

Three-dimensional Rayleigh–Taylor instability under the pole of a unidirectional curved substrate.

ME-466 Instability

A thin viscous film of initial average thickness H_i^* coats the inside of a cylinder of inner radius R whose axis is orthogonal to gravity (fig. 1). The film drains under the effect of gravitational acceleration g . The liquid's density, kinematic viscosity and surface tension are denoted by ρ , ν , and γ , respectively.

The lubrication equation can be expanded near the pole of the cylinder as

$$\frac{\partial H}{\partial t} + \nabla \cdot \left[\underbrace{\frac{H^3}{3}}_I \nabla \left(\underbrace{\frac{1}{\text{Bo}} \nabla^2 H}_{II} + \underbrace{H}_{III} + \underbrace{\frac{x^2}{2}}_{IV} \right) \right] = 0, \quad (1)$$

with $\nabla = (\partial/\partial x, \partial/\partial y)^T$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator, and the Bond number $\text{Bo} = \rho g H_i^* R / \gamma$. Here, the thickness is rescaled by the initial film thickness H_i^* , while the polar and axial coordinates x and y are rescaled by a typical length scale $\sqrt{R H_i^*}$.

1. Remind the hypotheses behind the lubrication equation.
2. Can you explain the physical meaning of the different underlined terms in equation (1)? Compare with previous exercises in lubrication theory. We have not encountered terms similar to IV in this course yet; what effect could it represent (guess)?
3. What does the Bond number measure ?
4. Find the basic flow $H_b(t)$, which is supposed invariant with respect to x and y , and with initial condition $H_b(t=0) = 1$,

$$H_b = T^{-1/2}. \quad (2)$$

Show that $T = 1 + 2t/3$.

5. Linearise eq. (1) around the basic flow, using the following perturbation ansatz,

$$H(x, y, T) = H_b(T) + \epsilon h(x, y, T) \quad (3)$$

and find

$$\frac{\partial h}{\partial T} + \frac{1}{2T^{3/2}} \left(\frac{\nabla^4 h}{\text{Bo}} + \nabla^2 h \right) + \frac{3}{2T} \left(x \frac{\partial h}{\partial x} + h \right) = 0. \quad (4)$$

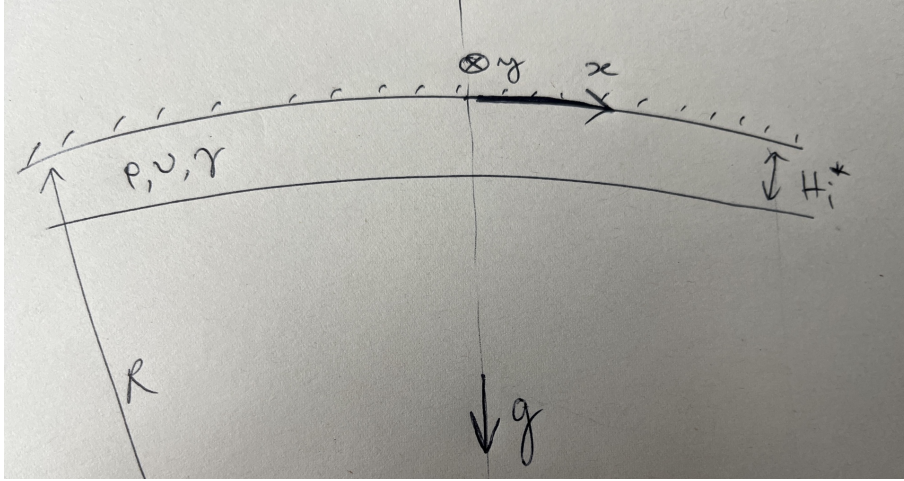


Figure 1: Cartoon of the problem at hand.

6. With respect to which variables can the perturbation h be decomposed in normal modes? Why?
7. Show that, by substituting the trial function

$$h(x, y, T) = A(T) \exp \left[i \left(\frac{\alpha x}{T^{3/2}} + \beta y \right) \right] + \text{c.c.} \quad (5)$$

into the linearised equation (4), one obtains an ordinary differential equation for $A(T)$,

$$\frac{dA}{dT} + \frac{1}{2} \left[\frac{1}{\text{Bo}} \left(\frac{\alpha^2}{T^{15/4}} + \frac{\beta^2}{T^{3/4}} \right)^2 - \left(\frac{\alpha^2}{T^{9/2}} + \frac{\beta^2}{T^{3/2}} \right) + \frac{3}{T} \right] A = 0. \quad (6)$$

((If this question is too calculatory for you, do the simplified case $\alpha = 0$.) Comment on the meaning of α and β . What do $\alpha = 0$ and $\beta = 0$ describe?

8. In a “frozen frame” mindset, if you suppose $A(T) = \exp[\sigma T]$, write the dispersion relation $\sigma(\alpha = 0, \beta)$ for different T . ((Here, σ will depend on T parametrically, but $d\sigma/dT$ is neglected!)) Is the “frozen” system stable or unstable (use a graphing calculator or Matlab)? Which terms are stabilising and which destabilising in this case? Is there an optimal wave number? Relate this to what you know about the Rayleigh–Taylor instability, is there a condition for the classical instability?
9. Find the solution to eq. (6) for $\alpha = 0$ with initial condition $A(T = 1) = 1$ ((calculatory)).
10. For small times t , meaning $T \sim 1$, the full solution can be expanded as

$$A(T \sim 1) \sim 1 + \frac{(\alpha^2 + \beta^2 - 3)\text{Bo} - (\alpha^2 + \beta^2)^2}{2\text{Bo}}(T - 1) + \mathcal{O}[(T - 1)^2]. \quad (7)$$

For which wave number pairs (α, β) is initial growth ensured? Is there a condition on Bo ? Which wave number pairs grow the fastest, initially?

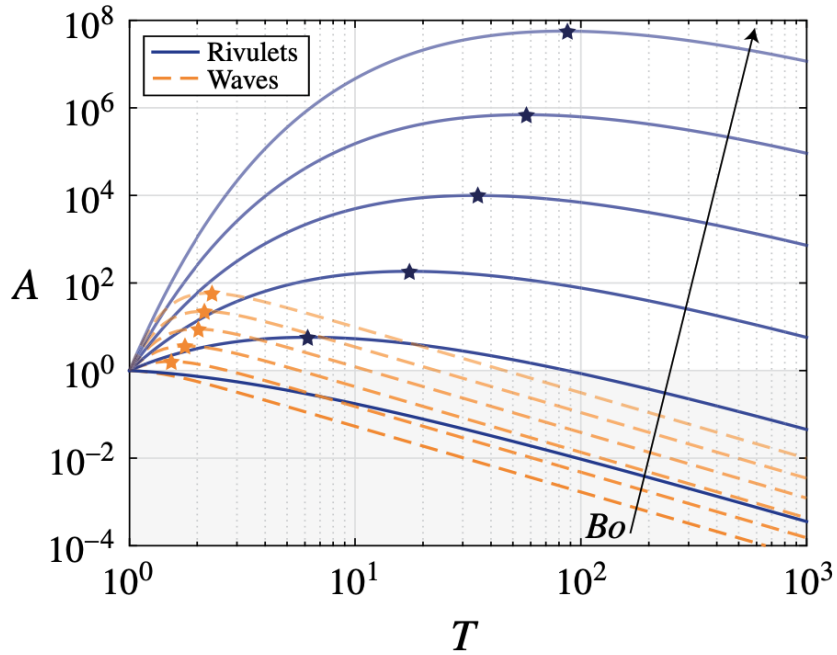


Figure 2: Amplitude A for waves ($\beta = 0$) and rivulets ($\alpha = 0$). They correspond to an initial perturbation with the optimal initial wave number, for Bond numbers $Bo = \{10, 30, 50, 70, 90, 110\}$. Stars denote the largest achieved amplitudes. Reprinted from Balestra et al. [2018].

11. The full solution to eq. (6) is presented in fig. 2. Comment on the amplification of waves ($\beta = 0$) and rivulets ($\alpha = 0$)? Could this result be used to predict the pattern that a generic, noisy initial condition would exhibit (see fig. 7 of Balestra et al. [2018] and discuss)? What do you notice about the long-time behaviour of $A(T)$?

References

- G. Balestra, N. Kofman, P.-T. Brun, B. Scheid, and F. Gallaire. Three-dimensional Rayleigh–Taylor instability under a unidirectional curved substrate. *Journal of Fluid Mechanics*, 837:19–47, 2018.