ME466- Instability

22^{nd} of January 2016

11h30-13h15

The objective of this exercise is to analyze the Bénard-Marangoni instability.

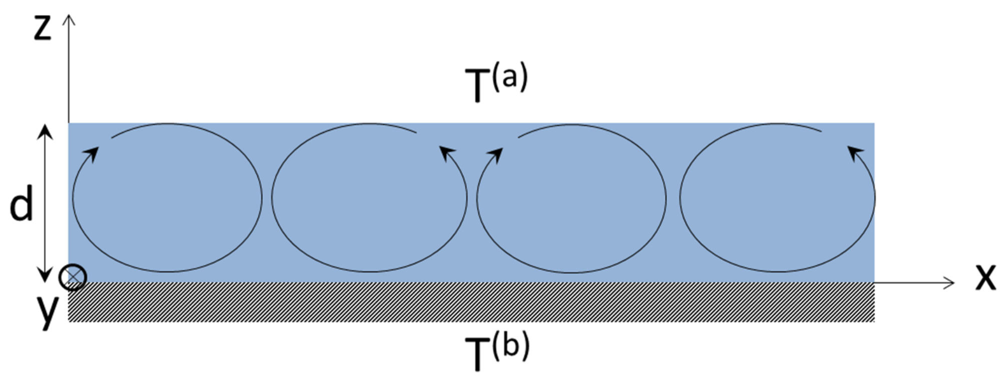

FIGURE 1 – Fluid layer heaten from below

We consider a layer of depth d of liquid of density ρ, kinematic viscosity ν, dynamic viscosity $\mu = \rho\nu$ in a gravitational field $\mathbf{g} = -ge_z$. We follow Boussinesq hypothesis and assume that ρ, μ (and therefore ν) are temperature indepedent in all internal stresses and accelerations. While the density is assumed to vary linearly with temperature $\rho = \rho^{(b)} - \alpha(T - T^{(b)})$ at first, surface tension and interface deformations are initially neglected. They are then progressively introduced. The base temperature profile (the base flow) is given by

$$T = T^{(b)} + \frac{T^{(a)} - T^{(b)}}{d} z, \tag{1}$$

where $z = 0$ is set at the bottom plate. The heat diffusivity of the fluid is designated by κ.

Part I : Instability in presence of a free surface

– 1. [1pts]
  What is the name of the instability of a layer of fluid heated from below and cooled from above?

– [2. 1pts]
  What is the physical mechanism that drives the instability?

– [3. 1pts]

Using the following nondimensionalization,

$$\begin{aligned} &\text{length} && d && (2)\\ &\text{time} && \frac{d^2}{\kappa} && (3)\\ &\text{pressure} && \frac{\rho\kappa^2}{d^2} && (4)\\ &\text{temperature} && \frac{\kappa\nu}{\alpha g d^3} && (5) \end{aligned}$$

what is the natural velocity scale ?

– [4. 1pts]
After some manipulations, we have shown in class that linearized equations around the base flow can be formulated for two unknowns only (the vertical velocity perturbation u'_z and the temperature fluctuation T') (you are NOT supposed to demonstrate them)

$$\frac{\partial \Delta u'_z}{\partial t} = Pr\Delta^2 u'_z + Pr\Delta_{\parallel} T', \quad (6)$$

$$\frac{\partial T'}{\partial t} = Ra u'_z + \Delta T', \quad (7)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\Delta_{\parallel} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and where $Ra = \frac{\alpha g d^3 (T^{(b)} - T^{(a)})}{\kappa\nu}$ and $Pr = \frac{\nu}{\kappa}$.
A normal mode decomposition exploiting the invariance of the base flow with x, y and t $\exp(i(k_x x + k_y y - \omega t)$ is then introduced with k_x and k_y the horizontal wave-numbers and ω the frequency. Noting $k^2 = k_x^2 + k_y^2$, $D = \frac{d}{dz}$ and using the hat ($\hat{}$) to designate z-dependent variables, this yields

$$-i\omega(D^2 - k^2)\hat{u}_z = Pr(D^4 - 2k^2 D^2 + k^4)\hat{u}_z - Prk^2\hat{T}, \quad (8)$$

$$-i\omega\hat{T} = Ra\hat{u}_z + (D^2 - k^2)\hat{T}, \quad (9)$$

For a perturbation with $k_x = 4\pi/h$, $k_y = 0$ and $\omega = 1$, what are the corresponding wavelengths in x and y and temporal period ?

– 5. [1pts]
When considering a free surface at the top boundary (in contrast to the two situations analyzed in class), both the temperature boundary condition and the dynamic/kinematic boundary condition have to be changed at the upper interface. Let us first focus on the dynamic/kinematic boundary condition.
In class, we have considered two pairs of boundary conditions on the upper and lower surfaces, free-stress on both sides and no-slip on both sides. Let us now consider the situation of mixed boundary conditions. We impose no slip at the bottom wall and free-stress at the upper one. Do you expect the instability threshold to differ if no-slip is enforced at the upper boundary and free-stress at the bottom one ? Why ?

– 6. [1pts]
The code used in class has been reported here. It computes the growth-rate ω_i as a function of the Rayleigh number and the perturbation wavenumber k.

```
1. Pr=1;     % Prandtl number
2. Ra=669; % Rayleigh number

%% Finite-difference operators:
%% n=1 is z=0+dz and n=N is z=1-dz

3. N=80;       %number interior points
4. dz=1/(N+1); % grid size
5. Z=zeros(N); % zero matrix
6. I=eye(N);   % identity matrix

% Second derivative operator

7. D2=-2*eye(N);
8. D2=full(spdiags(ones(N),1,D2));
9. D2=full(spdiags(ones(N),-1,D2));
10. D2=D2/dz^2;

% Fourth derivative operator

11. D4=6*eye(N);
12. D4=full(spdiags(-4*ones(N),1,D4));
13. D4=full(spdiags(-4*ones(N),-1,D4));
14. D4=full(spdiags(ones(N),2,D4));
15. D4=full(spdiags(ones(N),-2,D4));

%% Boundary conditions

%%rigid
16. D4(1,1)=7;
17. D4(N,N)=7;

%%stress free
%18. D4(1,1)=5;
%19. D4(N,N)=5;

20. D4=D4/dz^4;

%% Eigenvalue problem:  s M phi = L phi

%    [ D^2-k^2 I ; Z ] [uz']   [Pr(D^2-k^2 I)^2 ; -Pr k^2 I ] [uz']
%  s |               | |   | = |                            | |   |
%    [    Z      ; I ] [th']   [      Ra I      ; D^2-k^2 I ] [th']

21. kk=[0:0.01:10]; % all k-values

% iterate overa ll k-values
```

```
22. for ik=1:length(kk)
23.    k=kk(ik);
24.    M=[[D2-k^2*I Z];[Z I ]]; % define L operator
25.    L=[[Pr*(D4-2*k^2*D2+k^4*I) -Pr*k^2*I];[Ra*I D2-k^2*I]]; % define M operator
26.    eiv=eig(L,M); % solve the eigenvalue problem
27.    tx(ik)=max(real(eiv)); % store the largest real part of the computed eigenvalue
28. end;

29. figure;
30. hold all;
31. plot(kk,tx);
32. grid on;
33. xlabel('k');
34. ylabel('Im(\omega)');
35. %title(['Ra = ' num2str(Ra) ' , Pr = ' num2str(Pr) ]);
```

This code computes the growth-rate ω_i as a function of the Rayleigh number and the perturbation wavenumber k.

Lines 16 and 17 ensure the boundary conditions. Indeed, since the interior points range from $z_1 = 0 + dz$ to $z_N = 1 - dz$, the endpoints on the bottom $z_0 = 0$ and at the top $z_{N+1} = 1$ have both the velocity $\hat{u}_z$ and the temperature T equal to zero, i.e. $u_0 = 0$, $u_N = 0$, $T_0 = 0$ and $T_N = 0$. To impose a rigid boundary condition at the bottom for instance, one uses the centered second-order finite difference formula with ghost point $z_{-1} = -dz$

$$\left.\frac{d\hat{u}_z}{dz}\right|_{z=z_0} = \frac{u_1 - u_{-1}}{2dz}, \tag{10}$$

which gives

$$u_{-1} = u_1. \tag{11}$$

Therefore the second order derivation formula in z_1 writes

$$\left.\frac{d^2\hat{u}_z}{dz^2}\right|_{z=z_1} = \frac{u_2 - 2u_1 + u_0}{dz^2} = \frac{u_2 - 2u_1}{dz^2} \tag{12}$$

and the fourth order derivative formula in z_1 becomes

$$\left.\frac{d^4\hat{u}_z}{dz^4}\right|_{z=z_1} = \frac{u_3 - 4u_2 + 6u_1 - 4u_0 + u_{-1}}{dz^4} = \frac{u_3 - 4u_2 + 7u_1}{dz^4}, \tag{13}$$

as seen in lines 16 and 17.

Propose a single modification to determine the stability bounds of the mixed boundary conditions fluid system.

– 7. [1pts]

The figure generated by the modified code has been reported in figure 2. Can you read out the resulting instability threshold ? What the is the approximate critical wavenumber at threshold ? Complete the table and comment.

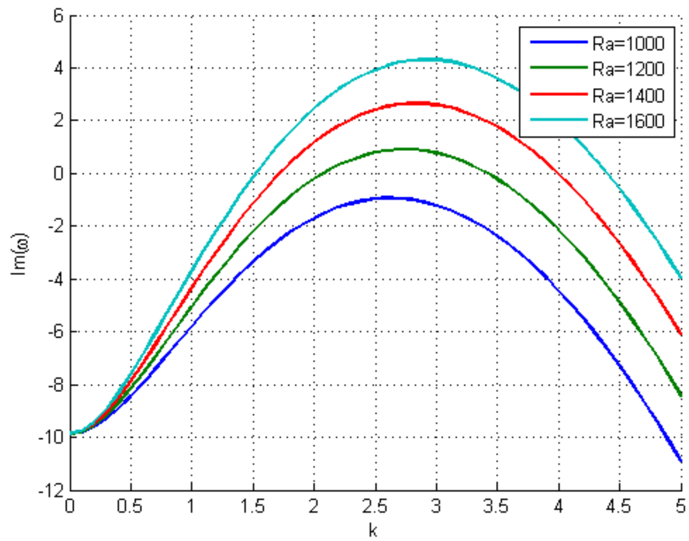

Figure 2 – Dispersion relations for different Ra numbers and mixed dynamic/kinematic boundary conditions

Boundary conditions	Critical Rayleigh number	Critical wavenumber
rigid/rigid	1707	3.16
free/free	657	2.22
rigid/free	?	?

Table 1 – Critical parameters for convection

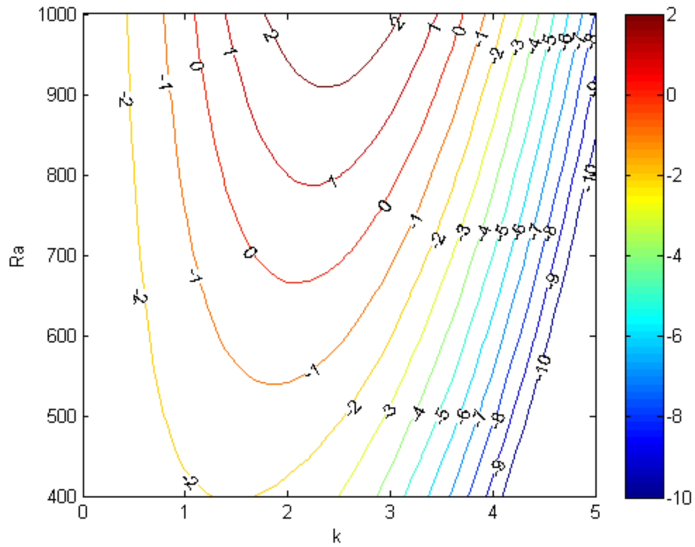

FIGURE 3 – Isocontours of growth-rate in the k-Ra plane for mixed dynamic/kinematic boundary conditions and no thermal flux perturbation on the free surface

– 8. [1pts]
  In reality, the Dirichlet boundary conditions for the temperature profile are rather unlikely. It is more reasonable to assume an insulating boundary at the free surface (a Biot number equal to zero)

$$\left.\frac{d\hat{T}}{dz}\right|_{z=1} = 0. \tag{14}$$

  Remembering that the first-order off-centered finite difference approximation of the first derivative writes $(T_{N+1} - T_{N-1})/2dz$, propose a very simple modification of the code to cope with this boundary condition.

– 9. [1pts]
  The figure generated by the modified code has been reported in figure 3. Can you read out the resulting instability threshold ? What the is the critical wavenumber at threshold ?

part II. Pure Bénard Marangoni instability

In this part only, the Archimedes force is neglected and ρ is taken constant. However, the surface tension γ is now taken into account.

– 10. [2pts]
  Which term disappears from equation 7 ? Which line should be modified in the code ? how ?

– 11. [2pts]

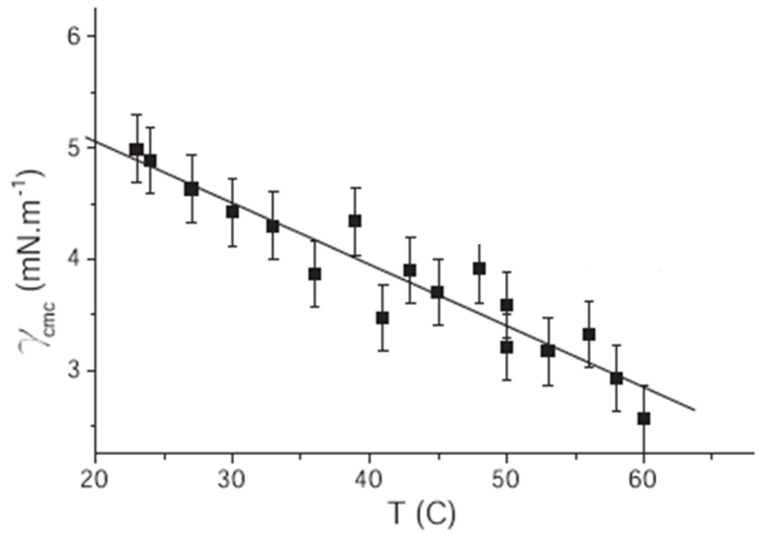

FIGURE 4 – surface tension dependance of water with temperature

In fact the inhomogeneous surface temperature creates strong surface tension variations which in turn create Marangoni stresses, according to the tangential stress discontinuity $\mathbf{t} \cdot \sigma \mathbf{n} = \nabla\gamma \cdot \mathbf{t}$, where σ designates the stress tensor $\sigma = -pI + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$. We further assume that the surface tension γ varies linearly with temperature, according to $\gamma = \gamma^a + \beta(T - T^{(a)})$. From figure 4, deduce an approximate value of β.

– `12. [1pts]`
Is the fluid going to be driven on the surface from the hot region to the cold one or the opposite ? Draw a sketch.

– `13. [1pts]` Show that the Marangoni condition writes

$$\mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \Bigg|_{z=1} = \beta \frac{\partial T}{\partial x} \Bigg|_{z=1} \quad (15)$$

$$\mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \Bigg|_{z=1} = \beta \frac{\partial T}{\partial y} \Bigg|_{z=1} \quad (16)$$

– `14. [1pts]` It is written here on the full variables before perturbation expansion. Why does it directly translate to the perturbations (noted in part I with ') ?

– `15. [2pts]` The nondimensional Marangoni number compares the driving force to the viscous friction force. Using the same characteristic velocity scale than in part I, show that it can be written as

$$Ma = \frac{\beta (T^{(b)} - T^{(a)}) d}{\mu \kappa} \quad (17)$$

– `16. [2pts]` Use incompressibility (what the name of the hypothesis that allows to claim it holds at leading order ?) and non-penetration to show that

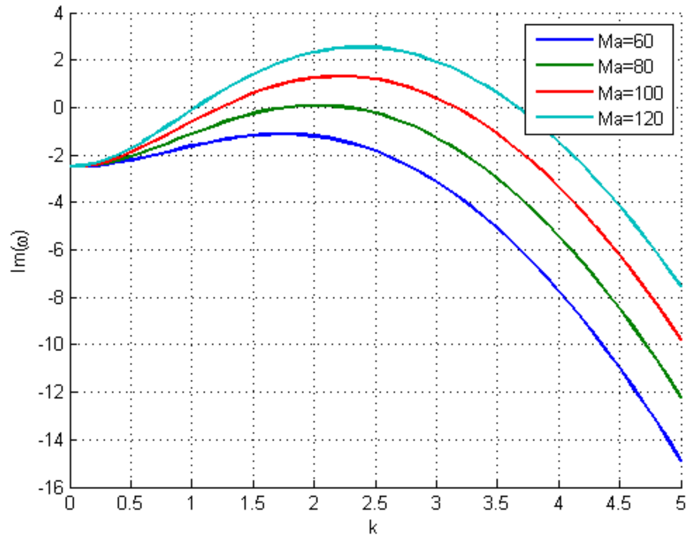

FIGURE 5 – Dispersion relations for different Ma numbers.

$$-\left.\frac{\partial^2 u'_z}{\partial z^2}\right|_{z=1} = Ma\Delta_{\parallel}T'. \tag{18}$$

– `17. [1pts]` Use the normal mode expansion to express this condition in "hat" ($\hat{}$) variables.

– `18. [2pts]` In matlab, the value of the ghost point u_{N+2} which was equal to u_N because of the condition of the $D^2u'_z|_{z=1} = 0$ becomes now $u_{N+2} + u_N = -Mak^2T_Ndz^2$. Explain why and add a single correction to the code to determine the eigenvalues.

– `19. [1pts]`the results of the above code are reprinted in figure 5. What is the critical value of the Ma number Ma_c for instability to happen.

– `20. [2pts]` Taking the values for Ma_c and Ra_c obtained in part I, deduce that there is a critical fluid layer size below which the Marangoni-Bénard sets in before the Rayleigh-Bénard instability. For water, it is typically 2cm and a little bit less for oil. Comment this result in light of everyday's life experience.

Free surface deformation

– `21. [2pts]` We now add a surface deformation to this system $\eta = 0+\epsilon\hat{\eta}\exp(i(k_xx+k_yy-\omega t)$. How are the boundary conditions modified to take this into account ?