism

Optimal transformation of vortices into streaks



Alfredson & Matsubara (1996), streaky structures in the boudary layer

Isolines of maximum transient amplification
β=0



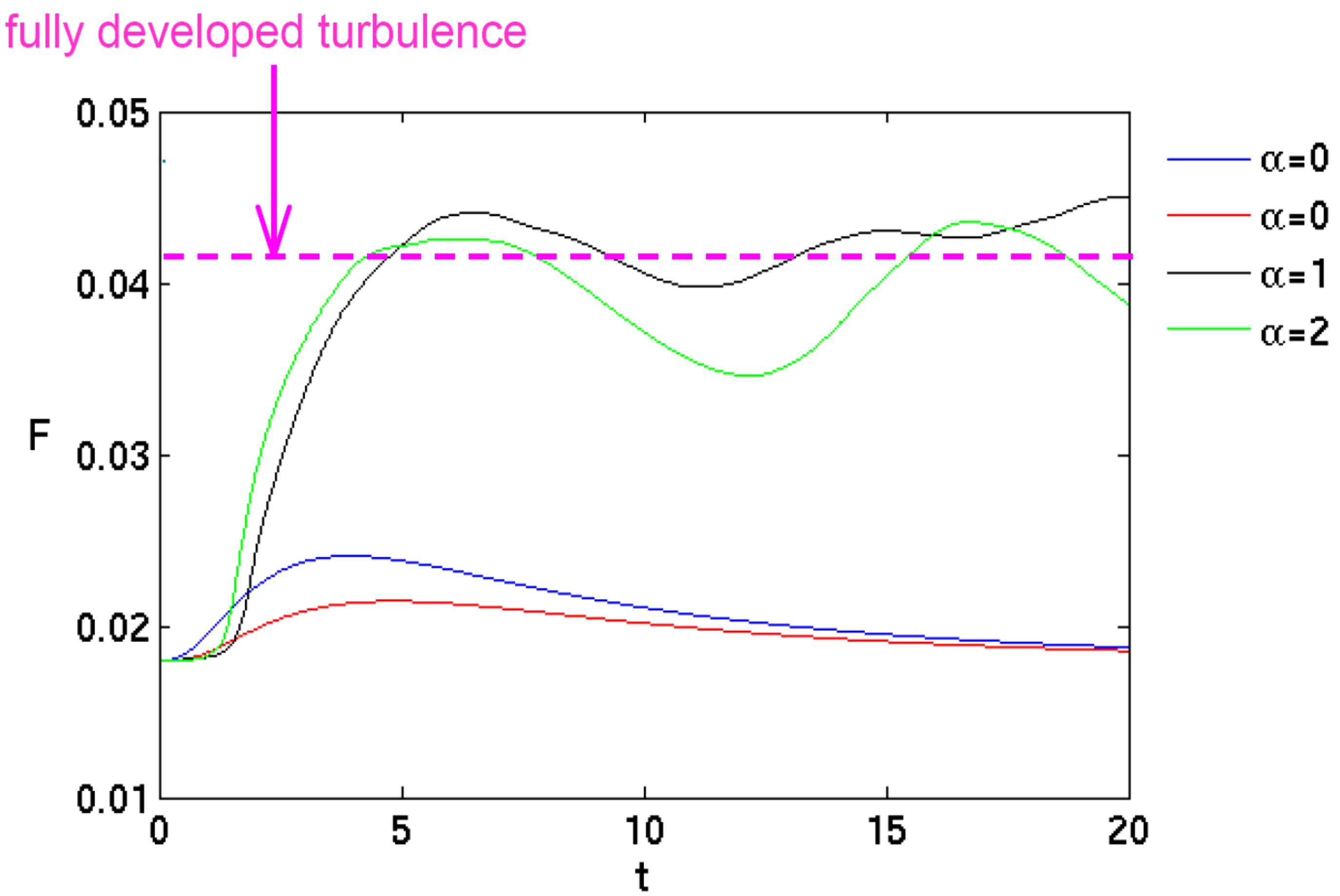
« By-pass transition »

Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

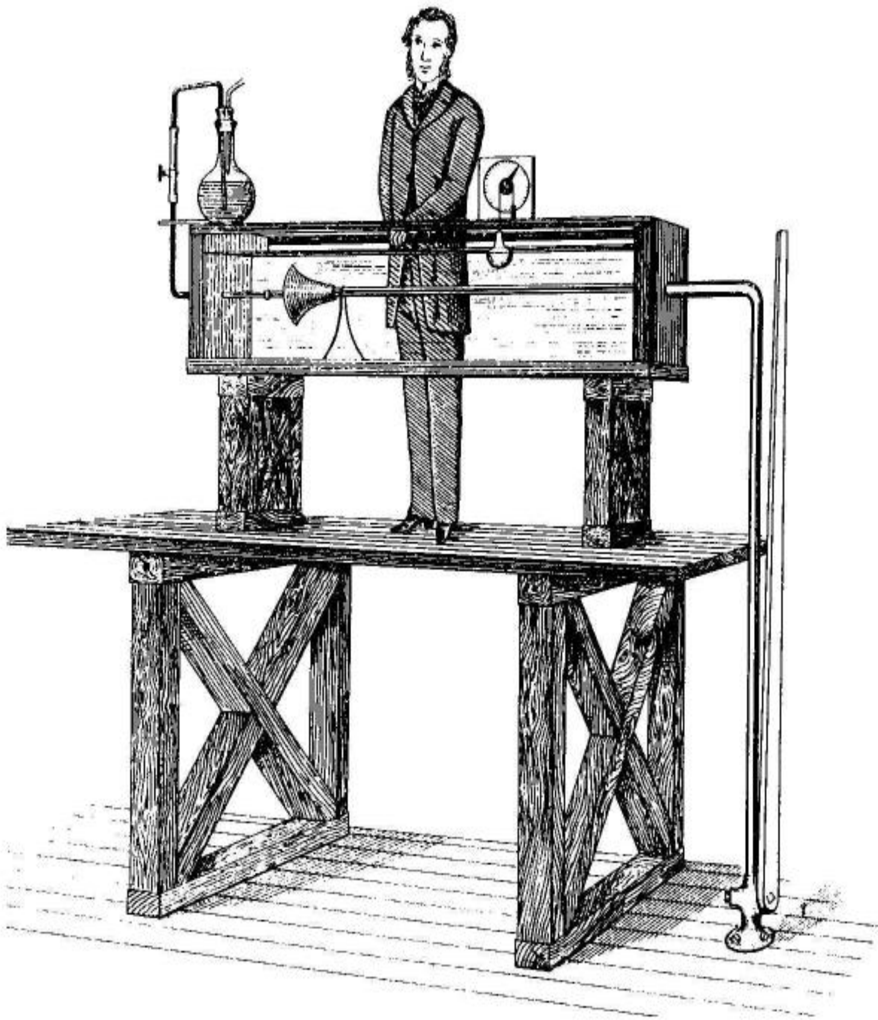
	$G_{\max}$ (10^{-3})	t_{max}	α	β
Plane Poiseuille	0.20 Re^2	0.076 Re	0	2.04
Couette	1.18 Re^2	0.117 Re	35/Re	1.6
Pipe	0.07 Re^2	0.048 Re	0	1
Boundary layer	1.50 Re^2	0.778 Re	0	0.65

Lift up mechanism in a square duct in the stable regime

Bye-pass transition to turbulence

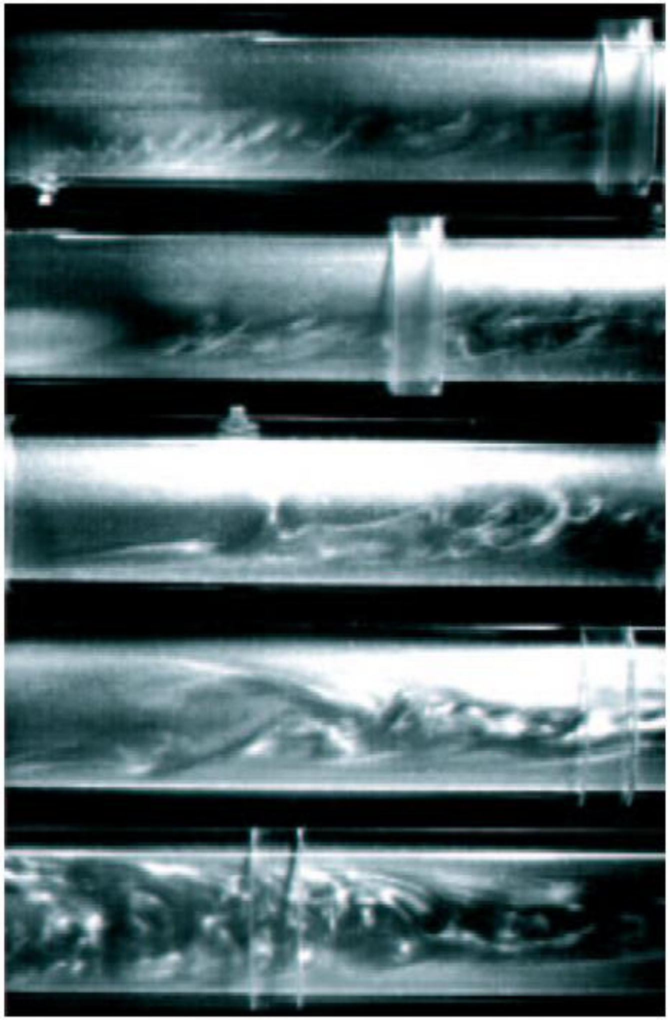


Bottaro et al. (2008)



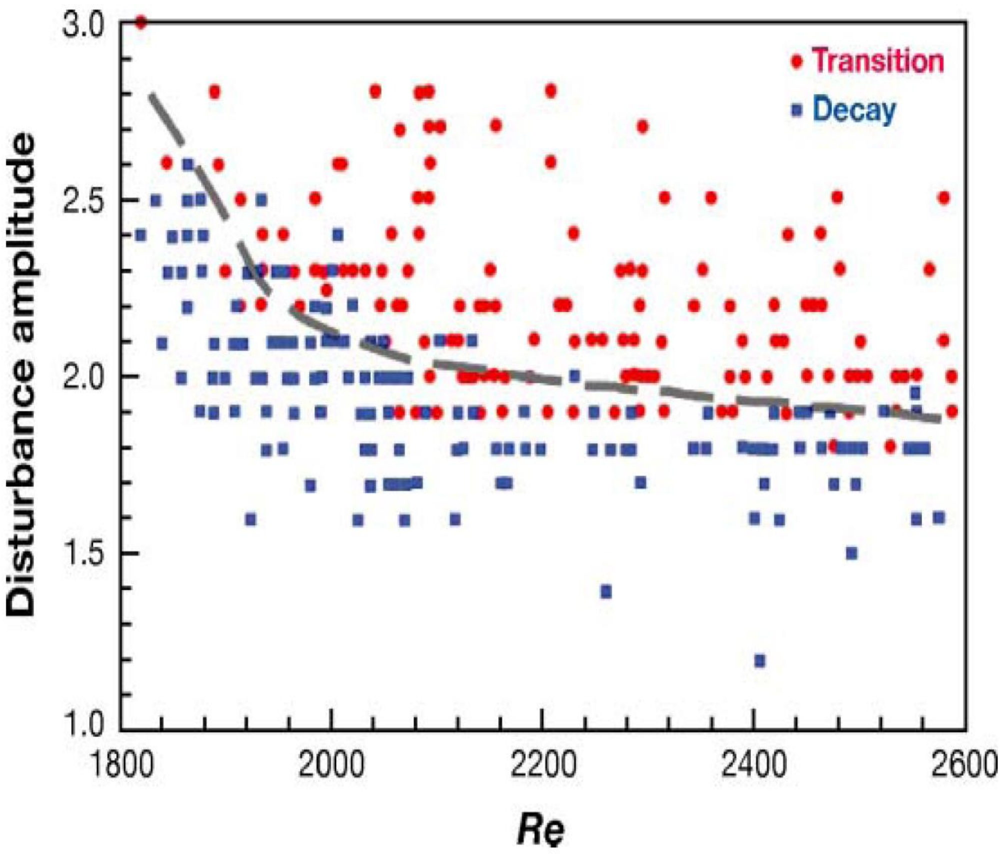
5.1 – *Osborne Reynolds en 1883 derrière son expérience à Manchester.*

Transition in Cylindrical Pipe Poiseuille flow



Mullin (2008)

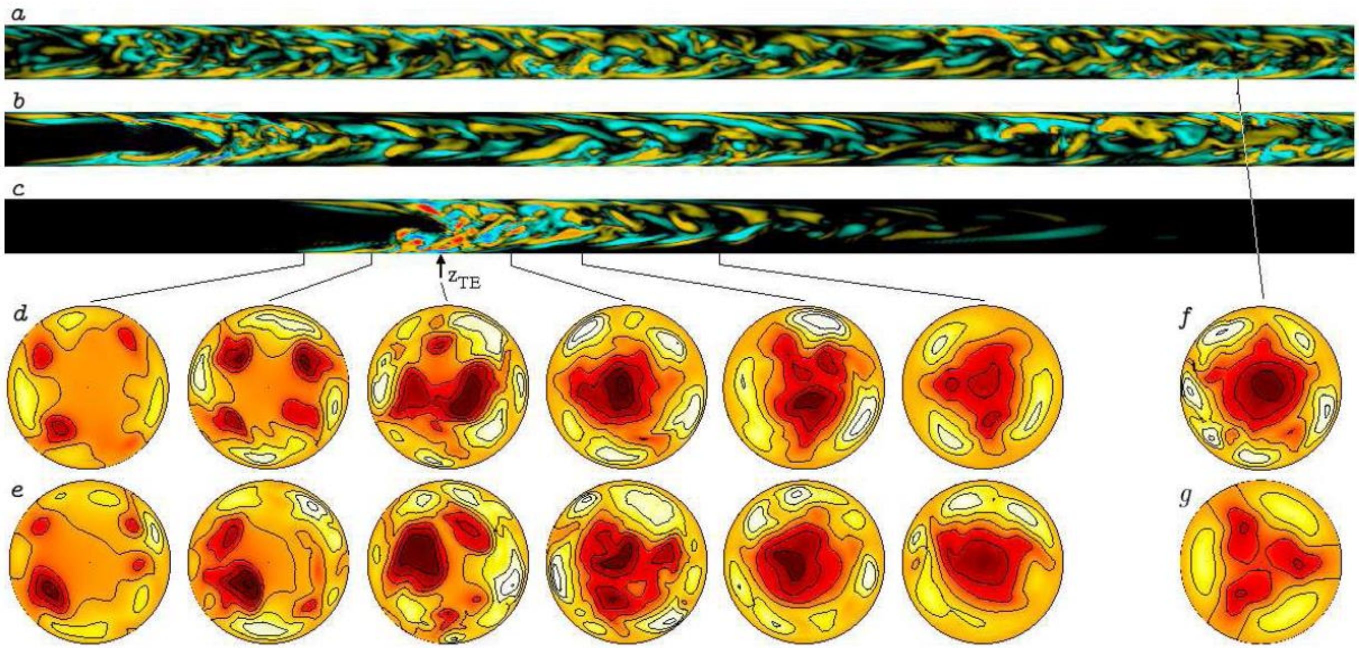
Transition in Cylindrical Pipe Poiseuille flow



Mullin (2008)

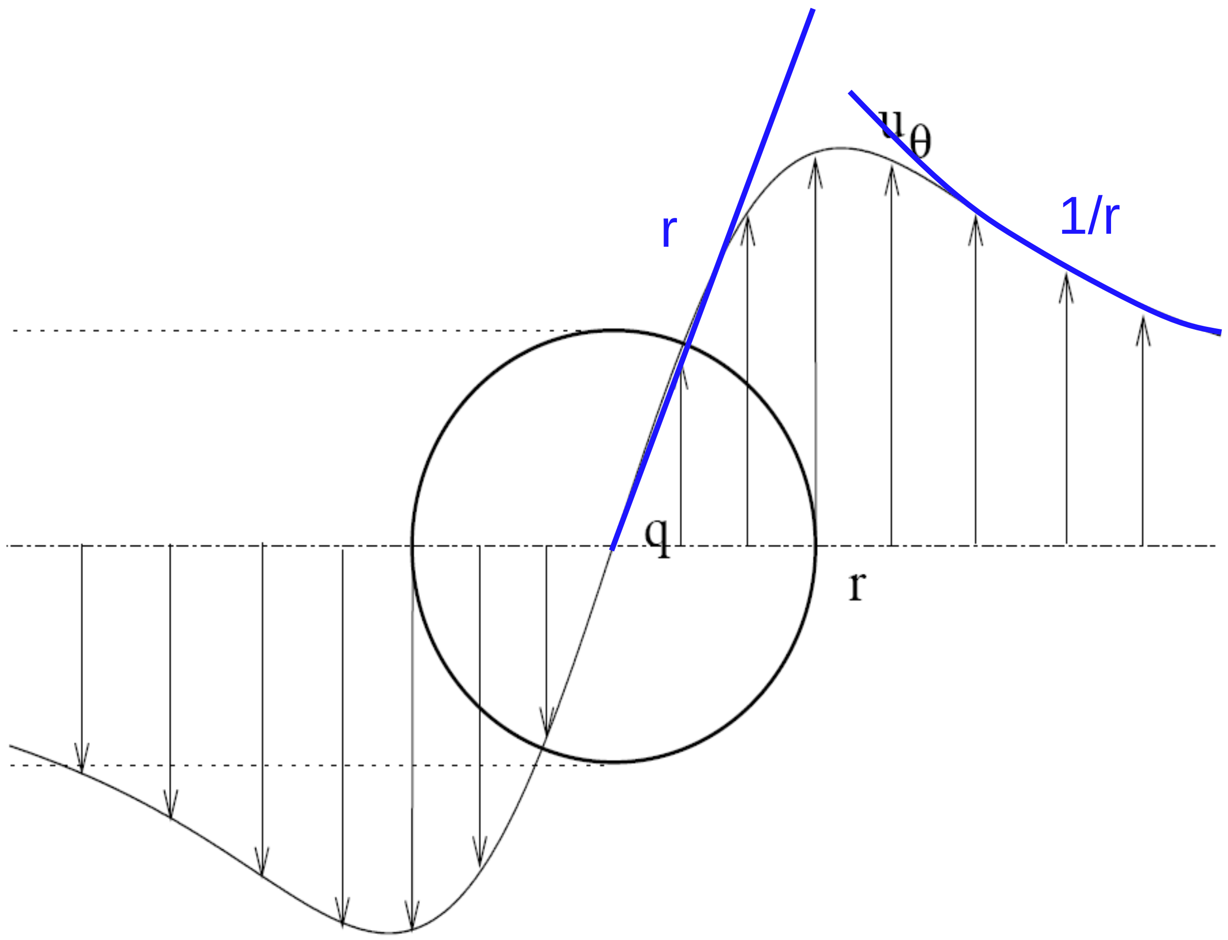
Transition in Cylindrical Pipe Poiseuille flow

Numerical study



Kerswell (2008)

Tourbillon de Lamb-Oseen



Stable lineairement!

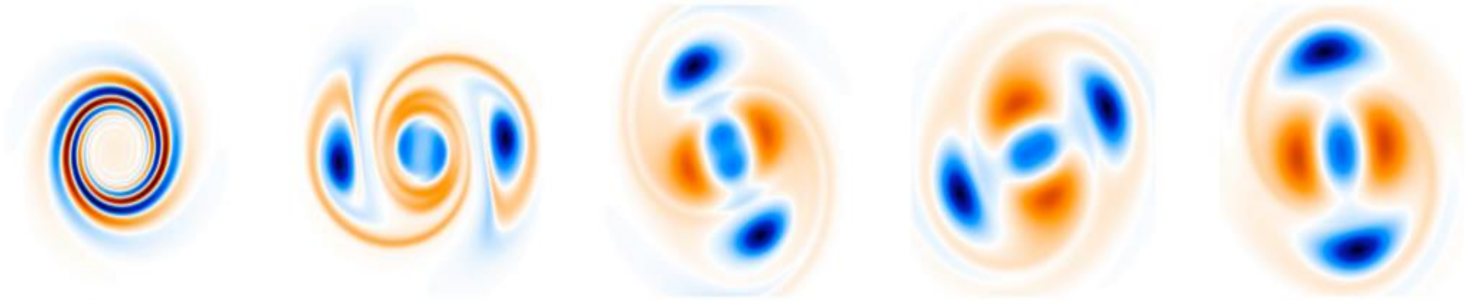
TOWARDS AN ELLIPTICAL STATE

Nonlinear evolution of the optimal perturbation with initial amplitude below a given threshold…

Re = 1000

TOWARDS AN ELLIPTICAL STATE

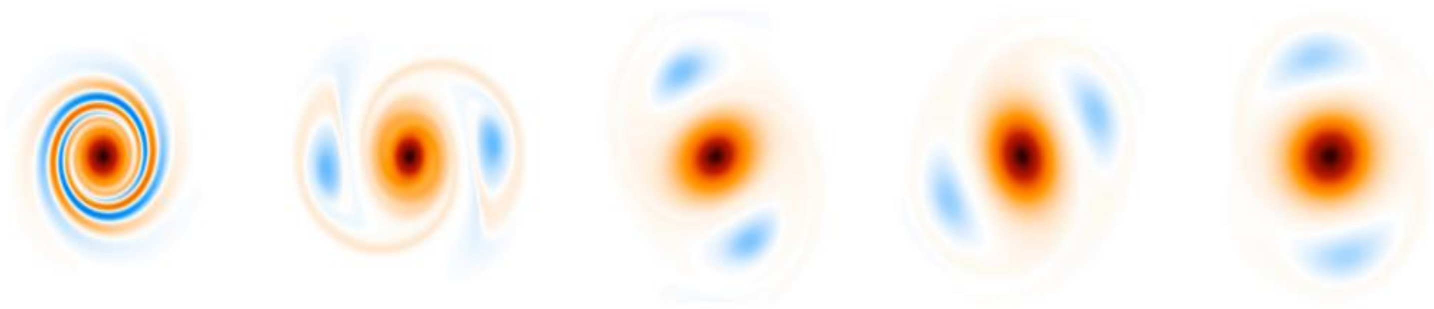
... and now *ABOVE* *the threshold*



Re = 1000

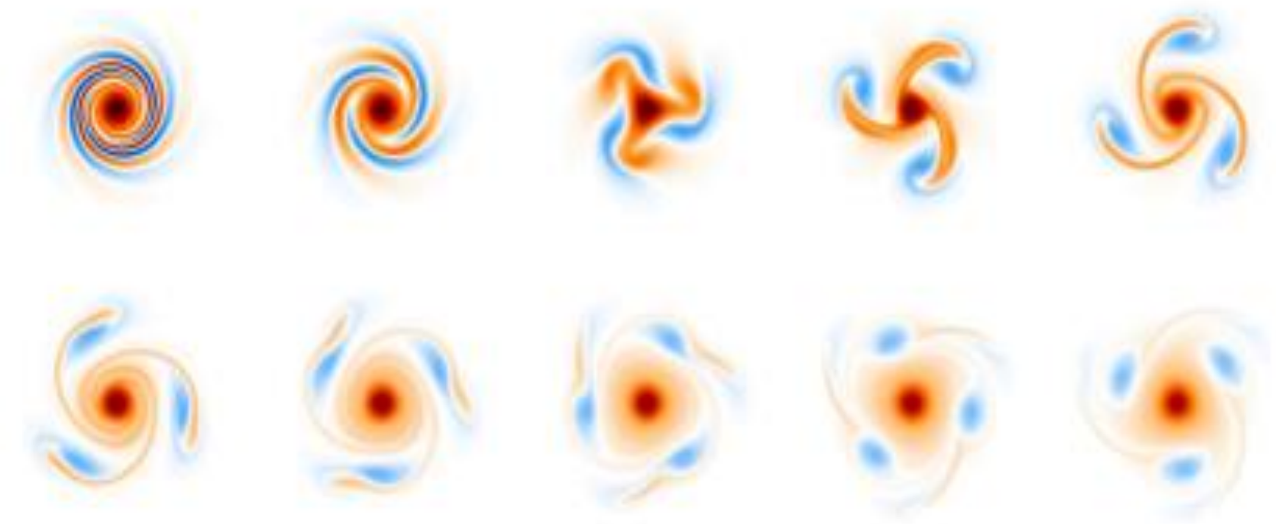
TOWARDS AN ELLIPTICAL STATE

Reconstructed flow

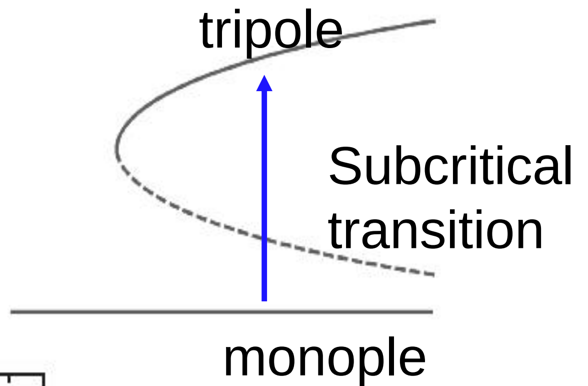


Re = 1000

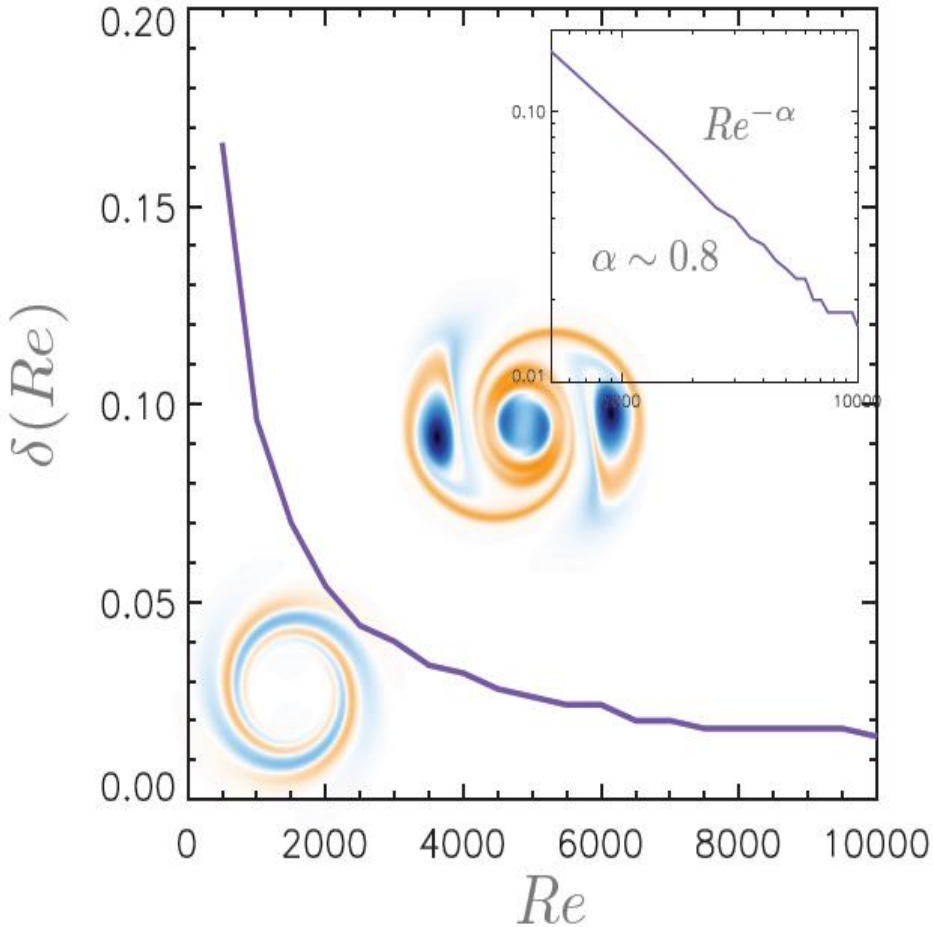
Tripolar vortices?



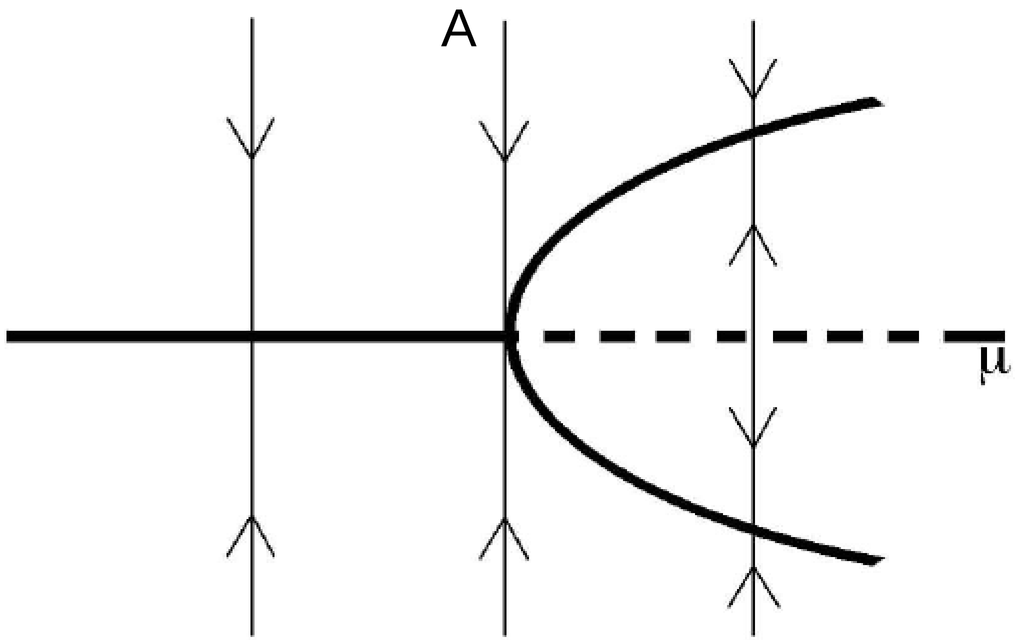
BASIN OF ATTRACTION'S SHRINKAGE



Lamb-Oseen vortex' bifurcation diagram

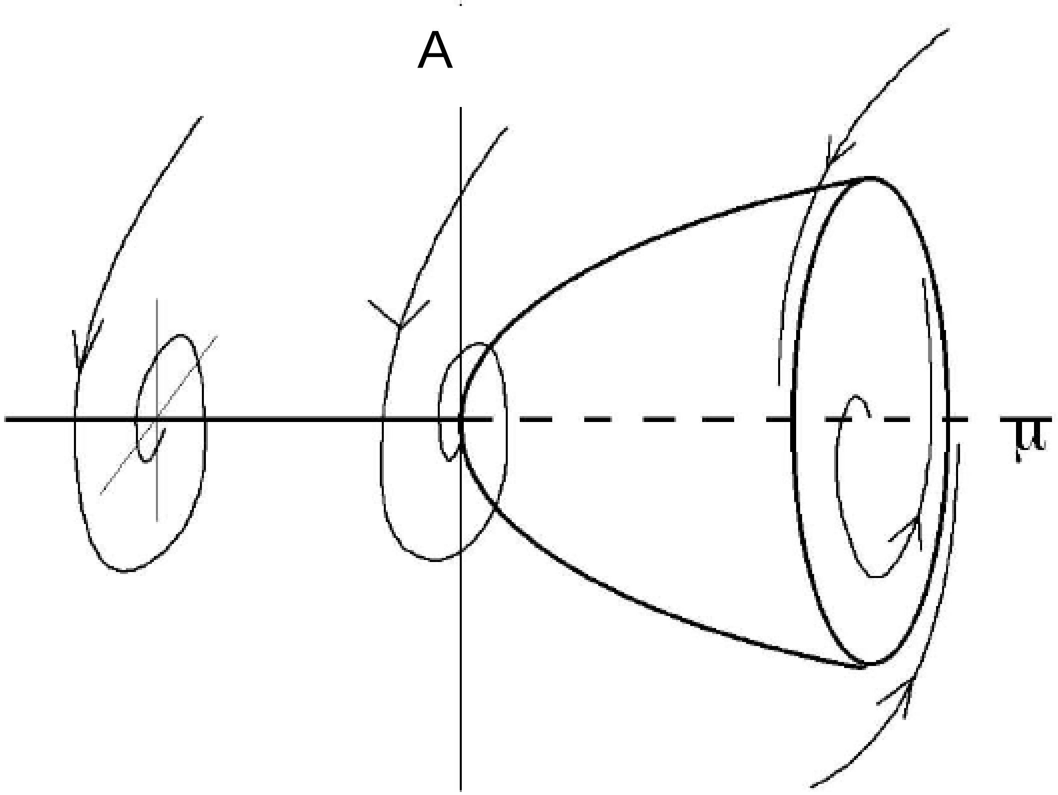


What about nonlinearities?



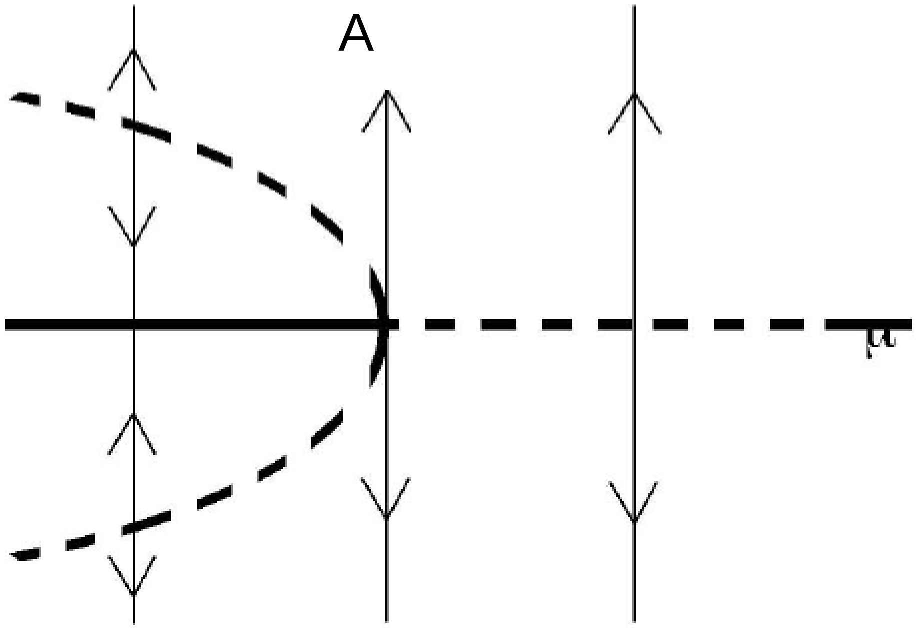
Supercritical fork bifurcation

What about nonlinearities?



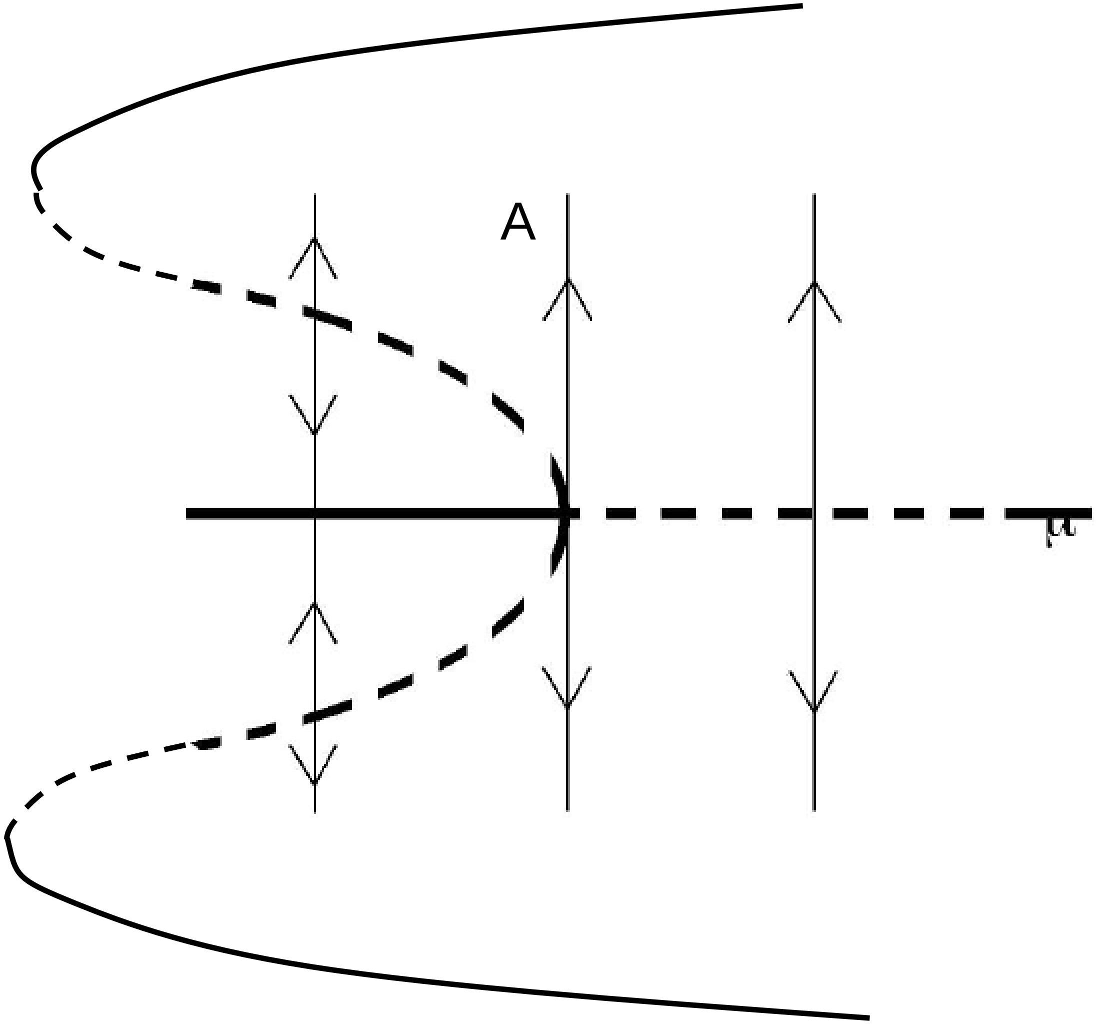
Hopf bifurcation

What about nonlinearities?



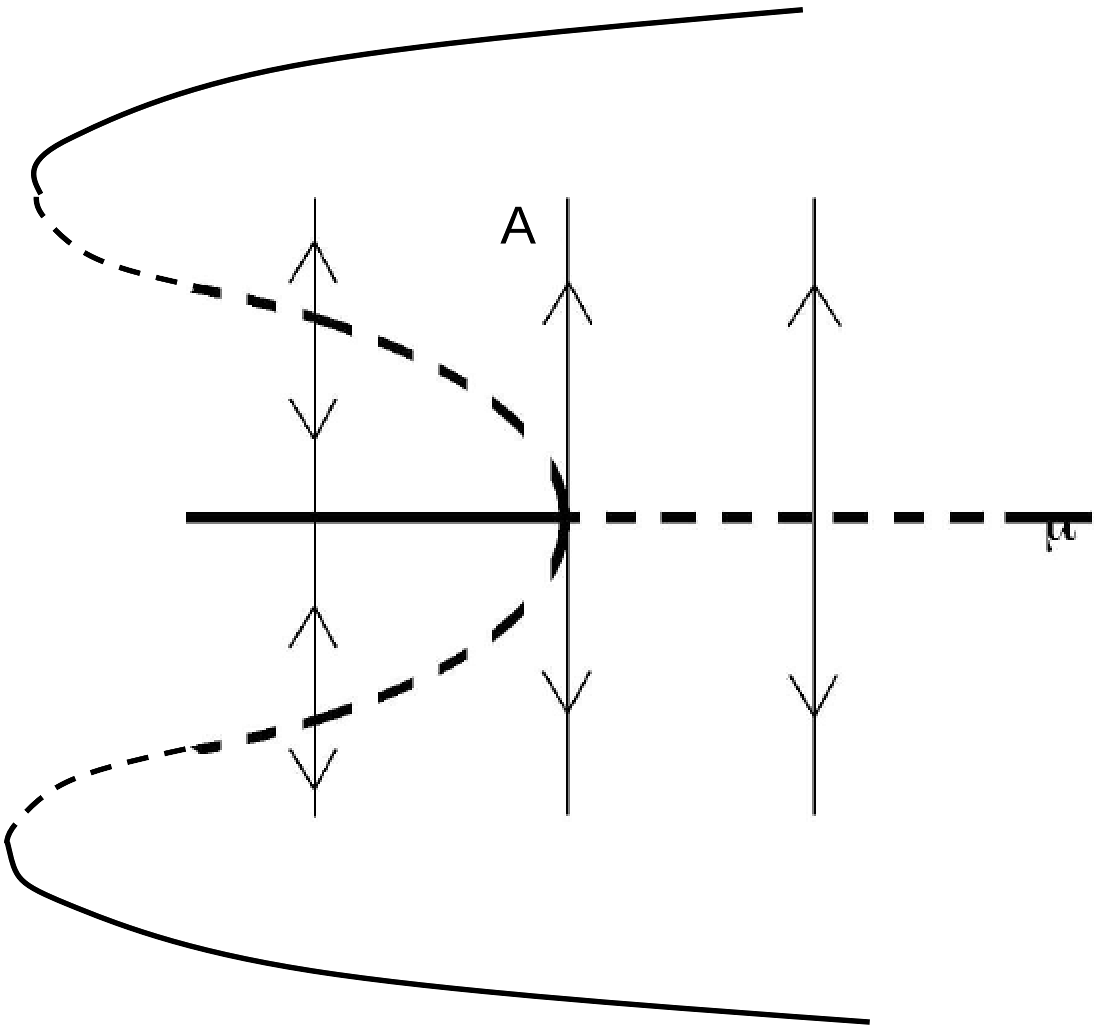
Subcritical fork bifurcation

What about nonlinearities?



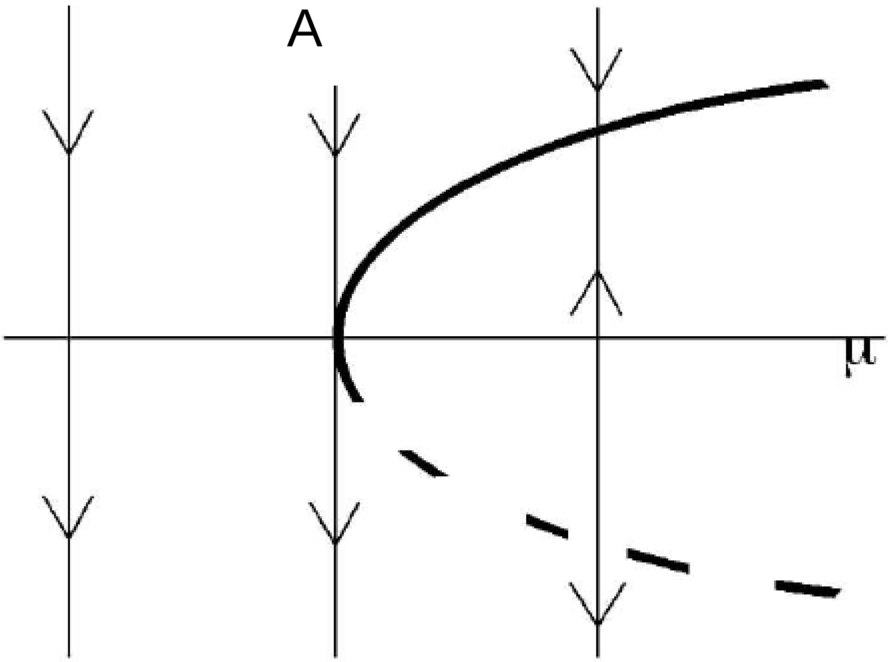
Subcritical bifurcation

What about nonlinearities?



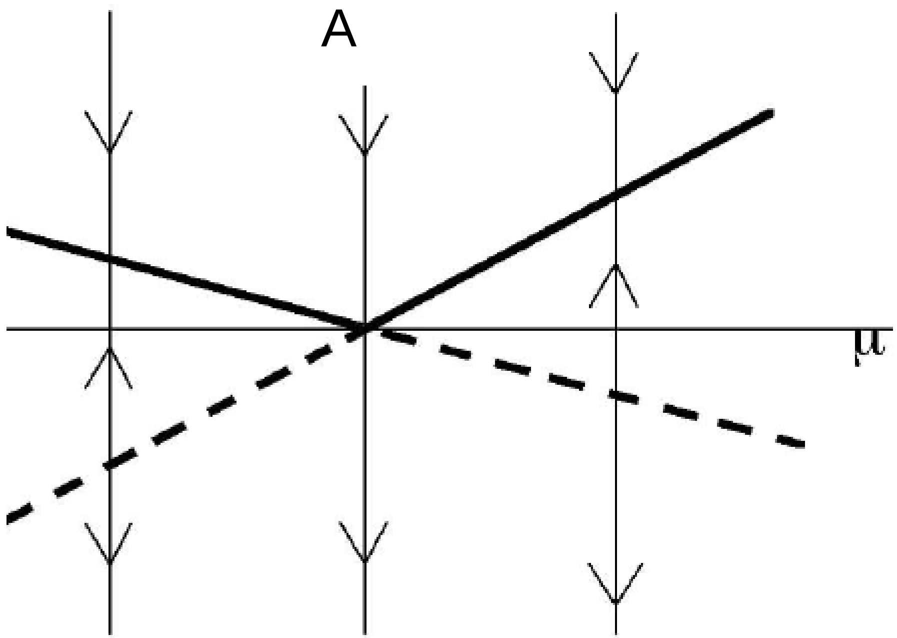
Hysteresis cycle!

What about nonlinearities?



Saddle Node bifurcation

What about nonlinearities?



Transcritical bifurcation