Formulaire du cours
Hydrodynamique

27 avril 2009

Equations de Navier-Stokes

Equations de Navier-Stokes pour un fluide incompressible :

$$\begin{aligned}
\operatorname{div}\mathbf{v} &= 0\\
\rho\frac{D\mathbf{v}}{Dt} &= -\mathbf{grad}\,p + \mu\,\Delta\mathbf{v} + \rho\mathbf{f}
\end{aligned}$$

Conditions aux limites à une surface libre :

$$\sigma_{ij\,liquide} n_j = \sigma_{ij\,gaz} n_j$$

Dynamique du tourbillon

Equation dynamique du tourbillon :

$$\frac{D}{Dt}\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\mathbf{grad}\,\mathbf{v} + \nu\Delta\,\boldsymbol{\omega}$$

Lemme 1 :

$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\omega}\times\mathbf{v} + \mathbf{grad}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)$$

Lemme 2 :

$$\mathbf{rot}\,\mathbf{a} = \frac{D}{Dt}\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{grad})\,\mathbf{v}$$

Lemme 3 :

$$\mathbf{a} = -\mathbf{grad}\left(\frac{p}{\rho} + V\right) - \nu\,\mathbf{rot}\,\mathbf{rot}\mathbf{v}$$

Equation de la circulation :

$$\frac{d\Gamma}{dt} = -\oint_{C_t}\nu(\mathbf{rot}\,\mathbf{rot}\,\mathbf{v})\cdot d\mathbf{l}$$

avec : $\Gamma = \oint_{C_t} \mathbf{v} \cdot d\mathbf{l}$

Théorème de Bernoulli :

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{\mathbf{v} \cdot \mathbf{v}}{2} + V = F(t)$$

Vorticité sur une paroi plane

$$\mathbf{t} = p\mathbf{n} + \mu(\mathbf{n} \times \boldsymbol{\omega})$$

Flux diffusif du tourbillon :

$$\boldsymbol{\zeta} = -\mathbf{n} \cdot \nabla \boldsymbol{\omega}$$

Potentiel Complexe

$$F(z) = \Phi + i\Psi$$

$$\frac{dF}{dz} = w = u - iv$$

$$\boldsymbol{\Gamma} = \int_C w(z)dz = \Gamma + iQ$$

Transformation conforme

$$\zeta = \zeta(z)$$

$$w_\zeta = G'(\zeta) = \frac{w(z)}{\zeta'(z)}$$

Transformation de Joukowski :

$$\zeta = z + \frac{a^2}{z}$$

Plan portant

identité :

$$\frac{1}{1-e^{-2i\theta}} = \frac{1-e^{2i\theta}}{4\sin^2\theta}$$

$$w_\zeta|_{\mathrm{plan}} = U_\infty \cos\alpha + \frac{U_\infty}{2\sin\theta}\left(\frac{\Gamma}{2\pi a U_\infty} - 2\sin\alpha\cos\theta\right)$$

$$F_N = \int_{-2a}^{2a} (p_i - p_e)\, d\xi$$

Théorie de la couche limite

$$\frac{\delta}{L} = O(\mathrm{Re}^{-1/2})$$

$$v \sim U\mathrm{Re}^{-1/2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x}(x,0,t) + \nu\frac{\partial^2 u}{\partial y^2}$$

Couche limite sur une plaque plane

$$s = y/\delta(x)$$

$$\psi = (U_\infty \nu x)^{1/2} f(s)$$

$$ff'' + 2f''' = 0$$

Équation intégrale de von Karman

$$\int_0^{h(x)} \frac{\partial}{\partial x}(u^2)dy = \frac{d}{dx}\int_0^{h(x)} u^2 dy - u^2(h(x))\frac{dh}{dx}$$

$$\frac{d\theta}{dx} = \frac{\tau_p}{\rho U_e^2} - \frac{1}{U_e}\frac{dU_e}{dx}(2\theta + \delta^*)$$

Théorie de l'instabilité

$$\psi' = F(y)e^{i\alpha(x-ct)}$$

$$p' = f(y)e^{i\alpha(x-ct)}$$

Équation d'Orr-Sommerfeld :

$$(\frac{d^2}{dy^2} - \alpha^2)^2 F = i\alpha \mathrm{Re}\,[(U-c)(\frac{d^2}{dy^2} - \alpha^2)F - F\,U'']$$

Turbulence

$$\overline{\mathbf{v}} = \frac{1}{T}\int_{t_0}^{t_0+T} \mathbf{v}dt$$

Équation stationnaire de Navier-Stokes moyennée :

$$\begin{aligned} \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} &= \frac{1}{\rho}\,\mathrm{div}(\overline{\boldsymbol{\sigma}} + \mathbf{R}) \\ \mathbf{R} &= -\rho\overline{\mathbf{v}' \otimes \mathbf{v}'} \end{aligned}$$

fermeture linéaire :

$$\mathbf{R} = -\frac{2}{3}\rho k\mathbf{I} + 2\mu_T \mathbf{d}$$

Equations de Navier-Stokes en coordonnées cylindriques

Équation de conservation de la masse

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Équations du mouvement

$$\begin{aligned}
\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}) &= -\frac{\partial p}{\partial r} + \mu(\triangle v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}) + \rho f_r \\
\rho(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}) &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\triangle v_\theta - \frac{v_\theta}{r^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}) + \rho f_\theta \\
\rho(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}) &= -\frac{\partial p}{\partial z} + \mu\triangle v_z + \rho f_z
\end{aligned}$$

Laplacien d'un champ scalaire $f(r,\theta,z)$

$$\triangle f = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Equations de Navier-Stokes en coordonnées sphériques

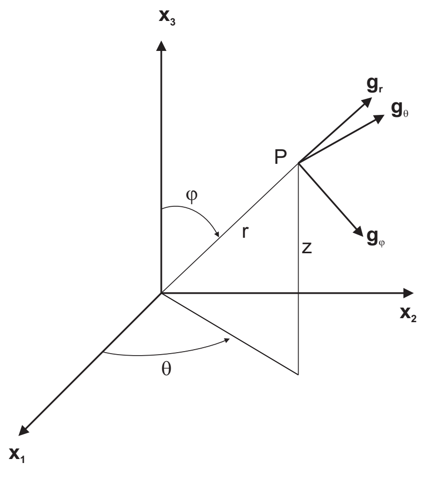

FIGURE 1 – Système de coordonnées sphériques

Équation de conservation de la masse

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,v_r) + \frac{1}{r\sin\varphi}\frac{\partial}{\partial \varphi}(v_\varphi \sin\varphi) + \frac{1}{r\sin\varphi}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{1}$$

Équations du mouvement

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2+v_\varphi^2}{r}\right)=-\frac{\partial p}{\partial r}$$
$$+\mu\left(\Delta v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}-\frac{2v_\varphi\cot\varphi}{r^2}-\frac{2}{r^2\sin\varphi}\frac{\partial v_\theta}{\partial\theta}\right)+\rho f_r \tag{2}$$
$$\rho\left(\frac{Dv_\varphi}{Dt}+\frac{v_r v_\varphi-v_\theta^2\cot\varphi}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\varphi}$$
$$+\mu\left(\Delta v_\varphi-\frac{v_\varphi}{r^2\sin^2\varphi}+\frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}-\frac{2\cos\varphi}{r^2\sin^2\varphi}\frac{\partial v_\theta}{\partial\theta}\right)+\rho f_\varphi \tag{3}$$
$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta+v_\theta v_\varphi\cot\varphi}{r}\right)=-\frac{1}{r\sin\varphi}\frac{\partial p}{\partial\theta}$$
$$+\mu\left(\Delta v_\theta-\frac{v_\theta}{r^2\sin^2\varphi}+\frac{2}{r^2\sin\varphi}\frac{\partial v_r}{\partial\theta}+\frac{2\cos\varphi}{r^2\sin^2\varphi}\frac{\partial v_\varphi}{\partial\theta}\right)+\rho f_\theta \tag{4}$$

avec l'opérateur laplacien défini par

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2} \tag{5}$$

et la dérivée matérielle telle que

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\varphi}{r}\frac{\partial}{\partial\varphi}+\frac{v_\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} \tag{6}$$

Les composantes du tenseur des taux de déformation sont

$$d_{rr} = \frac{\partial v_r}{\partial r},\quad d_{\varphi\varphi}=\frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_r}{r}, \tag{7}$$
$$d_{\theta\theta} = \frac{1}{r\sin\varphi}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{v_\varphi\cot\varphi}{r}, \tag{8}$$
$$d_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{r\sin\varphi}\frac{\partial v_\varphi}{\partial\theta}+\frac{1}{r}\frac{\partial v_\theta}{\partial\varphi}-\frac{v_\theta\cot\varphi}{r}\right), \tag{9}$$
$$d_{\theta r} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right), \tag{10}$$
$$d_{r\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial\varphi}+\frac{\partial v_\varphi}{\partial r}-\frac{v_\varphi}{r}\right) \tag{11}$$

Le rotationnel d'un vecteur $\boldsymbol{v}$ s'écrit

$$\begin{aligned}\boldsymbol{rot\,v} &= \frac{1}{r\sin\varphi}\left(\frac{\partial(v_\theta\sin\varphi)}{\partial\varphi}-\frac{\partial v_\varphi}{\partial\theta}\right)\boldsymbol{e}_r+\left(\frac{1}{r\sin\varphi}\frac{\partial v_r}{\partial\theta}-\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)\boldsymbol{e}_\varphi\\ &+ \frac{1}{r}\left(\frac{\partial(r\,v_\varphi)}{\partial r}-\frac{\partial v_r}{\partial\varphi}\right)\boldsymbol{e}_\theta\end{aligned} \tag{12}$$